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t-structures in the derived category of representations of quivers

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Abstract. Given a finite quiver without oriented cycles, we describe a family of algebras whose module category has the same derived category as that of the quiver algebra. This is done in the more general setting of t-structures in triangulated categories. A completeness result is shown for Dynkin quivers, thus reproving a result of Happel [H].

Keywords. Quivers; module category of algebras; t-structures; tilting; realization functor.

Introduction

The notion of a t-structure in a triangulated category is introduced in [1, 1–3]. Given a t-structure in a full thick subcategory D_U of D and a t-structure in the quotient, under some hypotheses, there is a notion of glueing (= 'recollement' [1, 1–4]) which produces a t-structure in D. We apply these constructions to the (bounded) derived category $D^b(\mathrm{mod}(\Lambda, \Omega))$ of the category $\mathrm{mod}(\Lambda, \Omega)$ of representations of a quiver (Λ, Ω). In § 2 we define 'data' in (Λ, Ω) and attach them to t-structures in $D^b(\mathrm{mod}(\Lambda, \Omega))$. Our main result, Theorem 3.3, gives a necessary and sufficient condition for the heart of this t-structure to give back the derived category $D^b(\mathrm{mod}(\Lambda, \Omega))$. This condition is easy to verify in practice. Some examples are given in § 5. When this condition is satisfied, Theorem 7.1 asserts that the heart of the t-structure can be identified to the category $\mathrm{m\acute{o}d}(B)$ where B is an algebra obtained as an n-step ($n = \#\Lambda$) tilting of the quiver algebra of $(\Lambda, \bar{\Omega})$ where $(\Lambda, \bar{\Omega})$ is another quiver obtained from (Λ, Ω) by changing the direction of some arrows.

In § 4 we prove that the intermediate t-structures that arise from disregarding tails of 'data' have the property that the realization functor (§ 3) is an equivalence if the final t-structure is so. This allows us to use the inductive arguments of § 6 leading to Theorem 7.1. The fact that if (Λ, Ω) is a Dynkin quiver then it is of finite representation type can be used to deduce from Theorem 7.1 a completeness result for this case (Theorem 8.5). This is related to a result of Happel [5, § 5].

1. Reflection functors R_α^+, R_α^-

Let k be a field assumed algebraically closed. Let (Λ, Ω) be a finite quiver (:= a finite set of vertices linked with arrows) without oriented cycles. Here Λ denotes the

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underlying set of vertices and Ω the orientation data for the edges of the graph. A representation V of (Λ, Ω) assigns a vector space V_α (over k) to each vertix $\alpha \in \Lambda$ and a linear map $V_{\overrightarrow{\alpha\beta}}: V_\alpha \to V_\beta$ to each arrow $\alpha \to \beta$. A morphism between two such representations V and W is a collection of linear maps $f(\alpha): V_\alpha \to W_\alpha$, $(\alpha \in \Lambda)$ satisfying $f(\beta) \cdot V_{\overrightarrow{\alpha\beta}} = W_{\overrightarrow{\alpha\beta}} \cdot f(\alpha)$ for each arrow $\alpha \to \beta$ of (Λ, Ω). Denote by $\operatorname{mod}(\Lambda, \Omega)$ the (abelian) category of finite dimensional representations of (Λ, Ω) $(:= \sum_\alpha \dim V(\alpha) < \infty)$.

(1.1) The category $\operatorname{mod}(\Lambda, \Omega)$ can be identified with the category of finitely generated (left) modules over a finite dimensional k-algebra $k[\Lambda, \Omega]$ called the quiver algebra of (Λ, Ω) (cf. [4], §4).

If $\alpha \in \Lambda$, α is said to be a sink (resp. source) if there are no arrows starting at α (resp. ending at α). If $\alpha \in \Lambda$, $(\Lambda, s_\alpha \Omega)$ is a new quiver obtained from (Λ, Ω) by reversing the arrows at α.

(1.2) The orientation Ω in (Λ, Ω) is called admissible if there exists an enumeration $\alpha_1, \ldots, \alpha_n$ of Λ such that α_1 is a sink of (Λ, Ω), α_2 is a sink of $(\Lambda, s_{\alpha_1}\Omega), \ldots, \alpha_i$ is a sink of $(\Lambda, s_{\alpha_{i-1}} \cdots s_{\alpha_2} \cdot s_{\alpha_1} \Omega)$ $(1 \leqslant i \leqslant n)$. In this case $\alpha_1, \ldots, \alpha_n$ is called an admissible enumeration. All the quivers that we encounter in this article shall have an admissible orientation.

