AN INFINITE SERIES INVOLVING THE PRODUCT OF BESSEL FUNCTIONS AND GENERALISED LAGUERRE POLYNOMIALS

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THE object of this paper is to show how the method of operational calculus can be used to obtain the sum of an infinite series involving the product of a Bessel function and a generalised Laguerre polynomial.

In the well-known generating function for $L_n^{\alpha}(y)$, the generalised Laguerre polynomial,

$$(yt)^{-\frac{1}{2}\alpha} J_\alpha (2\sqrt{yt}) = \sum_{n=0}^{\infty} \frac{t^n e^{-t}}{\Gamma(\alpha+n+1)} L_n^{\alpha}(y)$$

put

$$\sqrt{t} = -\frac{1}{\sqrt{p}}$$

where p is a symbolic operator given by the relation

$$\frac{1}{p^n} \doteqdot \frac{x^n}{\Gamma(n+1)}. \qquad R(n+1) > 0$$

We then get

$$\left(\frac{y}{p}\right)^{-\frac{1}{2}\alpha} J_\alpha\left(-2\sqrt{\frac{y}{p}}\right) = \sum_{n=0}^{\infty} \frac{p^{-n} e^{-\frac{1}{p}}}{\Gamma(\alpha+n+1)} L_n^{\alpha}(y). \tag{1}$$

Now Humbert* has shown that

$$p^{-\frac{2m-n}{2}} J_n\left(-2\sqrt{\frac{1}{p}}\right) \doteqdot x^{\frac{2m-n}{3}} J_{m,n}(3\sqrt[3]{x}),$$

where the function $J_{m,n}(x)$ is defined by the relation

$$J_{m,n}(x) = \frac{(x/3)^{m+n}}{\Gamma(m+1)\,\Gamma(n+1)}\; {}_0F_2\left(m+1, n+1, -\frac{x^3}{27}\right).$$

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* P. Humbert: "Les fonctions de Bessel du troisieme ordre," *Atti Pont. Accad. delle Scienze*, anno **83**, Sess. III, del Febbraio 1930, 128–146, and "Nouvelles remarques sur les Fonctions de Bessel du troisieme ordre," *ibid.*, anno **87**, Sess. IV del 18, Marzo 1934, 323–331.

In this if we set $m=0$ and $n=\alpha$ and use the known theorem* of operational calculus, that if

$$\phi(p) \doteq f(x),$$

then

$$\phi\left(\frac{p}{s}\right) \doteq f(sx) \qquad s=\text{const.} \neq 0$$

we have

$$\left(\frac{y}{p}\right)^{-\frac{1}{2}\alpha} J_\alpha\left(-2\sqrt{\frac{y}{p}}\right) \doteq (xy)^{-\frac{1}{3}\alpha} J_{0,\alpha}(3\sqrt[3]{xy}) \qquad (2)$$

which gives the original of the left-hand side of (1).

Term by term interpretation of the right-hand side of (1), by the help of the operational relation†

$$p^{-n} e^{-\frac{1}{p}} \doteq x^{\frac{1}{2}n} J_n(2\sqrt{x}),$$

gives that the original of the right-hand side is

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha+n+1)} x^{\frac{1}{2}n} J_n(2\sqrt{x}) L_n^\alpha(y). \qquad (3)$$

Hence by virtue of (2) and (3), we deduce from (1) the result

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha+n+1)} x^{\frac{1}{2}n} J_n(2\sqrt{x}) L_n^\alpha(y) = (xy)^{-\frac{1}{3}\alpha} J_{0,\alpha}(3\sqrt[3]{xy}).$$

This general result is easily verified by taking $y=0$. Since

$$L_n^\alpha(0) = \frac{\Gamma(\alpha+n+1)}{n!\,\Gamma(\alpha+1)},$$

we obtain‡ the known result

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}n} J_n(2\sqrt{x})}{n!} = 1$$

We now give some special interesting cases of our general formula.

Case 1.—For $\alpha=0$, $L_n^\alpha(y)$ reduces to ordinary Laguerre polynomial $L_n(y)$, hence

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^{\frac{1}{2}n} J_n(2\sqrt{x}) L_n(y) = J_{0,0}(3\sqrt[3]{xy})$$

* Carson: *Electric Circuit Theory and the Operational Calculus*, McGraw Hill, New York, 1926.

† B. Van der Pol: "On the operational solution of linear differential equation and an investigation of the properties of these solutions," *Phil. Mag.*, 1929, 8. 861–98.

‡ Watson: *Bessel Functions*, Cambridge University Press, 1922, 141.

Case 2.—Put $\alpha = 1$. Since

$$L_n^1(y) = (-)^n (n+1) e^{y/2} y^{-1} k_{2n+2}(y/2),$$

where $k_n(x)$ denotes Bateman Function, we have

$$\sum_{n=0}^{\infty} \frac{(-)^n}{n!} x^{\frac{1}{2}n} J_n(2\sqrt{x})\, k_{2n+2}(y/2) = y\, e^{-y/2} (xy)^{-\frac{1}{3}} J_{0,1}(3\sqrt[3]{xy}).$$

Case 3.—The generalised Laguerre polynomials are connected with Weber's parabolic cylinder functions by the relations

$$L_n^{-\frac{1}{2}}(y) = \frac{(-)^n}{2^n\, n!} e^{\frac{1}{2}y} D_{2n}(\sqrt{2y})$$

and

$$L_n^{\frac{1}{2}}(y) = \frac{(-)^n}{2^{n+\frac{1}{2}}\, n!} y^{-\frac{1}{2}} e^{\frac{1}{2}y} D_{2n+1}(\sqrt{2y}).$$

Hence on using the duplication formula

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\, \Gamma(n+\tfrac{1}{2}),$$

we get for $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$,

$$\sum_{n=0}^{\infty} \frac{(-2\sqrt{x})^n}{(2n)!} J_n(2\sqrt{x})\, D_{2n}(\sqrt{2y}) = \sqrt{\pi}\, e^{-\frac{1}{2}y} (xy)^{\frac{1}{6}} J_{0,-\frac{1}{2}}(3\sqrt[3]{xy})$$

and

$$\sum_{n=0}^{\infty} \frac{(-2\sqrt{x})^n}{(2n+1)!} J_n(2\sqrt{x})\, D_{2n+1}(\sqrt{2y})$$

$$= \sqrt{\frac{\pi}{2}}\, y^{\frac{1}{2}} e^{-\frac{1}{2}y} \left(xy\right)^{-\frac{1}{6}} J_{0,\frac{1}{2}}(3\sqrt[3]{xy}).$$

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