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A generalization of the Enright-Varadarajan Modules

R. Parthasarathy

For a semisimple Lie group admitting discrete series Enright and Varadarajan have constructed a class of modules. Denoted $D_{P,\lambda}$ (cf. [3]). Their infinitesimal description based on the theory of Verma modules parallels that of finite dimensional irreducible modules. The introduction of the modules $D_{P,\lambda}$ in [3] was primarily to give an infinitesimal characterization of discrete series but we feel that [3] my well be a starting point for a fresh approach towards dealing with the problem of classification of irreducible representations of a general semisimple Lie algebra.

In order to give more momentum to such an approach we first construct modules which broadly generalize those in [3]. We briefly describe them now.

Let $\mathfrak{g}_0$ be any real semisimple Lie algebra, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition and θ the associated Cartan involution. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the complexification. Let $U(\mathfrak{g}), U(\mathfrak{k})$ be the enveloping algebras of $\mathfrak{g}, \mathfrak{k}$ respectively and let $U^{\mathfrak{k}}$ be the centralizer of k in $U(\mathfrak{g})$. For each θ stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ we associate in this paper a class of irreducible $\mathfrak{k}$ finite $U(\mathfrak{g})$ modules having the following property: Like finite dimensional irreducible modules and like the Enright-Varadarajan modules $D_{P,\lambda}$, any member of this class comes with a special irreducible $\mathfrak{k}$-type occurring in it with multiplicity one, with an explicit description of the action of $U^{\mathfrak{k}}$ on the corresponding isotypical $\mathfrak{k}$-type. We obtain these modules by extending the techniques in [3].

To see in what way these modules are related to the θ invariant parabolic subalgebra $\mathfrak{q}$ we refer the reader to §2.

When our parabolic subalgebra $\mathfrak{q}$ is minimal in $\mathfrak{g}$ and when rank of $\mathfrak{g}$ = rank of $\mathfrak{k}$, the calsss of $U(\mathfrak{g})$ modules which we associate to this $\mathfrak{q}$ coincides with the class of modules $D_{P,\lambda}$ of [3] (with a slight difference in parametrization). On the other hand when $\mathfrak{q} = \mathfrak{g}$ is the maximal parabolic subalgebra, the class we obtain is just the class of all finite dimensional irreducible representations of $\mathfrak{g}$. If $\mathfrak{k}$ has trivial center, the trivial one dimensional $U(\mathfrak{g})$ module is not equivalent to any of the modules $D_{P,\lambda}$ of [3]. This gap is bridged by the introduction of our class of $U(\mathfrak{g})$ modules for every intermediate θ invariant parabolic subalgebra $\mathfrak{q}$ between $\mathfrak{q} = \mathfrak{g}$ and $\mathfrak{q} =$ a θ invariant Borel subalgebra of $\mathfrak{g}$.

We have to point out that the knowledge of [3] is a necessary prerequisite to read this paper. If an argument or construction needed at some stage of this paper is parallel to that in [3] then instead of repeating them, we simply refer to [3].

§1. θ-stable parabolic subalgebras

As in the introduction, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the complexified Cartan decomposition arising from a real one $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Let θ be the Cartan involution. Let $\mathfrak{b}$ the complexification of a fixed Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{k}_0$. Then the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$ is a θ stable Cartan subalgebra $\mathfrak{h}$, of $\mathfrak{g}$. We can write

$$\mathfrak{h} = \mathfrak{b} + \mathfrak{a} \tag{1.1.}$$

where $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{h}$. Let $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$. Let $\triangle$ be set of roots of $(\mathfrak{g}, \mathfrak{h})$. For α in $\triangle$, denote by $\mathfrak{g}^\alpha$ the corresponding root space.

(1.2) Lemma Let $\mathfrak{r}_\mathfrak{k}$ be a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{b}$. Let $\mathfrak{q}$ be a θ stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and assume that $\mathfrak{q}$ contains $\mathfrak{r}_\mathfrak{k}$. Then $\mathfrak{q}$ contains a θ stable Borel subalgebra $\mathfrak{r}\mathfrak{g}$ such that (i) $\mathfrak{h} \subseteq \mathfrak{r}$ and (ii) $\mathfrak{r}_\mathfrak{k} \subseteq \mathfrak{r}$.

Proof Let $\mathfrak{u}$ be the unipotent radical of $\mathfrak{q}$. Define a θ invariant element μ of $\mathfrak{h}^X(=$ $\mathrm{Hom}_\mathbb{C}(\mathfrak{h}), \mathbb{C}))$ by $\mu(H) =$ trace $(\mathrm{ad}(H)\mathfrak{u})$. Let H'_μ in $\mathfrak{h}$ be defined by $\lambda(H'_\mu) = (\lambda, \mu)$ for every λ in $\mathfrak{h}^X$. (Here and in the following the bilinear form is the nondegenerate one induced by the Killing form of $\mathfrak{g}$). Then

$$\theta(H'_\mu) = H'_\mu \text{ so } H'_\mu \in \mathfrak{b}. \tag{1.3}$$

Let

$$\triangle(\mathfrak{q}) = (\alpha \in \triangle \mid \alpha(H'_\mu) \geq 0\}. \tag{1.4}$$

Then one can see that

$$\mathfrak{q} = \mathfrak{h} + \sum_{\alpha \in \triangle(\mathfrak{q})} \mathfrak{g}^a. \tag{1.5}$$

Let $C_{\mathfrak{k}}$ be the open Weyl chamber in $i\mathfrak{b}_0$ for $(\mathfrak{k}, \mathfrak{b})$ defined by the Borel subalgebra $r_{\mathfrak{k}}$. Since we assumed that $r_{\mathfrak{k}} \subseteq \mathfrak{q}$, it follows from 1.5 that

$$H'_\mu \in \overline{C}_{\mathfrak{k}} = \text{the closure of } C_{\mathfrak{k}}. \tag{1.6}$$

Let α be in $\triangle$. If α is identically zero on $\mathfrak{b}$, it would follow that $\mathfrak{b}$ is not maximal abelian in $\mathfrak{k}$. Hence α is not identically zero on $\mathfrak{b}$. Let $C'_{\mathfrak{k}}$ be the open subset of $C_{\mathfrak{k}}$ got by deleting points of $C_{\mathfrak{k}}$ where some α belonging to $\triangle$ vanishes. Then $C'_{\mathfrak{k}}$ is the disjoint union

$$C'_{\mathfrak{k}} = U_{i=1\cdots N} C'_{\mathfrak{k},j} \tag{1.7}$$

of its connected components and one has

$$\overline{C}_{\mathfrak{k}} = U_{i=1\cdots N} \overline{C'}_{\mathfrak{k},j}. \tag{1.8}$$

Choose an index M between 1 and N such that

$$H'_\mu \in \overline{C'}_{\mathfrak{k}.M}. \tag{1.9}$$

Now choose an element X_j in $C'_{\mathfrak{k}.j}$ and consider the weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}d(X_j)$. We now define a Borel subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ by

$$\mathfrak{r}^j = \text{the sum of theeigen spaces for } ad(X_j) \tag{1.10}$$

with nonnegative eigenvalues.
If we define

$$P^j = \{\alpha \in \triangle \mid \alpha(X_j) > 0\} \tag{1.11}$$

then clearly P^j is a positive system of roots in $\triangle$ and $\mathfrak{r}^j = \mathfrak{h} + \sum_{\alpha \in P^j} \mathfrak{g}^\alpha$. Since X_j belongs to $\mathfrak{k}$ clearly both $\mathfrak{r}^j$ and P^j are θ stable. 1.9 implies that for every α in $P^M, \alpha(H'_\mu)$ is nonnegative. Hence from 1.4 and 1.5

$$\mathfrak{r}^M \subseteq \mathfrak{q}. \tag{1.12}$$

Also since X_M belongs to $C_{\mathfrak{k}}$, (1.10) implies that

(1.13) $$\mathfrak{r}_{\mathfrak{k}} \text{ is contained in } \mathfrak{r}^M.$$

(*q.e.d.*)

(1.14) Corollary *Let* $\mathfrak{r}_{\mathfrak{k}}$ *be as in Lemma 1.2. Let* $\mathfrak{r}$ *be a* θ *stable Borel subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{r}_{\mathfrak{k}}$*. Then* $\mathfrak{r}$ *equals one of the* N *Borel subalgebras* $\mathfrak{r}^j$ *of* (1.10).

Proof Since $\mathfrak{r}$ contains $\mathfrak{b}, \mathfrak{r}$ contains a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. $\mathfrak{h}$ is the unique Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. Hence $\mathfrak{r}$ contains $\mathfrak{h}$. In the proof of Lemma 1.2 take $\mathfrak{q} = \mathfrak{r}$. Then it is seen $\mathfrak{r} = \mathfrak{r}^M$. (*q.e.d.*)

Rather than starting with a Borel subalgebra $\mathfrak{r}_{\mathfrak{k}}$ of $\mathfrak{k}$ containing $\mathfrak{b}$, we want to start with an arbitrary θ invariant parabolic subalgebra of $\mathfrak{g}$ and recover the set up in Lemma 1.2. For this we prove the following lemma.

(1.15) Lemma *Let* $\mathfrak{q}$ *be an arbitrary* θ *stable parabolic subalgebra of* $\mathfrak{g}$*. Then* $\mathfrak{q}$ *contains a Borel subalgebra of* $\mathfrak{k}$.

