ON THE POLYNOMIAL $\pi_n(x)$

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The polynomial $\pi_n(x)$, suggested by A. Angelescu,[1] is defined by

$$\pi_n(x) = e^x \frac{d^n}{dx^n}\left[e^{-x} A_n(x)\right]$$

where

$$A_n(x) = (a_0, a_1, a_2, \cdots, a_n, x, 1)^n.$$

In a recent paper Mr. B. S. Sastry[2] has shown, among other results, that the polynomial $\pi_n(x)$ is the coefficient of $t^n/n!$ in the expansion of

$$\frac{1}{1-t}\, e^{-\frac{xt}{1-t}}\, \phi\left(-\frac{t}{1-t}\right)$$

where $\phi(z) = a_0 + \frac{a_1 z}{\underline{|1}} + \frac{a_2 z^2}{\underline{|2}} + \cdots$, *i.e.*, that

$$\frac{1}{1-t}\, e^{-\frac{xt}{1-t}}\, \phi\left(-\frac{t}{1-t}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \pi_n(x) \qquad (1)$$

a formula resembling

$$\frac{1}{1-t}\, e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n(x) \qquad (2)$$

which is the well-known generating function for the Laguerre polynomial $L_n(x)$. The object of this note is to investigate the following relations between the polynomials $\pi_n(x)$ and $L_n(x)$:

$$\text{(A)} \quad \sum_{r=0}^{n} \binom{n}{r} L_r(x) \frac{d^p}{dx^p}\left\{\pi_{n-r}(x)\right\} = \sum_{r=0}^{n} \binom{n}{r} \pi_r(x) \frac{d^p}{dx^p}\left\{L_{n-r}(x)\right\}$$

and

$$0 < p \leqslant n$$

$$\text{(B)} \quad \frac{\pi_n(u+v)}{n!} = \sum_{r=0}^{n} \frac{\pi_r(u)}{r!} \frac{L_{n-r}(v)}{(n-r)!} - \sum_{r=0}^{n-1} \frac{\pi_r(u)}{r!} \frac{L_{n-r-1}(v)}{(n-r-1)!}.$$

The result (A) is of a reciprocal nature while the second result may be taken as the addition theorem for the polynomial $\pi_n(x)$.

To prove the first result, we note that the relation (1), in consequence of (2), can be written in the form

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \pi_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n(x)\, \phi\left(-\frac{t}{1-t}\right). \tag{3}$$

Differentiating both sides of this p times with respect to x, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{d^p}{dx^p}\left\{\pi_n(x)\right\} = \phi\left(-\frac{t}{1-t}\right) \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{d^p}{dx^p}\left\{L_n(x)\right\}$$

which by virtue of (3) gives that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n(x) \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{d^p}{dx^p}\left\{\pi_n(x)\right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \pi_n(x) \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \frac{d^p}{dx^p}\left\{L_n(x)\right\} \tag{4}$$

If now we pick up the coefficients of t^n from either side of (4), we at once obtain the relation (A).

To establish the second result, write $u+v$ for x in (1). We have then

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \pi_n(u+v) = \frac{1}{1-t}\, e^{-\frac{ut}{1-t}}\, \phi\left(-\frac{t}{1-t}\right) e^{-\frac{vt}{1-t}}$$

$$= (1-t) \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \pi_n(u) \sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n(v),$$

where we have used (1) and (2). The result (B) follows by equating the coefficients of t^n on either side.

REFERENCES

1. *C. R. Acad. Sci. Roum.*, 1938, **2**, 199–201 ; *vide* also *Zbl. f. Math.*, 1938, **18**, 181, 356. I have not seen the original article of A. Angelescu.

2. *Proc. Ind. Acad. Sci.*, 1939, **10**, 176–80.