Quantum analogues of discrete series at roots of one

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Abstract. Quantum analogues of a coherent family of modules at roots of 1 are described for the case where the given family is the coherent extension of discrete series, by exploiting the Enright-Varadarajan algebraic construction of discrete series.

1. Coherent family

This talk is about arriving at a quantum analogue $\overline{\pi}$ for the quantum group U_λ at an ℓ-th root of 1 of a given module π for the enveloping algebra U of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. The quantum group U_λ at an ℓ-th root λ of unity is obtained from the quantum analogues of Serre relations and generators for $\mathfrak{g}$. In the later part of the talk, we will see how such an analogue $\overline{\pi}$ can be constructed when π belongs to the discrete class. For general π we described an algebraic expression giving the quantum analogue $\overline{\pi}$ in [P1,P2,P3] . In general $\overline{\pi}$ is a virtual module for U_λ. This description involved the notion of a coherent family of modules containing π. The basic problem to be solved here is the conjecture stated first in [P1], restated here just before the beginning of section 3. Apart from the above cited references, where only low rank cases are studied, no significant study in this direction seems to have been carried out. A study (incomplete) was begun in [P4] for the case of the exceptional rank two semisimple Lie algebra $\mathfrak{g}_2$ by making explicit the required initial calculations; relying on informal discussions with H.H. Andersen about decomposition of Weyl modules for quantum $\mathfrak{g}_2$ (at a root of 1), the author believes the conjecture to hold for $\mathfrak{g}_2$. The present study is the first case with no restriction on the rank and treating non-highest weight modules.

As already noted in [P3] just before [P3, sec.3], one may rely on character computattions for special cases like higest weight modules.

Let Λ be the weight lattice for $\mathfrak{g}$.($\Lambda \subset \mathfrak{h}^*, \mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$).

Definition 1.1. A family of virtual modules $\{\pi(\nu)\}_{\nu\in\Lambda}$ of U is called a coherent family (cf. [BV, 2.2]) if for every finite dimensional module F of U

$$\pi(\nu) \otimes F = \Sigma_{\mu\in\Delta(F)} m(\mu, F)\pi(\nu + \mu), \text{ (in the Grothendieck group)}$$

where the summation is over the weights $\Delta(F)$ of F and for $\mu \in \Delta(F), m(\mu, F)$ denotes the multiplicity of μ as a weight of F.

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2. A formula for $e^\nu \chi(St)$

The equation

$$x^k \cdot \frac{x^\ell - x^{-\ell}}{x - x^{-1}} = \frac{x^k - x^{-k}}{x - x^{-1}} \cdot x^\ell + \frac{x^{\ell-k} - x^{-\ell+k}}{x - x^{-1}} \cdot x^0$$

has a generalisation to any root system. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the simple roots and let $s_1, s_2, \cdots, s_n$ the simple reflections. These generate the Weyl group W. Let $\ell(?)$ denote the length function in W. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the fundamental weights. For $\sigma \in W$, let $I_\sigma = \{i \mid 1 \leq i \leq n, \ell(\sigma s_i) \leq \ell(\sigma)\}$. Put $\delta_\sigma = \Sigma_{i \in I_\sigma} \lambda_i$ and define $\epsilon_\sigma = \sigma(\delta_\sigma)$.

Let $\mathcal{R}$ be the ring of finite integral combinations $\Sigma_{\eta \in \Lambda} m_\eta e^\eta$. The character $\chi(?)$ of finite dimensional representations (both for U and U_λ) takes values in $\mathcal{R}^W$. Define the operator $c : \mathcal{R} \to \mathcal{R}^W$ by

$$c(e^\eta) = \frac{\Sigma_{\tau \in W} (-1)^{\ell(\tau)} e^{\tau\eta}}{\Sigma_{\tau \in W} (-1)^{\ell(\tau)} e^{\tau\rho}}$$

where ρ denotes half the sum of the positive roots. Define the operator $\widetilde{c} : \mathcal{R} \to \mathcal{R}^W$ by

$$\widetilde{c}(e^\eta) = \frac{\Sigma_{\tau \in W} (-1)^{\ell(\tau)} e^{\ell\tau\eta}}{\Sigma_{\tau \in W} (-1)^{\ell(\tau)} e^{\ell\tau\rho}}.$$