Proof Let $Ad(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$ and Q the parabolic subgroup with Lie algebra $\mathfrak{q}$. Let G^u be the compact form of $Ad(\mathfrak{g})$ with Lie algebra $\mathfrak{k}_0 + i\mathfrak{p}_0$. Note that G^u is θ- stable. It is well known that $G^u \cap Q$ is a compact form of a reductive Levi factor of Q (cf.[8, § 1.2]). But $G^u \cap Q$ is θ stable since G^u and Q are θ stable. Thus, going to the Lie algebra level, $\mathfrak{q}$ has a reductive Levi supplement which is θ stable. In this reductive Levi supplement we can surely find some θ stable Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$. Then, as in the proof of Lemma 1.2, we can find an element H'_μ in $\mathfrak{h}'$ such that $\theta(H'_\mu) = H'_\mu$ and such that $\mathfrak{q}$ is the sum of the nonnegative eigenspaces of $ad(H'_\mu)$. Since H'_μ lies in $\mathfrak{h}' \cap \mathfrak{k}$, clearly it follows that $\mathfrak{q}$ contains a Borel subalgebra of $\mathfrak{k}$. (*q.e.d*)

(1.16) Corollary *Let* $\mathfrak{r}$ *by any* θ *stable Borel subalgebra of* $\mathfrak{g}$*. Then* $\mathfrak{r} \cap \mathfrak{k}$ *is a Borel subalgebra of* $\mathfrak{k}$.

§2. The objects $\mathfrak{r}, \mathfrak{r}', PP'$ and the choice of P'' associated with a θ stable parabolic subalgerba $\mathfrak{q}$

Now let $\mathfrak{q}$ be a θ stable parabolic subalgebra of $\mathfrak{g}$. By (1.15) we can find a Borel subalgebra $\mathfrak{r}_\mathfrak{k}$ of $\mathfrak{k}$ contained in $\mathfrak{q}$. We fix a Cartan subalgebra $\mathfrak{b}_0$ of $\mathfrak{k}_0$ contained in $\mathfrak{r}_\mathfrak{k}$. Let $\mathfrak{a}_0$ be the centralizer of $\mathfrak{b}_0$ in $\mathfrak{p}_0$. Then $\mathfrak{h}_0 = \mathfrak{b}_0 + \mathfrak{a}_0$ is a θ stable Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{h} = \mathfrak{b} + \mathfrak{a}$ be its complexification. Note that $\mathfrak{h} \subseteq \mathfrak{q}$. By (1.12), we can find a θ stable Borel subalgebra r of $\mathfrak{g}$ such that $\mathfrak{r}_\mathfrak{k} \subset \mathfrak{r}$ and $\mathfrak{r} \subset \mathfrak{q}$. One has then $\mathfrak{h} \subset \mathfrak{r}$. There is a unique Borel subalgebra $\mathfrak{r}'$ of $\mathfrak{g}$ contained in $\mathfrak{q}$ such that

(2.1) $$\mathfrak{r} \cap \mathfrak{r}' = \mathfrak{h} + \mathfrak{u} \text{ where } \mathfrak{u} \text{ is the unipotent radical of} \mathfrak{q}.$$

Since $\theta(\mathfrak{r}')$ has the same property, we have $\theta(\mathfrak{r}') = \mathfrak{r}'$. Let $\mathfrak{r}'_k = \mathfrak{r}' \cap \mathfrak{k}$. Then by (1.16), $\mathfrak{r}'_\mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. We observe that $\mathfrak{r}'_\mathfrak{k}$ is the unique Borel subalgebra of $\mathfrak{k}$ such that

(2.2) $$\mathfrak{r}_\mathfrak{k} \cap \mathfrak{r}'_\mathfrak{k} = \mathfrak{b} + \mathfrak{u}_\mathfrak{k} \text{ where } \mathfrak{u}_\mathfrak{k} \text{ is the unipotent radical of } \mathfrak{q}_\mathfrak{k}(= \mathfrak{q} \cap \mathfrak{k}).$$

We denote by $W_\mathfrak{k}$ the Weyl group of $(\mathfrak{k}, \mathfrak{b})$ and by $W_\mathfrak{g}$ the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. $W_\mathfrak{k}$ is naturally embedded in $W_\mathfrak{g}$ as follows. If s belongs to $W_\mathfrak{k}$ then s normalizes $\mathfrak{b}$, hence also normalizes the centralizer of $\mathfrak{b}$ in $\mathfrak{g}$ which is precisely $\mathfrak{h}$. Thus s belongs to $W_\mathfrak{g}$.

We will now define two distinguished elements of the Weyl group $W_\mathfrak{k}$. Let t be the unique element of $W_\mathfrak{k}$ such that $t(P_\mathfrak{k}) = -P_\mathfrak{k}$. Next we denote by τ the unique element of the Weyl group $W_\mathfrak{k}$ such that $\tau(P_\mathfrak{k}) = P'_\mathfrak{k}$. The class of $U(\mathfrak{g})$ modules associated to $\mathfrak{q}$ will be parametrized by some subsets of h^X. We now prepare to describe these. Let $\triangle_\mathfrak{k}$ be the set of roots for $(\mathfrak{k}, \mathfrak{b})$. Whenever possible we will denote elements of $\triangle_\mathfrak{k}$ by φ while elements of $\triangle$(= the roots of $(\mathfrak{g}, \mathfrak{h})$) will be denoted by α. For a root φ in $\triangle_\mathfrak{k}$, denote by X_φ a nonzero root vector in $\mathfrak{k}$ of weight φ. For α in $\triangle$, we denote by E_α a nonzero root vector in $\mathfrak{g}$ of weight α. Let P and P' be the sets of positive roots in $\triangle$ defined respectively by $\mathfrak{r}$ and $\mathfrak{r}'$. Next let $P_\mathfrak{k}$ and $P'_\mathfrak{k}$ be the sets of positive roots in $\triangle_\mathfrak{k}$ defined respectively by $\mathfrak{r}_\mathfrak{k}$ and $P'_\mathfrak{k}$. Let δ and δ' denote half the sum of the roots in P and P' respectively and let $\delta_\mathfrak{k}$ and $\delta'_\mathfrak{k}$ denote half the sum of the roots in $P_\mathfrak{k}$ and $P'_\mathfrak{k}$ respectively.

Let P'' be a θ stable positive system of roots in $\triangle$ such that if $\mathfrak{r}''$ is the corresponding θ stable Borel subalgebra of $\mathfrak{g}$ then

(2.3) $$\mathfrak{r}'' \supseteq \mathfrak{r}'_{\mathfrak{k}} \text{ and}$$

(2.4) $$P'' \supseteq P' \cap -P.$$

(2.5) Remark If one takes $P'' = P'$ then (2.3) and (2.4) are clearly satisfied. If $\mathfrak{q}$ is a Borel subalgebra then $P' = P$ and P'' which satisfies (2.3) also satisfies (2.4). If $\mathfrak{q} = \mathfrak{g}$, then $P' = -P$; the only candidate which satisfies (2.3) and (2.4) is $P'' = P'$.

We can now describe the modules that we want to construct. As usual for α in P denote by H_α the element of $i\mathfrak{b}_0 + \mathfrak{a}_0$ such that $\lambda(H_\alpha) = 2(\lambda, \alpha)/(\alpha, \alpha)$ for every λ in $\mathfrak{h}^X$. Similarly for φ in $P_\mathfrak{k}$, denote by $H^\mathfrak{k}_\varphi$ the element of $i\mathfrak{b}_0$ such that $\lambda(H^\mathfrak{k}_\varphi) = 2(\lambda, \varphi)/(\varphi, \varphi)$ for every λ in $\mathfrak{b}^X$. (Note: The Killing form of $\mathfrak{g}$ induces a nondegenerate bilinear form on $\mathfrak{b}$ which in turn induces one on $\mathfrak{b}^X$).

Let $F(P'' : \mathfrak{q}, \mathfrak{r})$ be the set of all elements μ in $\mathfrak{h}^X$ with the following properties:

(2.6) $\mu(H_\alpha)$ is a nonnegative integer for every α in P''.

(2.7) $\mu(H^\mathfrak{k}_\varphi)$ is nonzero for every φ in $P_\mathfrak{k}$ and $\mu(H_\varphi)$ is nonzero for every α in $P \cap -P'$.

Example Suppose μ belonging to h^X is such that $\mu(H_\alpha)$ is a positive integer for every α in P''. Then one can show that μ belongs to $F(P'' : \mathfrak{q}, \mathfrak{r})$. The method of showing that $\mu(H^\mathfrak{k}_\varphi)$ is nonzero for every φ in $P_\mathfrak{k}$ can be found in the proof of (3.6).

We now use some definitions and notations from [3, §§ 2,5] (cf. also §§ 3,5 here). Let $U^\mathfrak{k}$ be the centralizer of $\mathfrak{k}$ in $U(\mathfrak{g})$. Let $\mu \in F(P'' : \mathfrak{q}, \mathfrak{r})$. Our aim is to construct a $\mathfrak{k}$-finite irreducible $U(\mathfrak{g})$ module, denoted $D_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ in which the irreducible $\mathfrak{k}$ type with highest weight $-t(\tau\mu + \tau\delta - \tau\delta_\mathfrak{k} - \delta_\mathfrak{k})$ (cf. 3.7) occurs with multiplicity one and such that on the corresponding isotypical $U(\mathfrak{k})$ submodule, elements of $U^\mathfrak{k}$ act by scalars given by the homomorphism $\chi_{P,-\mu-\delta}$ (cf. § 5).