For an introduction to triangulated categories and general notions associated with them and their study we refer to [7] and [6]. Derived categories and derived functors which we use here are as in [7] and [6]. Sometimes, we say triangles instead of distinguished triangles.

If α is a sink or a source one has 'reflection' functors $\operatorname{mod}(\Lambda, \Omega) \to \operatorname{mod}(\Lambda, s_\alpha \Omega)$ which were introduced by Gelfand, Bernstein and Ponomarev. We do not need the exact definitions here which can be found in [3, pp. 15, 16]. However, we state here in a convenient form the important fact that the derived functors of these reflection functors become isomorphisms between derived categories. This fact which was proved by Happel [5] in a more general context (namely, tiltings) was also independently proved by the author (unpublished). The form in which we state it here is closer to the latter version.

(1.3) It is convenient to denote an object V of $\operatorname{mod}(\Lambda, \Omega)$ by $\{V_\alpha, \ldots, V_{\overrightarrow{\alpha\beta}}, \ldots\}$ (see definition before 1.1). A complex $V^\cdot, d)$ (where d is the differential) of objects of $\operatorname{mod}(\Lambda, \Omega)$ will be denoted by $\{V^\cdot_\alpha, \ldots, V^\cdot_{\overrightarrow{\alpha\beta}}, \ldots\}$. Let E be the set of arrows of (Λ, Ω) whose end point is α. (Here, α is supposed to be a sink of (Λ, Ω).) Then we have a homomorphism

$$\bigoplus_{\overrightarrow{\beta\alpha} \in E} V^\cdot_\beta \to V^\cdot_\alpha \tag{1.3a}$$

by summing the $V^\cdot_{\overrightarrow{\beta\alpha}}$ for $\overrightarrow{\beta\alpha} \in E$. Note that 1.3a is a morphism of complexes of vector spaces. Let $M^\cdot$ be the mapping cone on the negative of the morphism 1.3a. By definition of the mapping cone construction, we have canonical morphism $M^\cdot[-1] \to$ left side of 1.3a $\{M^\cdot \to M^\cdot[1]$ is the translation functor$\}$. Now we define $V'^\cdot_\alpha = M^\cdot[-1]$ and $V'^\cdot_\gamma = V^\cdot_\gamma$ if $\gamma \neq \alpha$ $(\gamma \in \Lambda)$. Thus, we have a canonical morphism

$$V'^\cdot_\alpha \to \bigoplus_{\overrightarrow{\beta\alpha} \in E} V^\cdot_\beta,$$

which, by projecting into summands induces morphisms

$$V'^\cdot_\alpha \to V^\cdot_\beta \quad (\text{for } \overrightarrow{\beta\alpha} \in E).$$

We denote their negatives by $V'^{\cdot}_{\overrightarrow{\alpha\beta}}: V'^{\cdot}_{\alpha} \to V^{\cdot}_{\beta}(=V'^{\cdot}_{\beta})$. (Our sign conventions coincide with those in [V] but differ from [6].) Also, we define $V'^{\cdot}_{\overrightarrow{\mu\nu}}: V'^{\cdot}_{\mu} \to V'^{\cdot}_{\nu}$ to be equal to the given $V^{\cdot}_{\overrightarrow{\mu\nu}}$ if μ and ν are different from α. $\{V'^{\cdot}_{\mu}, \ldots, V'^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\}$ is a complex of objects of $\mathrm{mod}(\Lambda, s_\alpha\Omega)$. We define (1.3b) $R^+_\alpha\{V^{\cdot}_{\mu}, \ldots, V^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\} = \{V'^{\cdot}_{\mu}, \ldots, V'^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\}$ R^+_α is functorial for morphisms of chain complexes. It takes homotopy equivalent morphisms to homotopy equivalent morphisms and quasi-isomorphisms to quasi-isomorphisms. Thus, R^+_α gives rise to a functor between the derived categories of $\mathrm{mod}(\Lambda, \Omega)$ and $\mathrm{mod}(\Lambda, s_\alpha\Omega)$.

By the result [5, Theorem 1.6] of Happel.