Let St denote the (Steinberg) representation so that $\chi(St) = c(e^{\ell\rho})$. Let $\sigma_0 \in W$ denote the element of maximal length. The aforementioned generalisation is the second assertion below:

Observation 2.1. For any $\nu \in \Lambda$,

(1) there exist unique W-invariant elements $\chi_{\nu,\tau} \in \mathcal{R}^W (\tau \in W)$, such that

$$e^\nu = \Sigma_\tau \chi_{\nu,\tau} e^{\epsilon_\tau},$$

(2) there exist unique W-invariant elements $\eta_{\nu,\tau} \in \mathcal{R}^W (\tau \in W)$, such that

$$e^\nu \chi(St) = \Sigma_{\tau \in W} \eta_{\nu,\tau} e^{-\ell\epsilon_{\tau\sigma_0}}.$$

(It is also true that $e^\nu \chi(St)$ can be expressed uniquely as a linear combination of $e^{\ell\epsilon_\tau}$ with coefficients from $\mathcal{R}^W$.)

The main theorem of [P1] described an algebraic expression giving the quantum analogue $\overline{\pi}$ of a U-module π for quantum groups at roots of unity, at the level of a suitable Grothendieck group. This description involved not only the original U-module but in addition the coherent family of (virtual) U-modules to which π belongs.

We describe below the construction of $\overline{\pi}$. Let us recall some standard notations about a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ and its universal enveloping algebra U. For primitive ℓ-th roots of 1, Lusztig considered some Hopf algebras U_λ, called 'quantum groups at roots of unity' and defined a 'Frobenius' morphism $\psi : U_\lambda \to U$; ψ is a surjection and respects the Hopf-algebra structure.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Δ^+ a system of positive roots. Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the set of simple roots in Δ^+. Let $\Lambda \subseteq \mathfrak{h}^* (= \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}))$ be the integral weight lattice. Let $\mathcal{R}$ be the ring of finite integral combinations $\sum_{\eta \in \Lambda} m_\eta e^\eta$.

Finite dimensional representations of U have been quantized by Lusztig at all U_λ. Their 'weights' can be defined as elements of Λ; they admit a weight space decomposition. If F is an irreducible finite dimensional module for U, its quantization F' for U_λ is called a Weyl module; for $\mu \in \Lambda, \mu$ is a weight of F' iff it is a weight of F and then the multiplicity $m(\mu, F')$ equals $m(\mu, F)$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ (half the sum of the positive roots). If $\nu \in \Lambda$ is dominant integral let F_ν denote the irreducible finite dimensional representation of U with highest weight ν. We have then a representation $U_\lambda \to$ End (F'_ν) of the quantum group U_λ on the corresponding Weyl module F'_ν. We let St denote the Steinberg module $F'_{(\ell-1)\rho}$.

If $\pi : U \to$ End (V) is a representation of U, we define a representation $\widetilde{\pi} : U_\lambda \to$ End (V) by $\widetilde{\pi} = \pi \circ \psi$ where $\psi : U_\lambda \to U$ is the Frobenius morphism. Later, we will also use the notation $< \pi >^\psi$ to denote $\widetilde{\pi}$.

Given a coherent family $\{\pi(\nu)\}_{\nu \in \Lambda}$ of virtual representations of U we proceed to construct a coherent family $\{\overline{\pi}(\nu)\}_{\nu \in \Lambda}$ of virtual representations of U_λ such that for any $\nu^{''} \in \Lambda$

$$\overline{\pi}(\ell\nu^{''}) = \widetilde{\pi}(\nu^{''}) \otimes St$$

where St is the Steinberg representation.

To do this,

given any $\nu \in \Lambda$, write as in (2.1,2)

$$e^\nu.\chi(St) = \sum_i \eta^{'}_{\nu,i} \; e^{\ell\epsilon_i}$$

for some $\epsilon_i \in \Lambda$ and some W-invariant elements $\eta^{'}_{\nu,i} \in \mathcal{R}^W$, ($i \in$ some finite set of indices).