(2.8) Remark Fix $\mathfrak{q}$ and $\mathfrak{r}$. For any compatible choice of P'' and for any element μ in $F(P'' : \mathfrak{q}, \mathfrak{r})$, we will show (cf.3.6) that (i) $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every α in $P \cap -P'$ and (ii) $\tau\mu + \tau\delta - \tau\delta_\mathfrak{k} - \delta_\mathfrak{k}(H^\mathfrak{k}_\varphi)$ is a nonnegative integer for every φ in $P_\mathfrak{k}$. Now define $\overline{F}(\mathfrak{q}, \mathfrak{r})$ to consist of all μ in $\mathfrak{h}^X$ satisfying (i) and (ii) above. In general $\overline{F}(\mathfrak{q}, \mathfrak{r})$ properly contains $\cup_{P''} F(P'' : \mathfrak{q} : \mathfrak{r})$. Our constructions and proofs in §§ 3,4,5 go through perfectly well for any μ in $\overline{F}(\mathfrak{q},\mathfrak{r})$ and so we do have a $\mathfrak{k}$-finite irreducible $U(\mathfrak{g})$ module in which the irreducible $\mathfrak{k}$ type with highest weight $-t(\tau\mu+\tau\delta-\tau\delta_{\mathfrak{k}})$ occurs with multiplicity one and such that on the corresponding isotypical $U(\mathfrak{k})$ submodule elements of $U^{\mathfrak{k}}$ act by scalars given by $\chi_{P,-\mu-\delta}$. We have restricted ourselves to the subsets $F(P'' : \mathfrak{q},\mathfrak{r})$ rather than all of $\overline{F}(\mathfrak{q},\mathfrak{r})$ only because condition (ii) is the definition of $\overline{F}(\mathfrak{q},\mathfrak{r})$ is quite incomprehensible.

§3

Choose and fix an element μ in $F(P'' : \mathfrak{q},\mathfrak{r})$ as in § 2 (cf.(2.6) and (2.7)). For facts about Verma modules that we will be using we refer to [1,2,5,6].

Let M be any $U(\mathfrak{g})$ module. Let Q be a subset of $\triangle_{\mathfrak{k}}$. An element v of M is said to be Q extreme if $X_\varphi.v = 0$ for every φ in Q. For λ in $\mathfrak{b}^X, v$ is called a weight vector of weight λ with respect to $\mathfrak{b}$ if $H \cdot v = \lambda(H) \cdot v$ for all H in $\mathfrak{b}$. By $J(M)$ we denote the set of all λ in $\mathfrak{b}^X$ for which there exists a nonzero weight vector of weight λ in M, which is $P_{\mathfrak{k}}$ extreme where $P_{\mathfrak{k}}$ is the positive system of roots in $\triangle_{\mathfrak{k}}$ defined in §2. For φ in $\triangle_{\mathfrak{k}}, M$ is said to be X_φ free if $X_\varphi \cdot v = 0$ implies $v = 0$. For a subalgebra $\mathfrak{s}$ of $\mathfrak{g}, M$ is said to be $\mathfrak{s}$-finite if every vector of M lies in a finite dimensional $\mathfrak{s}$ submodule of M. For any η in $\pi_{\mathfrak{k}}$ let $m(\eta)$ denote the subalgebra of $\mathfrak{g}$ spanned by the elements $X_\eta, X_{-\eta}$ and H^k_η. For the notion of $U(\mathfrak{k})$ module of 'type $P_{\mathfrak{k}}$' we refer to [3, § 2].

Let P_0 be a positive system of roots of $\triangle$ and let $\lambda \in \mathfrak{h}^X$. The Verma module $V_{\mathfrak{g},P_0,\lambda}$ of $U(\mathfrak{g})$ is defined as follows: It is the quotient of $U(\mathfrak{g})$ by the left ideal generated by the elements $H - \lambda(H), (H \in \mathfrak{h})$ and $E_\alpha(\alpha \in P_0)$. The Verma modules of $U(\mathfrak{k})$ are defined similarly. We will suppress $\mathfrak{g}$ and write $V_{P_0,\lambda}$ for the Verma module $V_{\mathfrak{g},P_0,\lambda}$.

We have the inclusions $\mathfrak{h} \subseteq \mathfrak{r} \subseteq \mathfrak{q}$ (cf. § 2). Let π be the set of simple roots for P. The parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{r}$ are in one to one correspondence with subsets of π. The subset of π corresponding to $\mathfrak{q}$ is got as follows: Let σ in $\mathfrak{h}^X$ be defined by $\sigma(H) =$ trace $(adH) \mid \mathfrak{u}$. Then

$$\pi(\mathfrak{q}) = \{\alpha \in \pi \mid (\sigma,\alpha) = 0\}.$$

From standard facts about parabolic subalgebras (cf. [8, § 1.2]) we know that elements of $P \cap -P'$ are of the form $\sum m_i\alpha_i$ where m_i are nonnegative integers and α_i are in $\pi(\mathfrak{q})$. For α in $\triangle$ the element s_α of

$W_\mathfrak{g}$ is the reflection corresponding to α. It is given by $s_\alpha(\lambda) = \lambda - 2(\lambda,\alpha)/(\alpha,\alpha)\cdot\alpha$. We now define a $U(\mathfrak{g})$ module W_1 by

$$W_1 = V_{P,-\mu-\delta} \tag{3.3}$$

considered as a $U(\mathfrak{k})$ module it has some nice properties.

(3.4) Lemma *W_1 considered as a module for $U(\mathfrak{k})$ is a weight module with respect to $\mathfrak{b}$; i.e. W_1 is the sum of the weight spaces with respect to $\mathfrak{b}$. Denoting also $-\mu-\delta$ the restriction of $-\mu-\delta$ to b, all the weights are of the form $-\mu-\delta-\sum n_i\varphi_i$ where φ_i are elements of P and n_i are positive integers. Finally the weight spaces are finite dimensional and the weight space corresponding to $-\mu-\delta$ is one dimensional.*

Proof Since as a $U(\mathfrak{g})$ module W_1 is the sum of weight spaces with respect to $\mathfrak{h} = \mathfrak{b}+\mathfrak{a}$, the first statement is clear. Since no root α in $\triangle$ is identically zero on $\mathfrak{b}$, we can pick up an element H in $\mathfrak{b}$ such that for every α in $P, \alpha(H)$ is real and positive. As a $U(\mathfrak{g})$ module, the weights of W_1 with respect to $\mathfrak{h}$ are of the form $-\mu-\delta-\sum m_i\alpha_i$ ($\alpha_i \in P, m_i$ nonnegative integers). By considering the action of H it is clear that weight spaces of W_1 with respect to $\mathfrak{b}$ are finite dimensional and the weight space of $\mathfrak{b}$ with weight $-\mu-\delta$ is one dimensional. Finally since P is θ stable the restriction to $\mathfrak{b}$ of the weights with respect to $\mathfrak{h}$ are of the form $-\mu-\delta-\sum n_i\varphi_i$ where φ_i are in P and n_i nonnegative integers.

(q.e.d)

(3.5) Corollary *The $U(k)$ submodule of W_1 generated by the unique weight vector in W_1 of weight $-\mu-\delta$ is isomorphic to the $U(\mathfrak{k})$ Verma module $V_{\mathfrak{k},P_\mathfrak{k},-\mu-\delta}$. W_1 is $X_{-\varphi}$ free for every φ in $P_\mathfrak{k}$.*

Proof Let v_1 be the nonzero weight vector in W_1 of weight $-\mu-\delta$. v_1 is killed by every element of $[\mathfrak{r},\mathfrak{r}]$ hence in particular by every element of $[\mathfrak{r}_\mathfrak{k},\mathfrak{r}_\mathfrak{k}]$. On the other hand let $\bar{\mathfrak{r}}$ be the unique Borel subalgebra of $\mathfrak{g}$ such that $\bar{\mathfrak{r}}\cap\mathfrak{r} = \mathfrak{h}$ and let $\mathfrak{n}(\bar{\mathfrak{r}})$ be the unipotent radical of $\bar{\mathfrak{r}}$. If $\bar{\mathfrak{r}}_\mathfrak{k} = \bar{\mathfrak{r}}\cap\mathfrak{k}$, then $\bar{\mathfrak{r}}_\mathfrak{k}$ is the unique Borel subalgebra of $\mathfrak{k}$ such that $\bar{\mathfrak{r}}_\mathfrak{k}\cap\mathfrak{r}_\mathfrak{k} = \mathfrak{b}$. Let $U(\mathfrak{n}(\bar{\mathfrak{r}}))$ and $U(\mathfrak{n}(\bar{\mathfrak{r}}_\mathfrak{k}))$ denote the corresponding enveloping algebras considered as subalgebra of $U(\mathfrak{g})$. One knows that W_1 is $U(\mathfrak{n}(\bar{\mathfrak{r}}))$ free, [2]. Hence in particular it is $U(\mathfrak{n}(\bar{\mathfrak{r}}_\mathfrak{k}))$ free. The corollary now follows from [2,7.1.8]. *(q.e.d.)*

There is an ascending chain of $U(\mathfrak{k})$ Verma modules containing $V_{\mathfrak{k},P_\mathfrak{k},-\mu-\delta}$. This chain will give rise to a chain of $U(\mathfrak{g})$ modules, which is fundamental in the work [3].

Recall the two distinguished elements t and τ of $W_\mathfrak{k}$ from § 2. The highest weight of the special irreducible representation of $\mathfrak{k}$ which the $U(\mathfrak{g})$ module $D_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ will contain is described in the corollary to the lemma below.