PROPOSITION 1.4

Let $D^b(\mathrm{mod}(\Lambda, \Omega))$ *{resp.* $D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$*} denote the bounded derived category of* $\mathrm{mod}(\Lambda, \Omega)$ *{resp.* $\mathrm{mod}(\Lambda, s_\alpha\Omega)$*}. Then* $R^+_\alpha: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$ *is an equivalence of triangulated categories.*

(1.5) Dually, if α is a source, there is an analogous result. Let E be the set of arrows of (Λ, Ω) whose initial point is α. Then we have a homomorphism

$$V^{\cdot}_{\alpha} \to \bigoplus_{\overrightarrow{\alpha\beta} \in E} V^{\cdot}_{\beta}, \tag{1.5a}$$

where $V^{\cdot}$ is a complex of objects of $\mathrm{mod}(\Lambda, \Omega)$ as in 1.3. Let $M^{\cdot}$ be the cone on the negative of 1.5a. We now define $V'^{\cdot}_{\alpha} = M^{\cdot}$ and $V'^{\cdot}_{\mu} = V^{\cdot}_{\mu}$ if $\mu \neq \alpha (\mu \in \Lambda)$. The cone construction yields a canonical morphism $\bigoplus_{\overrightarrow{\alpha\beta} \in E} V^{\cdot}_{\beta} \to V'^{\cdot}_{\alpha}$, which by restriction to the summands induces morphisms

$$V'^{\cdot}_{\overrightarrow{\beta\alpha}}: V'^{\cdot}_{\beta} \to V'^{\cdot}_{\alpha} \quad (\text{for } \overrightarrow{\alpha\beta} \in E). \tag{1.5b}$$

If μ and ν are different from α, we define

$$V'^{\cdot}_{\overrightarrow{\mu\nu}}: V'^{\cdot}_{\mu} \to V'^{\cdot}_{\nu}$$

to be equal to the given $V^{\cdot}_{\overrightarrow{\mu\nu}}$ in $V^{\cdot}$.

We define

$$R^-_\alpha\{V^{\cdot}_{\mu}, \ldots, V^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\} = \{V'^{\cdot}_{\mu}, \ldots, V'^{\cdot}_{\overrightarrow{\mu\nu}}, \ldots\}. \tag{1.5c}$$

Similar to 1.4 we have

PROPOSITION 1.6

$R^-_\alpha: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega))$ *is an equivalence of triangulated categories.*

Remark 1.7 One can show that $R^-_\alpha \circ R^+_\alpha$ and $R^+_\alpha \circ R^-_\alpha$ are naturally equivalent to the identity functor.

(1.8) Let α be a sink of (Λ, Ω). Let (Λ_U, Ω_U) be the quiver obtained by deleting α from (Λ, Ω). We can regard an object of $\mathrm{mod}(\Lambda_U, \Omega_U)$ as an object of $\mathrm{mod}(\Lambda, \Omega)$ {resp. $\mathrm{mod}(\Lambda, s_\alpha\Omega)$} by extending by zero over α. This defines obvious functors $j_*: \mathrm{mod}(\Lambda_U, \Omega_U) \to \mathrm{mod}(\Lambda, \Omega)$ and $\tilde{j}_*: \mathrm{mod}(\Lambda_U, \Omega_U) \to \mathrm{mod}(\Lambda, s_\alpha\Omega)$ which are exact

We denote the derived functors by the same symbols. Thus we have defined

$$j_*: D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}(\Lambda, \Omega)), \tag{1.8a}$$

and

$$\tilde{j}_*: D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}(\Lambda, s_\alpha \Omega)). \tag{1.8b}$$

We define

$$j_!: D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) \to D^b(\mathrm{mod}(\Lambda, \Omega)), \tag{1.8c}$$

by $j_! = R_\alpha^- \circ \tilde{j}_*$ where $R_\alpha^-: D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega)) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is the isomorphism 1.5c. An object of $\mathrm{mod}(\Lambda, \Omega)$ gives rise to an object of $\mathrm{mod}(\Lambda_U, \Omega_U)$ by restriction. This gives rise to an exact functor $j^*: \mathrm{mod}(\Lambda, \Omega) \to \mathrm{mod}(\Lambda_U, \Omega_U)$ and its derived functor

$$j^*: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) \tag{1.9}$$

Lemma 1.10

(i) $j_!$ *is left adjoint to* j^*
(ii) j_* *is right adjoint to* j^*.