Having done this, the construction of $\overline{\pi}(\nu)$ goes as follows:

We denote by $\mathcal{F}$ the Grothendieck group of formal integral combinations of finite dimensional representations of U. If $\omega \in \mathcal{F}$, the character $\chi(\omega) \in \mathcal{R}^W$ has an obvious meaning and $\chi : \mathcal{F} \to \mathcal{R}^W$ is an isomorphism. We also have to introduce the corresponding Grothendieck group $\mathcal{F}'$ for U_λ-modules. Again if $\omega \in \mathcal{F}'$, the character $\chi(\omega) \in \mathcal{R}^W$ has an obvious meaning and $\chi : \mathcal{F}' \to \mathcal{R}^W$ is an isomorphism. Sometimes, if convenient, we use the same symbol to denote an element of $\mathcal{F}'$ and its character in $\mathcal{R}^W$.

THEOREM ([**P1**]). *Suppose a coherent family $\{\pi(\nu)\}_{\nu \in \Lambda}$ of virtual representations of U is given. Given $\nu \in \Lambda$, write $\nu = \nu' + \ell\nu^{''}$ where $\nu^{''} \in \Lambda$ and $2(\nu', \alpha)/(\alpha, \alpha) \in \{0, 1, \cdots, \ell - 1\}$ for each simple root α. Write as above $e^{\nu'}.\chi(St) = \sum_i \eta^{'}_{\nu',i} \; e^{\ell\epsilon_i}$.*

Choose $\rho(\nu', i) \in \mathcal{F}'$ whose character is $\eta_{\nu',i}$. Set

$$\overline{\pi}(\nu) = \sum_i \rho(\nu', i) \otimes \widetilde{\pi}(\nu^{''} + \epsilon_i)$$

(in the Grothendieck group of a suitable subcategory of representations of U_λ). Then $\{\overline{\pi}(\nu)\}_{\nu \in \Lambda}$ is a coherent family of virtual representations of U_λ with $\overline{\pi}(\ell\nu^{''}) = \widetilde{\pi}(\nu^{''}) \otimes St$.

However, from the algebraic expression for $\overline{\pi}$ above, one can not at all ascertain that this virtual module is an actual module. More specifically, we proposed the following conjecture in [**P1**].

Suppose the coherent family $\{\pi(\nu)\}_{\nu \in \Lambda}$ has the property that

i) $\pi(\nu)$ has infinitesimal character parametrized by the W- orbit of ν
ii) $\pi(\nu)$ is zero or irreducible when ν is dominant with respect to a fixed positive system, and $\pi(\nu) \neq 0$ if ν is dominant regular.

Then one should expect that for dominant ν (with respect to the positive system in (ii) above) $\overline{\pi}(\nu)$ is represented in the Grothendieck group by a U_λ- module (not just a virtual module). The purpose of the next section is to explain how this can be verified when the coherent family is discrete series in the positive chamber.

3. Application: the discrete series and the generalized Enright-Varadarajan modules.

Let G be a connected real semisimple Lie group with finite center and let K be a maximal compact subgroup of G. Assume that $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the complexification of the Lie algebra of G (resp. K).