(3.6) Lemma (i) $-\mu-\delta(H_\alpha)$ *is a nonnegative integer for every* α *in* $P\cap -P'$ *and* (ii) $\tau\mu+\tau\delta-\tau\delta_\mathfrak{k}(H^\mathfrak{k}_\varphi)$ *is a nonnegative integer for every* φ *in* $P_\mathfrak{k}$.

Proof By (2.4), (2.7) and (2.8), one sees that $-\mu(H_\alpha)$ is a positive integer for every α in $P\cap -P'$. The elements of $P\cap -P'$ are nonnegative integral linear combination of elements of $\pi(\mathfrak{q})$. Since $\delta(H_\alpha)=1$ for every α in $\pi(\mathfrak{q})$ it now follows that $-\mu-\delta(H_\alpha)$ is a nonnegative integer for every α in $P\cap -P'$.

To prove (ii) first suppose φ lies in $P'_\mathfrak{k}\cap P_\mathfrak{k}$. We will show that $\tau\mu-\delta_\mathfrak{k}(H^\mathfrak{k}_\varphi)$ and $\tau\delta-\tau\delta_\mathfrak{k}(H^\mathfrak{k}_\varphi)$ are both nonnegative integers. For this it is enough to show that $\tau\mu(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $P_\mathfrak{k}$ and that $\tau\delta(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $\tau P_\mathfrak{k}$. By (2.6) there exists a finite dimensional representation of $\mathfrak{g}$ having a weight vector v of weight μ with respect to the Cartan subalgebra $\mathfrak{h}$ and such that v is annihilated by $[\mathfrak{r}'',\mathfrak{r}'']$ (cf. (2.3)). Since $\mathfrak{r}'_\mathfrak{k}\subseteq\mathfrak{r}''$, v is in particular annihilated by $[\mathfrak{r}'_\mathfrak{k},\mathfrak{r}'_\mathfrak{k}]$. It is clear from this that $\mu(H^\mathfrak{k}_\varphi)$ is a nonnegative integer for every φ in $P'_\mathfrak{k}$. In view of (2.7), $\mu(H^\mathfrak{k}_\varphi)$ is then a positive integer for every φ in $P'_\mathfrak{k}$. Note that $\tau P'_\mathfrak{k}=P_\mathfrak{k}$. Hence $\tau\mu(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $P_\mathfrak{k}$. It remains to show that $\tau\delta(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $\tau P_\mathfrak{k}$. For this consider the representation ρ of $\mathfrak{g}$ having a weight vector v of weight δ with respect to the Cartan subalgebra $\mathfrak{h}$ and annihilated by $[\mathfrak{r},\mathfrak{r}]$. Clearly then v is annihilated by $[\mathfrak{r}_\mathfrak{k},\mathfrak{r}_\mathfrak{k}]$, hence $\delta(H^\mathfrak{k}_\varphi)$ is a nonnegative integer for every φ in $P_\mathfrak{k}$. To show that $\delta(H^\mathfrak{k}_\varphi)$ is nonzero we give the following reason: one can easily see that the stabilizer of v in $\mathfrak{g}$ is exactly $\mathfrak{r}$. If $\delta(H^\mathfrak{k}_\varphi)$ is zero for some φ in $P_\mathfrak{k}$, then $X_{-\varphi}$ would stabilize v. But $X_{-\varphi}$ does not belong to $\mathfrak{r}$. Hence $\delta(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $P_\mathfrak{k}$, so that $\tau\delta(H^\mathfrak{k}_\varphi)$ is a positive integer for every φ in $\tau P_\mathfrak{k}$.

Now suppose φ lies in $P_\mathfrak{k}\cap -P'_\mathfrak{k}$. Let $\mathfrak{r}(\mathfrak{q})$ be the maximal reductive subalgebra of $\mathfrak{q}$ defined by $\mathfrak{r}(\mathfrak{q})=\mathfrak{h}+\sum_{\alpha\in P\cap -P'}(\mathfrak{g}^\alpha+\mathfrak{g}^{-\alpha})$. By (ii) $-\mu-\delta(H_\alpha)$ is a nonnegative integer for every α in $P\cap -P'$. Hence, if $\mathfrak{n}_{\mathfrak{r}(\mathfrak{q})}=\sum_{\alpha\in P\cap -P'}\mathfrak{g}^\alpha$, there exists a finite dimensional representation of $\mathfrak{r}(\mathfrak{q})$ and a weight vector for $\mathfrak{h}$ of weight $-\mu-\delta$ annihilated by all of

$\mathfrak{n}_{\mathfrak{r}(\mathfrak{q})}$, hence in particular by $\mathfrak{k} \cap \mathfrak{n}_{\mathfrak{r}(\mathfrak{q})}$. Observe that $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$ is precisely the set of roots in $P_{\mathfrak{k}}$, whose corresponding root spaces span $\mathfrak{k} \cap \mathfrak{n}_{\mathfrak{r}(\mathfrak{q})}$. Thus there exists a finite dimensional representation of $\mathfrak{b} + \sum_{\varphi \in P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}} (\mathbb{C} \cdot X_\varphi + \mathbb{C} \cdot X_\varphi)$ with a weight vector for $\mathfrak{b}$ of weight $-\mu - \delta$ annihilated by X_φ for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$. Hence we conclude that $-\mu - \delta(H^{\mathfrak{k}}_\varphi)$ is a nonnegative integer for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$. Since $-\tau(P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}) = P_{\mathfrak{k}} \cap P'_{\mathfrak{k}}, \tau(\mu + \delta)(H^{\mathfrak{k}}_\varphi)$ is a nonnegative integer for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$. On the other hand $\tau\delta_{\mathfrak{k}} = \delta'_{\mathfrak{k}} =$ half the sum of the roots in $P'_{\mathfrak{k}}$, while $\delta_{\mathfrak{k}} + \delta'_{\mathfrak{k}}(H^{\mathfrak{k}}_\varphi) = 0$ for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$. Thus $\tau\mu + \tau\delta - \tau\delta_{\mathfrak{k}} - \delta_{\mathfrak{k}}(H^{\mathfrak{k}}_\varphi)$ is a nonnegative integer for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$.

This completes the proof of (3.6). *(q.e.d.)*

(3.7) Corollary $-t(\tau\mu + \tau\delta - \tau\delta_{\mathfrak{k}} - \delta_{\mathfrak{k}})(H^{\mathfrak{k}}_\varphi)$ *is a nonnegative integer for every* φ *in* $P_{\mathfrak{k}}$.

Proof Clear since $-tP_{\mathfrak{k}} = P_{\mathfrak{k}}$. *(q.e.d.)*

Let $\pi_{\mathfrak{k}}$ be the set of simple roots of $P_{\mathfrak{k}}$. For φ in $P_{\mathfrak{k}}$, let s_φ be the reflection $s_\varphi(\lambda) = \lambda - \lambda(H_\varphi)\varphi$ of $\mathfrak{b}^X$. If φ lies in $\pi_{\mathfrak{k}}, s_\varphi$ is called a simple reflection. For w in $W_{\mathfrak{k}}$, the length $N(w)$ of w is the smallest integer N such that w is a product of N simple reflections. A reduced word for w is an expression of w as a product of $N(w)$ simple reflections. Choose any reduced word for the element τt of $W_{\mathfrak{k}}$. Following the notation in [5, §4.15], we write it as

$$\tau t = s_1 s_2 \cdots s_m \tag{3.8}$$

where $s_i = s_{\eta_i}, \eta_i = \varphi_{j_i}, \varphi_{j_i} \in \pi_{\mathfrak{k}}$. For λ in $\mathfrak{b}^X$ and w in $W_{\mathfrak{k}}$ write $w'(\lambda) = w(\lambda + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$. Having chosen the element μ in $F(P'' : \mathfrak{q}, \mathfrak{r})$ we now define elements μ_i of $\mathfrak{b}^X$ as follows:

$$\mu_{m+1} = -t(\tau\mu + \tau\delta - \tau\delta_{\mathfrak{k}} - \delta_{\mathfrak{k}}) \text{ and}$$

$$\mu_i = (s_i s_{i+1} \cdots s_m)' \mu_{m+1} (i = 1, \cdots, m)$$

(3.9) Note that $\mu_1 = (\tau t)' \mu_{m+1} = -\mu - \delta$ and that μ_1 and μ_{m+1} are independent of the reduced expression (3.8). We now define the positive integers e_i by

$$e_i = \mu_{i+1} + \delta_k(H^{\mathfrak{k}}_{\eta_i}) \cdot (i = 1, \cdots, m). \tag{3.10}$$

With μ_i defined as above, the following inclusion relations between Verma modules are well known [2.6]:

$$V_{\mathfrak{k},P_{\mathfrak{k}},\mu_1} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\mu_2} \subseteq \cdots \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\mu_{m+1}}. \tag{3.11}$$

Define elements $v_1, v_2, \cdots, v_{m+1}$ of $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_{m+1}}$ as follows: μ_{m+1} is the unique nonzero weight vector of $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_{m+1}}$ of weight μ_{m+1}. For $i = 1, 2, \cdots, m, v_i = X^{e_i}_{-\eta_i} \cdot v_{i+1}$. Then one knows that v_i is of weight μ_i and that $V_{\mathfrak{k},P_{\mathfrak{k},\mu_i}} = U(\mathfrak{k})v_i$. Associated to the reduced word (3.8) and μ in $F(P'' : \mathfrak{q}, \mathfrak{r})$ is a fundamental chain of $U(\mathfrak{g})$ modules: $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$. It will turn out that W_1 and W_{m+1} are independent of the reduced expression (3.8). They are defined as follows: W_1 is defined to be $V_{P.-\mu-\delta}$ as in (3.3). Then W_{m+1} is given by the following lemma.