This lemma will be proved along with Lemma 1.15 below.

We can regard a k-vector space V as an object of $\mathrm{mod}(\Lambda, \Omega)$ by setting $V_\beta = 0$ for $\beta \neq \alpha (\beta \in \Lambda)$, $V_\alpha = V$ and $V_{\overrightarrow{\beta\alpha}} = 0$ for all arrows. This defines an obvious exact functor $i_*: \mathrm{mod}\, k \to \mathrm{mod}(\Lambda, \Omega)$ and its derived functor

$$i_*: D^b(\mathrm{mod}\, k) \to D^b(\mathrm{mod}(\Lambda, \Omega)). \tag{1.11}$$

Since an object V of $\mathrm{mod}(\Lambda, \Omega)$ {resp. $\mathrm{mod}(\Lambda, s_\alpha\Omega)$} assigns a vector space V_β for each $\beta \in \Lambda$, we have an exact functor $i^!: \mathrm{mod}(\Lambda, \Omega) \to \mathrm{mod}\, k$ {resp. $\tilde{i}^!: \mathrm{mod}(\Lambda, s_\alpha\Omega) \to \mathrm{mod}\, k$} defined by $V \mapsto V_\alpha$ and derived functors

$$i^!: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}\, k) \tag{1.12}$$

and

$$\tilde{i}^!: D^b(\mathrm{mod}(\Lambda, s_\alpha\Omega)) \to D^b(\mathrm{mod}\, k). \tag{1.13}$$

We define

$$i^*: D^b(\mathrm{mod}(\Lambda, \Omega)) \to D^b(\mathrm{mod}\, k) \tag{1.14}$$

by $i^*(X) = (\tilde{i}^! \circ R_\alpha^+)(X[1])$

Lemma 1.15

(i) i^* *is left adjoint to* i_*
(ii) $i^!$ *is right adjoint to* i_*.

Before proving 1.10 and 1.15, we observe the easy relations

$$i^* \circ i_* = \text{identity} = i^! \circ i_* \tag{1.15a}$$

and

$$j^* \circ j_* = \text{identity} = j^* \circ j_!. \tag{1.15b}$$

Proof of 1.10 and 1.15. Let $M \in D^b(\mathrm{mod}(\Lambda,\Omega))$ and $N_U \in D^b(\mathrm{mod}(\Lambda_U,\Omega_U))$. Then one sees from definitions that $\mathrm{Hom}(j^*M, N_U) \approx \mathrm{Hom}(M, j_*N_U)$ which implies 1.10 (ii). Also,

$$\begin{aligned}\mathrm{Hom}(N_U, j^*M) &= \mathrm{Hom}(N_U, \tilde{j}^* R_\alpha^+ M)\\ &= \mathrm{Hom}(\tilde{j}_* N_U, R_\alpha^+ M)\\ &= \mathrm{Hom}(R_\alpha^- \tilde{j}_* N_U, M) \qquad \text{(by 1.7)}\\ &= \mathrm{Hom}(j_! N_U, M) \qquad \text{(by 1.8c)}\end{aligned}$$

which proves 1.10 (i).

Let $M \in D^b(\mathrm{mod}(\Lambda,\Omega))$ and $V \in D^b(\mathrm{mod}\, k)$. Then it is trivial to see that $\mathrm{Hom}(i_* V, M) = \mathrm{Hom}(V, i^! M)$. Also,

$$\begin{aligned}\mathrm{Hom}(M, i_* V) &= \mathrm{Hom}(R_\alpha^+ M, R_\alpha^+ i_* V) \quad (1.4)\\ &= \mathrm{Hom}(R_\alpha^+ M, \tilde{i}_*(V[-1]))\\ &= \mathrm{Hom}(\tilde{i}^! R_\alpha^+ M, V[-1])\\ &= \mathrm{Hom}(i^* M, V). \qquad (1.14).\end{aligned}$$

This proves 1.15. (q.e.d.)

(1.16) The functors i^*, i_*, $i^!$, $j_!$, j^*, j_* satisfy the formalism of [1, § 1.4.3] namely the properties 1.4.3.1 thru 1.4.3.5 as can be checked easily using the definitions. That is all that one needs to apply 'recollement' [1, § 1.4].