Discrete series representations of G arise as a particular case of a more general construction called "generalized Enright-Varadarajan modules" (see [**P5**]). Let $\mathfrak{p}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Choose a θ-stable Borel subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ such that $\mathfrak{p} \supseteq \mathfrak{r}$. Let Π be the corresponding (θ-stable) system of positive roots. Write $\Pi = \Pi_{\mathfrak{m}} \cup \Pi_{\mathfrak{u}}$ so that $\Pi_{\mathfrak{m}}$ corresponds to the reductive part of $\mathfrak{p}$ and $\Pi_{\mathfrak{u}}$ corresponds to the nilradical of $\mathfrak{p}$. Then write $\Pi' = (-\Pi_{\mathfrak{m}}) \cup \Pi_{\mathfrak{u}}$. It is also a positive system of roots whose corresponding Borel algebra is denoted $\mathfrak{r}'$. We have implicitly fixed a Cartan subalgebra $\mathfrak{h}$ which is θ-stable and such that $\mathfrak{b} \stackrel{def.}{=} \mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ and such that $\mathfrak{h} \subseteq \mathfrak{r}$. We fix a Borel subalgebra $\mathfrak{r}_{\mathfrak{k}}$ of $\mathfrak{k}$ by $\mathfrak{r}_{\mathfrak{k}} \stackrel{def.}{=} \mathfrak{r} \cap \mathfrak{k}$. The corresponding positive system of roots of $\mathfrak{k}$ with respect to $\mathfrak{b}$ is denoted $\Pi_{\mathfrak{k}}$. Similarly the positive system for $\mathfrak{k}$ defined by $\mathfrak{r}'_{\mathfrak{k}}$ ($\stackrel{def.}{=} \mathfrak{r}' \cap \mathfrak{k}$) is denoted $\Pi'_{\mathfrak{k}}$. Finally, denote by $\overline{F}(\mathfrak{p}, \mathfrak{r})$ the subset of $\mathfrak{h}^\star$ consisting of linear forms μ on $\mathfrak{h}$ such that

(1) $\mu(H_\alpha)$ is a positive integer for $\alpha \in -\Pi_{\mathfrak{m}}$, and
(2) $(-\mu - \delta + \delta_{\mathfrak{k}})(H_\phi^{\mathfrak{k}})$ is a positive integer for $\forall \phi \in (-\Pi'_{\mathfrak{k}})$.

Consider the Verma module $V_{\mathfrak{g},\Pi,-\mu-\delta}$ for $\mathfrak{g}$ with highest weight $-\mu-\delta$ with respect to Π and the Verma modules $V_{\mathfrak{k},\Pi_{\mathfrak{k}},-\mu-\delta}$ for $\mathfrak{k}$ with $\Pi_{\mathfrak{k}}$-highest weight given by the restriction of $-\mu - \delta$ to $\mathfrak{b}$. Evidently, $V_{\mathfrak{k},\Pi_{\mathfrak{k}},-\mu-\delta}$ can be canonically identified with the $\mathcal{U}(\mathfrak{k})$-module generated by the highest weight vector of $V_{\mathfrak{g},\Pi,-\mu-\delta}$. There is a unique $\Pi_{\mathfrak{k}}$-dominant integral weight η such that $V_{\mathfrak{k},\Pi_{\mathfrak{k}},-\mu-\delta} \subseteq V_{\mathfrak{k},\Pi_{\mathfrak{k}},\eta}$. The $\mathfrak{k}$-module $V_{\mathfrak{k},\Pi_{\mathfrak{k}},\eta}$ and the $\mathfrak{g}$-module $V_{\mathfrak{g},\Pi,-\mu-\delta}$ can both be simultaneously imbedded in a $\mathfrak{g}$-module $W_{\mathfrak{p},\mu}$, compatible with the prolongment $V_{\mathfrak{k},\Pi_{\mathfrak{k}},-\mu-\delta} \subset V_{\mathfrak{k},\Pi_{\mathfrak{k}},\eta}$ and having nice properties. Some of the important properties of the inclusions $V_{\mathfrak{g},\Pi,-\mu-\delta} \subseteq W_{\mathfrak{p},\mu}$ and $V_{\mathfrak{k},\Pi_{\mathfrak{k}},\eta} \subseteq W_{\mathfrak{p},\mu}$ are the following (see [**P5**]):

(3.1)

(i) $W_{\mathfrak{p},\mu}$ has a unique irreducible quotient $\mathfrak{g}$-module $D_{\mathfrak{p},\mu}$ which is $\mathfrak{k}$-finite,
(ii) the irreducible finite dimensional $\mathfrak{k}$-module $F_{\mathfrak{k},\eta}$ with $\Pi_{\mathfrak{k}}$- highest weight η occurs with multiplicity one in $D_{\mathfrak{p},\mu}$,
(iii) if $\chi_{\Pi,-\mu}$ denotes the algebra homomorphism from $U(\mathfrak{g})^{\mathfrak{k}}$ into $\mathbf{C}$ defining the scalar by which $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ acts on the highest weight vector of $V_{\mathfrak{g},\Pi,-\mu-\delta}$, then the same homomorphism defines the action of $\mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ on $F_{\mathfrak{k},\eta} \subseteq D_{\mathfrak{p},\mu}$.