(3.12) Lemma *There exists a* $U(\mathfrak{g})$ *module* $W_{m+1} = U(\mathfrak{g}) \cdot v_{m+1}$ *such that* (a) W_1 is a $U(\mathfrak{g})$ *submodule of* W_{m+1}, (b) v_1 *belongs to* $U(\mathfrak{k})v_{m+1}$, (c) v_{m+1} *is a* $P_{\mathfrak{k}}$ *extreme weight vector (with respect to* $\mathfrak{b}$*) of weight* μ_{m+1} (d) W_{m+1} *is* $X_{-\varphi}$ *free for all* φ in $P_{\mathfrak{k}}$ *and* (e) W_{m+1} *is a sum of* $U(\mathfrak{k})$ *submodules of type* $P_{\mathfrak{k}}$.

Proof Start with the conclusion of $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_1}$ in W_1 given by Corollary 3.5 and the inclusion of $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_1}$ in $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_{m+1}}$ given by 3.11. By 3.5 we know that W_1 is $X_{-\varphi}$ free for every φ in $P_{\mathfrak{k}}$. Now [3, Lemma 4] gives us the module W_{m+1} with the properties required in the lemma. (One easily sees that the results of [3 § 2] do not depend on the assumption there that rank of $\mathfrak{g}$ = rank of $\mathfrak{k}$). *(q.e.d)*

(3.13) Remark If V and $\overline{V}$ are Verma modules for, say, $U(\mathfrak{k})$ then the space of $U(\mathfrak{k})$ homomorphisms of V into $\overline{V}$ has dimension equal to zero or one. Thus the inclusion of $V_{\mathfrak{k},P_{\mathfrak{k}},\mu_{m+1}}$ given by (3.11) is independent of the reduced expression (3.8) for τt. Hence also the $U(\mathfrak{g})$ module W_{m+1} and the inclusion of W_1 in W_{m+1} with the properties listed in Lemma 3.12 can be chosen to be independent of the reduced expression (3.8).

Having defined W_1 and W_{m+1} as above, now for any given reduced word for τt such as (3.8), we define submodules $W_2, W_3, \cdots, W_m$ of W_{m+1} by

$$W_i = U(\mathfrak{g})v_i \tag{3.14}$$

where v_i are the elements of W_{m+1} defined after (3.11). We have

$W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ because v_i belongs to $U(\mathfrak{k})v_{i+1}, (i = 1, \cdots, m)$. The properties of this chain of $U(\mathfrak{g})$ modules are summarized below from the work of [3, § 3]:

(3.15) $W_1 = V_{P,-\mu-\delta}$ and each W_i is the sum of its weight spaces with respect to $\mathfrak{b}$. Moreover as a $U(\mathfrak{k})$ module W_i is the sum of $U(\mathfrak{k})$ submodules of type $P_{\mathfrak{k}}$.

(3.16) Each W_i is a cyclic $U(\mathfrak{g})$ module with a cyclic vector v_i, which is a $P_{\mathfrak{k}}$ extreme weight vector of weight μ_i with respect to $\mathfrak{b}, i = 1, \cdots, m+1$.

(3.17) The $P_{\mathfrak{k}}$ extreme vectors of weight μ_i in W_i are scalar multiples of v_i; for $i = 1, \cdots m+1$, the vector v_i does not belong to W_{i-1}.

(3.18) Each W_i is $X_{-\varphi}$ free for every φ in $P_{\mathfrak{k}}$ and W_{i+1}/W_i is $m(\eta_i)$ finite $(i = 1, \cdots, m)$.

(3.19) $v_i = X_{-\eta_i}^{e_i} v_{i+1} (i = 1, \cdots, m)$.

(3.20) Let w be in $W_{\mathfrak{k}}$. Let $i = 1, \cdots, m$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_i)$. Then $N(w)$ equals at least $m + 1 - i$.

We will not prove the properties (3.15) to (3.20) here since they are essentially proved in [3, Lemma 5]. Though (3.20) has the same form as [3, Lemma 5, vi] its proof is different in our case. It is important to first know the case $i = 1$ of (3.20) to carry over the inductive arguments of [3, § 3] to our situation. To this end we prove the following lemma. Before that we make the following remark.

(3.21) Remark Let $H'_{\mathfrak{q}}$ be the element of $\mathfrak{h}$ defined by $(H'_{\mathfrak{q}}, H) =$ trace (ad $H \mid \mathfrak{u}$), for every H belonging to $\mathfrak{h}$, where $\mathfrak{u}$ is the unipotent radical of $\mathfrak{q}$. Since $\mathfrak{q}$ and $\mathfrak{h}$ are θ invariant $\theta(H'_{\mathfrak{q}}) = H'_{\mathfrak{q}}$; hence $H'_{\mathfrak{q}}$ belongs to $\mathfrak{b}$. One can easily prove the following: For every α in $P \cap -P'$, $\alpha(H'_{\varphi})$ equals zero; for every α in $P \cap P', \alpha(H'_{\mathfrak{q}})$ is a positive real number; and for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}, \varphi(H'_{\mathfrak{q}})$ equals zero while for every φ in $P_{\mathfrak{k}} \cap P'_{\mathfrak{k}}, \varphi(H'_{\mathfrak{q}})$ is a positive real number. (Observe that any φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$ is the restriction to $\mathfrak{b}$ of some α in $P \cap -P'$).

Now we come to the lemma which is basic to carry over the inductive arguments of [3, § 3].

(3.22) Lemma *Let w be in $W_{\mathfrak{k}}$. Suppose $w'(\mu_{m+1})$ belongs to $J(W_1)$. Then $N(w)$ is greater than or equal to m.*

Proof Since $w'(\mu_{m+1})$ belongs to $J(W_1)$ it is in particular a weight of W_1 of for $\mathfrak{b}$. Hence by (3.4), $w'(\mu_{m+1})$ is of the form $\mu_1 - \sum n_i\alpha_i \mid \mathfrak{b}$, where n_i are nonnegative integers and α_i are in P. That is $w(\mu_{m+1}+\delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}} = \mu_1 - \sum n_i\alpha_i \mid \mathfrak{b} = \tau t(\mu_{m+1}+\delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}} - \sum n_i\alpha_i\mathfrak{b}$. Thus

$$\tau t(\mu_{m+1}+\delta_{\mathfrak{k}}) - w(\mu_{m+1}+\delta_{\mathfrak{k}}) = \sum n_i\alpha_i \mid \mathfrak{b}.$$

Write $\mu'_{m+1} = -t\mu_{m+1}$. Hence

$$-\tau(\mu'_{m+1}+\delta_{\mathfrak{k}}) + wt(\mu'_{m+1}+\delta_{\mathfrak{k}}) = \sum n_i\alpha_i \mid \mathfrak{b} \tag{3.23}$$

where n_i are nonnegative integers and α_i are in P. The left side of the equality in (3.23) is the sum of $wt(\mu'_{m+1}+\delta_{\mathfrak{k}}) - (\mu'_{m+1}+\delta_{\mathfrak{k}})$ and $(\mu'_{m+1}+\delta_{\mathfrak{k}}) - \tau(\mu'_{m+1}+\delta_{\mathfrak{k}})$. We claim that (3.23) implies

$$P_{\mathfrak{k}} \cap -wtP_{\mathfrak{k}} \text{ is contained in } P_{\mathfrak{k}} \cap -\tau P_{\mathfrak{k}}. \tag{3.24}$$

To see this enumerate the elements of $P_{\mathfrak{k}} \cap -wtP_{\mathfrak{k}}$ in a sequence $(\epsilon_1, \epsilon_2, \cdots, \epsilon_k)$ such that ϵ_1 is a simple root of $P_{\mathfrak{k}}$ and ϵ_{i+1} is a simple root of $s_{\epsilon_i}\cdots s_{\epsilon_1}P_{\mathfrak{k}}(i = 1, \cdots, k-1)$. Then $wt = s_{\epsilon_k}\cdots s_{\epsilon_1}$ (cf. (5, 4.15.10] and [7, 8.9.13]). By induction on i one can show that $(\mu'_{m+1}+\delta_{\mathfrak{k}}) - s_{\epsilon_i}\cdots s_{\epsilon_1}(\mu'_{m+1}+\delta_{\mathfrak{k}})$ can be written as $\sum_{j=1}^{i} d_{j,i}\epsilon_j$ where $d_{j,i}$ are positive integers. Thus $(\mu'_{m+1}+\delta_{\mathfrak{k}}) - wt(\mu'_{m+1}+\delta_{\mathfrak{k}})$ can be written as $d_1\epsilon_1 + d_2\epsilon_2 + \cdots + d_k\epsilon_k$ where d_j are positive integers. Similarly $(\mu'_{m+1}+\delta_{\mathfrak{k}}) - \tau(\mu'_{m+1}+\delta_{\mathfrak{k}})$ can be written as $d'_1\epsilon'_1 + d'_2\epsilon'_2 + \cdots + d'_h\epsilon'_h$ where d'_i are positive integers and $(\epsilon'_1, \cdots, \epsilon'_h)$ is an enumeration of $P_{\mathfrak{k}} \cap -\tau P_{\mathfrak{k}}$. With these observations we can write

$$\begin{aligned} &-\tau(\mu'_{m+1}+\delta_{\mathfrak{k}}) + wt(\mu'_{m+1}+\delta_{\mathfrak{k}}) \\ &= (d'_1\epsilon'_1 + \cdots + d'_h\epsilon'_h) - (d_1\epsilon_1 + \cdots + d_k\epsilon_k) \end{aligned} \tag{3.25}$$

where $d'_1, \cdots, d'_h, d_1, \cdots, d_k$ are positive integers. Let $H'_{\mathfrak{q}}$ be the element of $\mathfrak{h}$ defined by $(H'_{\mathfrak{q}}, H) = \text{trace}(\text{ad } H \mid \mathfrak{u})$, where $\mathfrak{u}$ is the unipotent radical of $\mathfrak{q}$. Then $H'_{\mathfrak{q}}$ belongs to $\mathfrak{b}$. We can apply remark (3.21) to (3.25) and conclude that $[-\tau(\mu'_{m+1}+\delta_{\mathfrak{k}}) + wt(\mu'_{m+1}+\delta_{\mathfrak{k}})](H'_{\mathfrak{q}})$ is a strictly negative real number unless (3.24) holds. But by looking at the right hand side of (3.23) and applying remark (3.21), we see that $[-\tau(\mu'_{m+1}+\delta_{\mathfrak{k}}) + wt(\mu'_{m+1}+\delta_{\mathfrak{k}})](H'_{\mathfrak{q}})$ is a nonnegative real number.