(1.16a) We recall the definition of a 't-structure' [1, § 1.3]. It is a data $(D^{\leq 0}, D^{\geq 0})$ where $D^{\leq 0}$ and $D^{\geq 0}$ are full subcategories of $D^b(\mathrm{mod}(\Lambda,\Omega))$ satisfying axioms (i), (ii) and (iii) below. For any integer n write $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$

(i) $\mathrm{Hom}(X, Y) = 0$ for $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$.
(ii) $D^{\leq 0} \subseteq D^{\leq 1}$ and $D^{\geq 0} \supseteq D^{\geq 1}$.
(iii) For any $X \in D^b(\mathrm{mod}(\Lambda,\Omega))$ there exists a distinguished triangle $A \to X \to B$ such that $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

PROPOSITION 1.16b

Let $(D_U^{\leq 0}, D_U^{\geq 0})$ be a t-structure in $D^b(\mathrm{mod}(\Lambda_U,\Omega_U))$ and $(D_F^{\leq 0}, D_F^{\geq 0})$ a t-structure in $D^b(\mathrm{mod}\, k)$. Define

$$\tilde{D}^{\leq 0} = \{K \in D^b(\mathrm{mod}(\Lambda,\Omega)) \mid j^*K \in D_U^{\leq 0} \text{ and } i^*K \in D_F^{\leq 0}\}$$

and

$$\tilde{D}^{\geq 0} = \{K \in D^b(\mathrm{mod}(\Lambda,\Omega)) \mid j^*K \in D_U^{\geq 0} \text{ and } i^!K \in D_F^{\geq 0}\}.$$

Then $(\tilde{D}^{\leq 0}, \tilde{D}^{\geq 0})$ is a t-structure in $D^b(\mathrm{mod}(\Lambda,\Omega))$.

The proposition is [1, Theorem 1.4.10].

(1.17) Let $(D_{U,\mathrm{nat}}^{\leq 0}, D_{U,\mathrm{nat}}^{\geq 0})$ {resp. $(D_{\mathrm{nat}}^{\leq 0}, D_{\mathrm{nat}}^{\geq 0})$} be the natural t-structure of $D^b(\mathrm{mod}(\Lambda_U,\Omega_U))$ {resp. $D^b(\mathrm{mod}(\Lambda,\Omega))$}. For the definition of natural t-structure see [1, 1.3.2]. Let $(D_{F,\mathrm{nat}}^{\leq 0}, D_{F,\mathrm{nat}}^{\geq 0})$ be the natural t-structure of $D^b(\mathrm{mod}\, k)$.

Remark 1.17a. The functors $i^!$, i_*, j^*, j_* are exact with respect to the natural t-structures (i.e. $D_{\ldots}^{\leqslant 0}$ goes into $D_{\ldots}^{\leqslant 0}$ and $D_{\ldots}^{\geqslant 0}$ goes into $D_{\ldots}^{\geqslant 0}$). This is an easy consequence of the definitions.

(1.18) In the context of Proposition 1.16b, if $D_U^{\leqslant 0} \subseteq D_{U,\mathrm{nat}}^{\leqslant 0}$ and $D_F^{\leqslant 0} = D_{F,\mathrm{nat}}^{\leqslant 0}$, then $\tilde{D}^{\leqslant 0}$ can also be described by

$$\tilde{D}^{\leqslant 0} = \{K \in D_{\mathrm{nat}}^{\leqslant 0} \mid j^* K \in D_U^{\leqslant 0}\}.$$

In addition one also has

$$\tilde{D}^{\leqslant 0} = \{K \in D^b(\mathrm{mod}(\Lambda, \Omega)) \mid j^* K \in D_U^{\leqslant 0} \text{ and } i^! K \in D_F^{\leqslant 0}\}.$$

These assertions will be proved below. But first we deduce from the last assertion and 1.16b (definition of $\tilde{D}^{\geqslant 0}$) the following

(1.18a) When $D_U^{\leqslant 0} \subseteq D_{U,\mathrm{nat}}^{\leqslant 0}$ and $D_F^{\leqslant 0} = D_{F,\mathrm{nat}}^{\leqslant 0}$, the functor $i^!$ is exact with respect to the t-structures $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$ and $(D_F^{\leqslant 0}, D_F^{\geqslant 0})$ in $D^b(\mathrm{mod}(k))$.