The construction of the inclusion $V_{\mathfrak{g},\Pi,-\mu-\delta} \subseteq W_{\mathfrak{p},\mu}$ which arises in [**P5**] actually introduces, as a preliminary step, some intermediate inclusions $W_1 = V_{\mathfrak{g},\Pi,-\mu-\delta} \subseteq W_2 \subseteq W_3 \subseteq \cdots \subseteq W_{n+1} = W_{\mathfrak{p},\mu}$.

More specifically,

(3.2)

(i) Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be simple roots (not necessarily distinct) of $\mathfrak{k}$ relative to $\mathfrak{b}$.
(ii) Let $s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} \cdots s_{\alpha_n}$ be a reduced expression of the element of the Weyl group of $\mathfrak{k}$ which takes $\Pi_{\mathfrak{k}}$ to $-\Pi'_{\mathfrak{k}}$.
(iii) Let $\mathfrak{m}_{\alpha_i} = \mathfrak{b} + \mathbb{C}X_{\alpha_i} + \mathbb{C}X_{-\alpha_i}$ where X_β denotes the rootspace corresponding to a root β.
(iv) Then the successive quotients W_{i+1}/W_i are $\mathfrak{m}_{\alpha_i}$-finite.

Let ν be Π-dominant integral and let $F(\nu)$ be the finite dimensional irreducible representation for $\mathfrak{g}$ with highest weight ν. Let $\mu \in \overline{F}(\mathfrak{p},\mathfrak{r})$ and assume $\mu+\nu \in \overline{F}(\mathfrak{p},\mathfrak{r})$ so that we have the irreducible $(\mathfrak{g},K)$-modules $D_{\mathfrak{p},\mu}$ and $D_{\mathfrak{p},\mu+\nu}$ as above. We have a canonical inclusion

$$\varphi : D_{\mathfrak{p},\mu} \hookrightarrow D_{\mathfrak{p},\mu+\nu} \otimes F(\nu) \tag{3.3}$$

which is a consequence of the inclusion of $\mathfrak{g}$-Verma modules

$$V_{\mathfrak{g},\Pi,-\mu-\delta} \hookrightarrow V_{\mathfrak{g},\Pi,-\mu-\nu-\delta} \otimes F(\nu).$$

The last inclusion exists also for the quantum group U_q over the function field $\mathbb{C}(q)$. We write this inclusion as follows:

$$V_{U_q,\Pi,-\mu-\delta} \hookrightarrow V_{U_q,\Pi,-\mu-\nu-\delta} \otimes F_{U_q}(\nu).$$

Moreover, if $\mathcal{A}$ denotes the Laurent polynomial ring $\mathbb{C}[q,q^{-1}] \subseteq \mathbb{C}(q)$ the above mentioned inclusion exists for $U_{\mathcal{A}} \subseteq U_q$, the $\mathcal{A}$-form of U_q, defined by Lusztig, analogous to Kostant's $\mathbb{Z}$-form of U. We write this inclusion as follows:

$$V_{U_{\mathcal{A}},\Pi,-\mu-\delta} \hookrightarrow V_{U_{\mathcal{A}},\Pi,-\mu-\nu-\delta} \otimes F_{U_{\mathcal{A}}}(\nu).$$

If λ is a root of 1, denote by $\varphi_\lambda : \mathcal{A} \to \mathbb{C}$ the evaluation of Laurent polynomials at $q=\lambda$. Finally, "$\cdot \otimes_{\mathcal{A}} \mathbb{C}$" (where $x \in \mathcal{A}$ acts on $\mathbb{C}$ by multiplication by $\varphi_\lambda(x)$) yields an inclusion

$$\vartheta : V_{U_\lambda,\Pi,-\mu-\delta} \hookrightarrow V_{U_\lambda,\Pi,-\mu-\nu-\delta} \otimes F_{U_\lambda}(\nu). \tag{3.4}$$