Thus we have proved the validity of (3.24). Now (3.24) implies that $N(wt)$ is less than or equal to $N(\tau)$. But note that $N(wt) = N(t) - N(w)$, while $N(\tau) = N(t) - N(\tau t) = N(t) - m$. Hence $N(w)$ is greater than or equal to m. *(q.e.d.)*

(3.22) Enables us to carry over the inductive arguments in [3, § 3] without any further change and obtain the properties (3.15) to (3.20).

§ 4. The $\mathfrak{k}$-finite quotient $U(\mathfrak{g})$ module of W_{m+1}

.

The difference between the special situation in [3] and our more general situation becomes more apparent in this section which parallels [3, § 4].

Start with an arbitrary reduced word (3.8) for τt and let $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ be a fundamental chain of $U(\mathfrak{g})$ modules satisfying (3.15) through (3.20). Recall $W_1 = V_{P-\mu-\delta}$. Recall the subset $\pi(\mathfrak{q}) \subseteq \pi$ corresponding to the parabolic subalgebra $\mathfrak{q}$. For α in π and λ in $\mathfrak{h}^X$ define $s^X_\alpha(\lambda) = s_\alpha(\lambda + \delta) - \delta$. By lemma 3.6, $-\mu - \delta(H_\alpha)$ is a nonnegative integer for every α in $P \cap -P'$, hence in particular for every α in $\pi(\mathfrak{q})$. Thus one has the inclusion of the Verma modules $V_{P,s^X_\alpha(-\mu-\delta)} \subseteq V_{P,-\mu-\delta}$ for every α in $\pi(\mathfrak{q})$. We now define a $U(\mathfrak{g})$ submodule.

$$W_0 = \sum_{\alpha \in \pi(\mathfrak{q})} V_{P,s^X_\alpha(-\mu-\delta)} \text{ of } W_1. \tag{4.1}$$

As is well known the Verma modules have unique proper maximal submodules. Let I be the proper maximal $U(\mathfrak{g})$ submodule of $V_{P,-\mu-\delta}$.

Then each $V_{P,s^X_\alpha(-\mu-\delta)}, (\alpha \in \pi(\mathfrak{q}))$, is contained in I. Hence

$$v_1 \text{ does not belong to } W_0. \tag{4.2}$$

Now fix some $i, (i = 1, \cdots m)$. Define a $U(\mathfrak{g})$ submodule (relative to some reduced word (3.8) for $\tau t)\overline{W}_i$ of W_{m+1} as follows: Let $W_{i,0}$ be the $U(\mathfrak{g})$ submodule of all vectors in W_{m+1} that are $m(\eta_i)$ finite mod W_{i-1}; once $W_{i,0}, \cdots, W_{i,p-1}$ are defined, $W_{i,p}$ is the $U(\mathfrak{g})$ submodule of all vectors in W_{m+1} that are $m(\eta_{i+p})$ finite mod $W_{i,p-1}, p = 1, 2, \cdots, m-i$. We have $W_{i,0} \subseteq \cdots \subseteq W_{i,m-i}$. We then define $\overline{W}_i = W_{i,m-i}$. Define

$$\overline{W} = W_m + \overline{W}_1 + \overline{W}_2 + \cdots + \overline{W}_m. \tag{4.3}$$

Thus for each reduced expression (3.8) for τt, we have defined a $U(\mathfrak{g})$ submodule $\overline{W}$ of W_{m+1}.

(4.4) Proposition *For any reduced word (3.8) for τt define the $U(\mathfrak{g})$ submodule $\overline{W}$ of W_{m+1} as above. Then v_{m+1} does not belong to $\overline{W}$. If $\lambda \in \mathfrak{b}^X$ is such that W_{m+1} has a nonzero $P_{\mathfrak{k}}$ extreme weight vector (with respect to $\mathfrak{b}$) of weight λ which is nonzero mod $\overline{W}$, then $(\tau t)'\lambda$ is a $P_{\mathfrak{k}}$ extreme weight of W_1/W_0.*

Proof We refer to the proof of [3, Lemma 9].

Since we do not hae a full chain of $U(\mathfrak{g})$ modules corresponding to a reduced word for t as in [3] but only a shorter chain corresponding to a reduced word for τt, we have to work more to obtain a $\mathfrak{k}$-finite quotient $U(\mathfrak{g})$ module of W_{m+1}. We now define

(4.5) $W_X = \sum \overline{W}$, the summation being over all reduced expressions (3.8) for τt.

(4.6) Lemma *v_{m+1} does not belong to W_X. Let $\lambda \in \mathfrak{b}^X$ be such that there is a $P_{\mathfrak{k}}$ extreme vector in W_{m+1} of weight λ which is nonzero mod W_X. Then $\tau t)'\lambda(H_\varphi^{\mathfrak{k}})$ is a nonnegative integer for every φ in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$.*

Proof v_{m+1} is a $P_{\mathfrak{k}}$ extreme weight vector in W_{m+1} of weight μ_{m+1}. From (3.7) and the definition of μ_{m+1}, we know that $\mu_{m+1}(H_\varphi^{\mathfrak{k}})$ is a nonnegative integer for every φ in $P_{\mathfrak{k}}$. Now suppose v_{m+1} belongs to W_X. Since $W_X = \sum \overline{W}, W_X$ is a quotient of the abstract direct sum $\oplus\overline{W}$, the summation being ower all reduced words (3.8) for τt. We can then apply [3, Lemma 7] and conclude that for some reduced word (3.8) for τt, the corresponding $\overline{W}$ has a nonzero $P_{\mathfrak{k}}$ extreme vector of weight μ_{m+1}. This vector has to be a nonzero scalar multiple of v_{m+1} in view of (3.17). Hence v_{m+1} belongs to that $\overline{W}$. But this contradicts (4.4). This proves the first assertion in (4.6).

Next let λ be as in the lemma. Let c be the reductive component of $\mathfrak{q}$ defined by $c = \mathfrak{h} + \sum_{\alpha\in P\cap -P'}(\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$. We *calim that W_1/W_0 is $\mathfrak{c}$-finite.* For this it is enough to show that the image $\overline{v}_1$ in W_1/W_0 of v_1 is c-finite. For any α in $\pi(\mathfrak{q})$ the submodule $V_{\mathfrak{g},P,s_\alpha^X(\mu_1)}$ of W_1 coincides with $U(\mathfrak{g})X_{-\alpha}^{\mu_1(H_\alpha)+1} \cdot v_1$ (cf. [2, 7.1.15]). Thus we have $W_0 = \sum_{\alpha\in\pi(\mathfrak{q})} U(\mathfrak{g})X_{-\alpha}^{\mu_1(H_\alpha)+1} \cdot v_1$. Hence the annhilator in $U(\mathfrak{g})$ of $\overline{v}_1$ contains $U(\mathfrak{g})X_{-\alpha}^{\mu_1(H_\alpha)+1}$ for every α in $\pi(\mathfrak{q})$. This suffices in view of [2, 7.2.5] to conclude that $\overline{v}_1$ is $\mathfrak{c}$-finite. Thus W_1/W_0 is $\mathfrak{c}$-finite.

Let $\mathfrak{c}_\mathfrak{k} = \mathfrak{c} \cap \mathfrak{k}$. Then in particular W_1/W_0 is $\mathfrak{c}_\mathfrak{k}$-finite. But note that $\mathfrak{c}_\mathfrak{k} = \mathfrak{b} + \sum_{\varphi \in P_\mathfrak{k} \cap -P'_\mathfrak{k}} (\mathbb{C} \cdot X_\varphi + \mathbb{C} \cdot X_{-\varphi})$.

Now choose some reduced word (3.8) for τt and relative to it define $\overline{W}$ as in (4.3). Note that $\overline{W} \subseteq W_X$. For λ as in the lemma, choose a $P_\mathfrak{k}$ extreme weight vector v in W_{m+1} which is nonzero mod W_X and is of weight λ. Then v is in particular nonzero mod $\overline{W}$. Hence from (4.4), $(\tau t)'\lambda$ is a $P_\mathfrak{k}$ extreme weight of W_1/W_0. Since W_1/W_0 is $\mathfrak{c}_\mathfrak{k}$-finite, it now follows that $(\tau t)'\lambda(H_\varphi^\mathfrak{k})$ is a nonnegative integer for every φ in $P_\mathfrak{k} \cap -P'_\mathfrak{k}$. *(q.e.d)*

For our proof of the $\mathfrak{k}$-finiteness of $W_{m+1}/W_\mathfrak{k}$, we need one more lemma.

(4.7) Lemma *Let η be in $\mathfrak{b}^X$. Suppose $\eta(H_\varphi^\mathfrak{k})$ is nonnegative for every φ in $P_\mathfrak{k}$. Let s be in $W_\mathfrak{k}$. Suppose $\tau ts)'\eta(H_\varphi^\mathfrak{k})$ is nonnegative for every φ in $P_\mathfrak{k} \cap -P'_\mathfrak{k}$. Then $N(\tau t) = N(\tau ts) + N(s^{-1})$.*

Proof $(\tau ts)'\eta = \tau ts(\eta + \delta_\mathfrak{k}) - \delta_\mathfrak{k}$. Since $\eta(H_\varphi^\mathfrak{k})$ is nonnegative for every φ in $P_\mathfrak{k}$, $\tau ts(\eta + \delta_\mathfrak{k})(H_\varphi^\mathfrak{k})$ is negative for every φ in $-\tau ts P_\mathfrak{k}$. Also $-\delta_\mathfrak{k}(H_\varphi^\mathfrak{k})$ is negative for every φ in $P_\mathfrak{k}$. Hence $(\tau ts)'\eta(H_\varphi^\mathfrak{k})$ is negative for every φ in $(-\tau ts P_\mathfrak{k}) \cap P_\mathfrak{k}$. Hence the assumption implies

$$P_\mathfrak{k} \cap -\tau ts P_\mathfrak{k} \subseteq \text{ complement of} P_\mathfrak{k} \cap -P'_\mathfrak{k} \text{ in } P_\mathfrak{k}. \tag{4.8}$$

Note that $tP_\mathfrak{k} = -P_\mathfrak{k}$ and $\tau P_\mathfrak{k} = P'_\mathfrak{k}$. So, $-P'_\mathfrak{k} = \tau t P_\mathfrak{k}$. So, the complement of $P_\mathfrak{k} \cap -P'_\mathfrak{k}$ in $P_\mathfrak{k}$ is $P_\mathfrak{k} \cap -\tau t P_\mathfrak{k}$. Hence from (4.8) we have

$$P_\mathfrak{k} \cap (-\tau ts P_\mathfrak{k}) \subseteq P_\mathfrak{k} \cap (-\tau t P_\mathfrak{k}). \tag{4.9}$$

Let $(\epsilon_1, \epsilon_2, \cdots, \epsilon_j, \epsilon_{j+1}, \cdots, \epsilon_m)$ be an enumeration of the elements of $(-\tau t P_\mathfrak{k}) \cap P_\mathfrak{k}$ such that ϵ_1 is a simple root of $P_\mathfrak{k}, \epsilon_2$ is a simple root of $s_{\epsilon_1} P_\mathfrak{k}, \cdots, \epsilon_{i+1}$ is a simple root of $s_{\epsilon_i} s_{\epsilon_{i-1}} \cdots s_{\epsilon_1} P_\mathfrak{k} (i = 1, \cdots, m-1)$. Because of (4.9) we can further assume $(\epsilon_1, \cdots, \epsilon_j)$ is an enumeration of $(-\tau ts P_\mathfrak{k}) \cap P_\mathfrak{k}$. Let

$$\varphi'_i = s_{\epsilon_1 \dots} s_{\epsilon_{i-1}}(\epsilon_i)(i = 1, \cdots, m)(\varphi'_1 = \epsilon_1).$$

Then φ'_i belongs to $\pi_\mathfrak{k}$. One can show that $\tau t = s_{\epsilon_m} \cdots s_{\epsilon_1}$ and a reduced word for τt is

$$\tau t = s_{\varphi'_1} s_{\varphi'_2} \cdots s_{\varphi'_m} \tag{4.10}$$

(cf. [5,4.15.10] and [7,8.9.13]). Similarly $\tau ts = s_{\epsilon_j} \cdots s_{\epsilon_1}$ and a reduced word for τts is

$$\tau ts = s_{\varphi'_1} \cdots s_{\varphi'_j}. \tag{4.11}$$

Note that $N(\tau t) = m$ and $N(\tau ts) = j$. Now from (4.10) and (4.11) it is clear that $s^{-1} = s_{\varphi'_{j+1}} \cdots s_{\varphi'_m}$ is a reduced word for s^{-1}. These observations substantially prove the lemma. *(q.e.d.)*

(4.12) Remark With the data assumed in Lemma 4.7 we have actually proved more than what is asserted in (4.7): There exists a reduced word $\tau t = s_{\varphi'_1} \cdots s_{\varphi'_j} s_{\varphi'_{j+1}} \cdots s_{\varphi'_m}$ for τt such that $s^{-1} = s_{\varphi'_{j+1}} \cdots s_{\varphi'_m}$.

The following proposition gives the $\mathfrak{k}$-finite $U(\mathfrak{g})$ module quotient of W_{m+1}.

(4.13) Propostion *The $U(\mathfrak{g})$ module W_{m+1}/W_X is $\mathfrak{k}$-finite.*

Proof Let $\overline{v}_{m+1}$ be the image of v_{m+1} in W_{m+1}/W_X. Sine $U(\mathfrak{g})\overline{v}_{m+1} = W_{m+1}/W_X$, it suffices to prove that $U(\mathfrak{k}) \cdot \overline{v}_{m+1}$ has finite dimension over $\mathbb{C}$. For this again, by well known facts [2, 7.2.5] it suffices to prove that the annihilaor of $\overline{v}_{m+1}$ in $U(\mathfrak{k})$ contains $X_{-\varphi}^{e(\varphi)}$ for every φ in $\pi_{\mathfrak{k}}$, where $e(\varphi) = \mu_{m+1}(H_{\varphi}^{\mathfrak{k}})+1$ (observe that in view of (3.7), $\mu_{m+1}(H_{\varphi}^{\mathfrak{k}})$ is a nonnegative integer for every φ in $\pi_{\mathfrak{k}}$). Thus it suffices to show that for every φ in $\pi_{\mathfrak{k}}$,

$$X_{-\varphi}^{e(\varphi)} \cdot v_{m+1} \text{ belongs to } W_X. \tag{4.14}$$

Suppose (4.14) *is not true*. Choose a φ in $\pi_{\mathfrak{k}}$, such that $X_{-\varphi}^{e(\varphi)} v_{m+1}$ does not belong to W_X. Then $X_{-\varphi}^{e(\varphi)} v_{m+1}$ is a $P_{\mathfrak{k}}$ extreme vector of weight $s'_{\varphi}(\mu_{m+1})$ in W_{m+1} which is nonzero mod W_X. Hence by (4.6), $(\tau ts_{\varphi})'\mu_{m+1}(H_{\varphi'}^{\mathfrak{k}})$ is a nonnegative integer for every φ' in $P_{\mathfrak{k}} \cap -P'_{\mathfrak{k}}$. We can now apply(4.7) and (4.12) and conclude that there exists a reduced word.

$$\tau t = s_{\varphi'_1} s_{\varphi'_2} \cdots s_{\varphi'_{m-1}} s_{\varphi'_m} (\varphi'_i \in \pi_{\mathfrak{k}}) \tag{4.15}$$

for τt such that

$$\varphi'_m = \varphi. \tag{4.16}$$

Take the reduced word (4.15) for τt in (3.8) and consider the corresponding modules W_m and $\overline{W}$. By definition $W_m \subseteq \overline{W}$. But in the fundamental chain $W_1 \subseteq \cdots \subseteq W_m \subseteq W_{m+1}$ associated to the reduced word (4.15) for τt, the module W_m is simply $U(\mathfrak{g}) \cdot X_{-\varphi}^{e(\varphi)} v_{m+1}$. This is clear from the definitions (cf.(3.14) and the definition of v_i after (3.11) and (4.16). Thus it follows that $X_{-\varphi}^{e(\varphi)} v_{m+1} \in \overline{W} \subseteq W_X$. But this is a contradiction to the hypothesis. Thus (4.14) is true and proved and with that also the $\mathfrak{k}$-finiteness of W_{m+1}/W_X. *(q.e.d)*

§ 5

Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{h}$ its centralizer in $\mathfrak{g}$, so that $\mathfrak{h}$ is a θ stable Cartan subalgebra of $\mathfrak{g}$. Let P be a system of positive roots for $(\mathfrak{g}, \mathfrak{h})$ such that $\theta(P) = P$. Let

$$\mathfrak{n}^+ = \sum_{\alpha \in P} \mathfrak{g}^\alpha$$

and

$$\mathfrak{n}^- = \sum_{\alpha \in P} \mathfrak{g}^{-\alpha}.$$

The following fact is standard if $\mathfrak{b} = \mathfrak{h}$, but it remains true in our general case.

(5.1) Lemma *Let $U^{\mathfrak{b}}$ be the centralizer of $\mathfrak{b}$ in $U(\mathfrak{g})$. If the set P of positive roots satisfies $\theta P = P$, we have a unique homomorphism*

(5.2) $$\beta_P : U^{\mathfrak{b}} \to U(\mathfrak{h})$$

such that for any y in $U^{\mathfrak{b}}$.