For a given μ choose ν such that $\mu+\nu = \ell\psi$ where $\psi \in \overline{F}(\mathfrak{p},\mathfrak{r})$. Consider the associated chain of inclusions

$$W_1 = V_{\mathfrak{g},\Pi,-\psi-\delta} \subseteq W_2 \subseteq W_3 \subseteq \cdots \subseteq W_{n+1} = W_{\mathfrak{p},\psi}.$$

This gives rise to a chain of inclusions of $U \otimes U_\lambda \otimes U_\lambda$-modules

$$W_1 \otimes St \otimes F_{U_\lambda}(\nu) = V_{\mathfrak{g},\Pi,-\psi-\delta} \otimes St \otimes F_{U_\lambda}(\nu) \subseteq W_2 \otimes St \otimes F_{U_\lambda}(\nu) \subseteq W_3 \otimes St \otimes F_{U_\lambda}(\nu)$$

$$\subseteq \cdots \subseteq W_{n+1} \otimes St \otimes F_{U_\lambda}(\nu) = W_{\mathfrak{p},\psi} \otimes St \otimes F_{U_\lambda}(\nu).$$

As already remarked, when we have a module M for U we get a module $\widetilde{M}$ for U_λ by pulling by the Frobenius morphism $\psi : U_\lambda \to U$. Thus we can view the same chain as a chain of inclusions of $U_\lambda \otimes U_\lambda \otimes U_\lambda$-modules

$$\widetilde{W}_1 \otimes St \otimes F_{U_\lambda}(\nu) = \widetilde{V}_{\mathfrak{g},\Pi,-\psi-\delta} \otimes St \otimes F_{U_\lambda}(\nu) \subseteq \widetilde{W}_2 \otimes St \otimes F_{U_\lambda}(\nu) \subseteq \widetilde{W}_3 \otimes St \otimes F_{U_\lambda}(\nu)$$

$$\subseteq \cdots \subseteq \widetilde{W}_{n+1} \otimes St \otimes F_{U_\lambda}(\nu) = \widetilde{W}_{\mathfrak{p},\psi} \otimes St \otimes F_{U_\lambda}(\nu).$$

Now $V_{U_\lambda,\Pi,-\mu-\nu-\delta} = \widetilde{W}_1 \otimes St$. Write $\overline{W}_{1,\mu}$ for the image of the inclusion (3.4) in $\widetilde{W}_1 \otimes St \otimes F_{U_\lambda}(\nu)$. Let $\overline{W}_{2,\mu}, \overline{W}_{3,\mu}, \cdots, \overline{W}_{n+1,\mu}$ be the unique maximal U_λ- submodules of $\widetilde{W}_2 \otimes St \otimes F_{U_\lambda}(\nu), \widetilde{W}_3 \otimes St \otimes F_{U_\lambda}(\nu), \cdots, \widetilde{W}_{n+1} \otimes St \otimes F_{U_\lambda}(\nu)$ respectively, whose intersection with $\widetilde{W}_1 \otimes St \otimes F_{U_\lambda}(\nu)$ is contained in $\overline{W}_{1,\mu}$. One then tries to imitate the construction of [**P5**] to get a quotient of $\overline{W}_{n+1,\mu}$ by using the chain of inclusions $\overline{W}_{1,\mu} \subseteq \overline{W}_{2,\mu} \subseteq \overline{W}_{3,\mu} \subseteq \cdots \subseteq \overline{W}_{n+1,\mu}$ in place of the chain $W_1 = V_{\mathfrak{g},\Pi,-\mu-\delta} \subseteq W_2 \subseteq W_3 \subseteq \cdots \subseteq W_{n+1} = W_{\mathfrak{p},\mu}$ which was used in [**P5**]. This quotient of $\overline{W}_{n+1,\mu}$ can also be described to be the image of the composite morphism $\overline{W}_{n+1,\mu} \hookrightarrow \widetilde{W}_{n+1} \otimes St \otimes F_{U_\lambda}(\nu) \longrightarrow \widetilde{D}_{\mathfrak{p},\mu+\nu} \otimes St \otimes F_{U_\lambda}(\nu)$.

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