(5.3) $$y \equiv \beta_P(y) \ (\mathbf{mod}\ U(\mathfrak{g})\mathfrak{n}^+).$$

Proof We have

(5.4) $$U(\mathfrak{g}) = U(\mathfrak{n}^- + \mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}^+$$

and this decomposition is stable under adH for every H in $\mathfrak{h}$ i.e. $adH(U(\mathfrak{n}^- + \mathfrak{h})) \subseteq U(\mathfrak{n}^- + \mathfrak{h})$ and $adH(U(\mathfrak{g})\mathfrak{n}^+) \subseteq U(\mathfrak{g})\mathfrak{n}^+$. For y in $U^{\mathfrak{b}}$, let $y = y_0 + y_1$ be its decomposition with respect to (5.4). Define

$\beta_P(y) = y_0$. *We claim* $\beta_P(y)$ *belongs to the subalgebra* $U(\mathfrak{h})$ of $U(\mathfrak{n}^- + \mathfrak{h})$. Since y is in $U^{\mathfrak{b}}$, y_0 and y_1 are also in $U^{\mathfrak{b}}$. Let $S(\mathfrak{n}^- + \mathfrak{h})$ and $S(\mathfrak{h})$ denote the symmetric algebras and λ the symmetrizer map of $S(\mathfrak{n}^- + \mathfrak{h})$ onto $U(\mathfrak{n}^- + \mathfrak{h})$. Then for H in $\mathfrak{b}, \lambda^{-1}(y_0)$ is annihilated by adH (extended as a derivation to $S(\mathfrak{n}^- + \mathfrak{h})$). It is enough to show that $\lambda^{-1}(y_0)$ belongs to $S(\mathfrak{h})$. Using (1.14), one can show that there exists an element X_P in $\mathfrak{b}$ such that $\alpha(X_P)$ is a nonzero real number for every α in $\triangle$(= the roots of $(\mathfrak{g}, \mathfrak{h})$) and such that P consists of precisely those α in $\triangle$ such that $\alpha(X_P)$ is positive. It is then clear that in $S(\mathfrak{n}^- + \mathfrak{h})$, the null space for adX_P is just $S(\mathfrak{h})$. Since $adX(\lambda^{-1}(y_0)) = 0$ for every X in $\mathfrak{b}$, in particular $adX_P(\lambda^{-1}(y_0)) = 0$. Hence $\lambda^{-1}(y_0)$ belongs to $S(\mathfrak{h})$, so that $\beta_P(y)$ belongs to $U(\mathfrak{h})$.

Now suppose y and y' are in $U^{\mathfrak{b}}$. Let $y = y_0 + y_1$ and $y' = y'_0 + y'_1$ be their decomposition with respect to (5.4), so that $\beta_P(y) = y_0$ and $\beta_P(y') = y'_0$. Then $yy' = y_0y'_0 + y_0y'_1 + y_1y'_0 + y_1y'_1$. Clearly $y_0y'_0$ belongs to $U(\mathfrak{h})$ and $y_0y'_1 + y_1y'_1$ belongs to $U(\mathfrak{g})\mathfrak{n}^+$. Also $y_1y'_0 \in U(\mathfrak{g})\mathfrak{n}^+ \cdot U(\mathfrak{h}) \subseteq U(\mathfrak{g})U(\mathfrak{h})\mathfrak{n}^+$. Thus $y_0y'_0$ is the component of yy' in $U(\mathfrak{n}^- + \mathfrak{h})$ with respect to (5.4). We already know that this component is in $U(\mathfrak{h})$. Thus β_P is a homomorphism of algebras. *(q.e.d)*

The centralizer $U^{\mathfrak{k}}$ of $\mathfrak{k}$ in $U(\mathfrak{g})$ is contained in $U^{\mathfrak{b}}$. As usual interpret elements of $S(\mathfrak{h})$ as polynomials on $\mathfrak{h}^X$. For any φ in $\mathfrak{h}^X$, define a homomorphism $\chi_{P,\varphi}$ of $U^{\mathfrak{k}}$ into $\mathbb{C}$ as follows:

$$\chi_{P,\varphi}(y) = \beta_P(y)(\varphi) \quad (y \in U^{\mathfrak{k}}). \tag{5.5}$$

The main results of the previous sections can now be formulated. Let $\mathfrak{b}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$ and $\mathfrak{b}$ its complexification. Let $\mathfrak{q}$ be a θ stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$. The centralizer $\mathfrak{h}$ of $\mathfrak{b}$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{q}$ contains $\mathfrak{h}$. Let $\mathfrak{r}$ be a θ stable Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$ (cf. (1.15) and (1.2)). Let P be the set of positive roots for $(\mathfrak{g}, \mathfrak{h})$ corresponding to $\mathfrak{r}$. Define the θ stable Borel subalgebra $\mathfrak{r}' \subseteq \mathfrak{q}$ by (2.1). Choose a θ stable positive system P'' of roots of $(\mathfrak{g}, \mathfrak{h})$ having properties (2.3) and (2.4). Denote by $F(P'' : \mathfrak{q}, \mathfrak{r})$ the set of all elements μ in $\mathfrak{h}^X$ having properties (2.6) and (2.7). Now choose a μ in $F(P'' : \mathfrak{q}, \mathfrak{r})$ and recall the objects associated to it in §§ 3,4.

We can now state

(5.6) Theorem *Let* $\mathfrak{q}$ *be a* θ *stable parabolic subalgebra. Let* $\mu \in F(P'' : \mathfrak{q}, \mathfrak{r})$. *Let* $W_{P'':\,\mathfrak{q},\mathfrak{r}} = W_{m+1}/W_X$ (cf. (3.12) and (4.5)). Then $W_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ *is a* $\mathfrak{k}$ *finite* $U(\mathfrak{g})$ *module having the following properties:*

(i) $W_{P'':\ \mathfrak{q},\mathfrak{r}}(\mu) = U(\mathfrak{g})\overline{v}_{m+1}$, *where* $\overline{v}_{m+1}$ *is the image of the vector* v_{m+1} of W_{m+1}. *The irreducible finite dimensional representation of* $\mathfrak{k}$ *with highest weight* $-t(\tau\mu + \tau\delta - \tau\delta_{\mathfrak{k}} - \delta_{\mathfrak{k}})$ *occurs with multiplicity one in* $W_{P":\ \mathfrak{q},\mathfrak{r}}(\mu)$. *The corresponding isotypical* $U(\mathfrak{k})$ *submodule of* $W_{P'':\ \mathfrak{q},\mathfrak{r}}$ *is* $U(\mathfrak{k})\overline{v}_{m+1}$; *on this space elements of* $U^{\mathfrak{k}}$ *act by scalars given by the homomorphism* $\chi_{P,-\mu-\delta}$.

(ii) *If* τ_λ *is an irreducible finite dimensional representation of* $\mathfrak{k}$ *with highest weight* λ *with respect to* $P_{\mathfrak{k}}$, *then the multiplicity of* τ_λ *in* $W_{P'':\ \mathfrak{q},\mathfrak{r}}(\mu)$ *is finite; it is zero if* λ *is not of the form* $(t\tau)'(-\mu - \delta - \sum_{\varphi\in P} m_\varphi\varphi) \mid \mathfrak{b}$ *where* m_φ *are nonnegative integers.*

Proof By (4.13), we know that $W_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ is nonzero and $\mathfrak{k}$-finite. By (4.6) the vector v_{m+1} of W_{m+1} does not belong to W_X. The image of v_{m+1} in $W_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ is $P_{\mathfrak{k}}$ extreme of weight $(t\tau)'(-\mu-\delta) = -t(\tau\mu+\tau\delta-\tau\delta_{\mathfrak{k}}-\delta_{\mathfrak{k}})$ (which is dominant by (3.7)) and this image generates an irreducible $\mathfrak{k}$-module with highest weight $-t(\tau\mu + \tau\delta + \tau\delta_{\mathfrak{k}} - \delta_{\mathfrak{k}})$ with respect to $P_{\mathfrak{k}}$.

Based on the preceding sections one can complete the proof of the theorem in the same way as [3, Theorem 1].

It is easy to conclude from (5.6) that $W_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ has a unique proper maximal $U(\mathfrak{g})$ submodule and hence $W_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ has a unique nonzero quotient $U(\mathfrak{g})$ module which is irreducible. We denote this $U(\mathfrak{g})$ module by $D_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$. The following theorem is now immediate from (5.6).

(5.7) Theorem *Let* $\mu \in F(P'' : \mathfrak{q}, \mathfrak{r})$. *Up to equivalence there exists a unique* $\mathfrak{k}$*-finite irreducible* $U(\mathfrak{g})$ *module* $D_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ *having the following property: The finite dimensional irreducible* $U(\mathfrak{k})$ *module with highest weight* $-t(\tau\mu+\tau\delta-\tau\delta_{\mathfrak{k}}-\delta_{\mathfrak{k}})$ *(with respect to* $P_{\mathfrak{k}}$*) occurs with multiplicity one in* $D_{P'':\mathfrak{q},\mathfrak{r}}(\mu)$ *and the action of* $U^{\mathfrak{k}}$ *on the corresponding isotypical* $U(\mathfrak{k})$ *submodule is given by the homomorphism* $\chi_{P,-\mu-\delta}$.

The uniqueness follows from the well known theorem of Harish Chandra [4]: An irreducible $\mathfrak{k}$- finite $U(\mathfrak{g})$ module M is completely determined by a nonzero isotypical $U(\mathfrak{k})$ submodule of M and the action of $U^{\mathfrak{k}}$ on it.

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Tata Inst. Fund. Research
School of Math.
Bombay, 400005 India