Proof of the assertions 1.18. Suppose $K \in D^b(\mathrm{mod}(\Lambda, \Omega))$ and $j^* K \in D_U^{\leqslant 0}$. We will show that $i^* K \in D_F^{\leqslant 0} \Leftrightarrow i^! K \in D_F^{\leqslant 0}$ and further when this is so, $K \in D_{\mathrm{nat}}^{\leqslant 0}$. The last assertion follows since $D_{\mathrm{nat}}^{\leqslant 0} = \{K \in D^b(\mathrm{mod}(\Lambda, \Omega)) \mid j^* K \in D_{U,\mathrm{nat}}^{\leqslant 0}$ and $i^! K \in D_{F,\mathrm{nat}}^{\leqslant 0}\}$. Recall the definitions of $i^* K$, $i^! K$ and R_α^+ {see resp. 1.14, 1.12 and 1.3b}. Write $K = \{V_\mu^{\cdot}, \ldots, V^{\cdot}(\overrightarrow{\mu\nu}), \ldots\}$ in the notation of 1.3. Then we have a distinguished triangle

$$L \to i^! K \to i^* K \to L[1] \tag{1.19}$$

in $D^b(\mathrm{mod}\, k)$ where L is the left side of 1.3a (i.e. the direct sum of the parts of K over vertices adjacent to α). From the assumptions $j^* K \in D_U^{\leqslant 0}$ and $D_U^{\leqslant 0} \subseteq D_{U,\mathrm{nat}}^{\leqslant 0}$, it follows that $L \in D_{F,\mathrm{nat}}^{\leqslant 0}$ ($= D_F^{\leqslant 0}$ by assumption) and aportioti $L[1] \in D_F^{\leqslant 0}$. The distinguished triangles $(L, i^! K, i^* K)$ and $(i^! K, i^* K, L[1])$ (1.19) imply that $i^! K \in D_F^{\leqslant 0} \Leftrightarrow i^* K \in D_F^{\leqslant 0}$, this completes the proof of the second assertion in 1.18. In particular, one deduces that $\tilde{D}^{\leqslant 0} \subseteq D_{\mathrm{nat}}^{\leqslant 0}$. Conversely suppose $K \in D_{\mathrm{nat}}^{\leqslant 0}$ and $j^* K \in D_U^{\leqslant 0}$. Then $i^! K \in D_{F,\mathrm{nat}}^{\leqslant 0}$ ($= D_F^{\leqslant 0}$) and hence also $i^* K \in D_F^{\leqslant 0}$. Then, by what is already proved $K \in \tilde{D}^{\leqslant 0}$. This completes the proof of the first assertion in 1.18.

2. The t-structure associated to a data

Applying the previous constructions inductively, we will produce some t-structures in $D^b(\mathrm{mod}(\Lambda, \Omega))$. Recall that we assume that (Λ, Ω) does not have oriented cycles. In addition we assume that there is at most one arrow between any two vertices. Let n be the cardinality of Λ.

DEFINITION 2.1

A *data* in (Λ, Ω) is a collection of n sequences $\alpha_{p1}, \alpha_{p2}, \ldots, \alpha_{p\nu_p}$ ($1 \leqslant p \leqslant n$, $1 \leqslant \nu_p$) of not necessarily distinct vertices of (Λ, Ω). We assume that they satisfy the following conditions: α_{11} is a sink of (Λ, Ω); α_{12} is a sink of $(\Lambda, s_{11}\Omega)$ {here and in the sequel, $s_{p\mu}\Omega$ is abbreviation for $s_{\alpha_{p\mu}}\Omega$}$\ldots$; $\alpha_{1\mu}$ is a sink of $(\Lambda, s_{1(\mu-1)} \cdots s_{12} s_{11} \Omega)$ ($1 \leqslant \mu \leqslant \nu_1$). Define quivers (Λ_1,Ω_1) and (Λ_1,Ω_1') by $(\Lambda_1,\Omega_1)=(\Lambda,\Omega)$ and $(\Lambda_1,\Omega_1')=(\Lambda,s_{1(\nu_1-1)}\cdots s_{11}\Omega)$. Let (Λ_p,Ω_p) and (Λ_p,Ω_p') $(1\leqslant p\leqslant n)$ be the quivers defined inductively as follows. If (Λ_q,Ω_q) and (Λ_q,Ω_q') have been defined $(\Lambda_{q+1},\Omega_{q+1})$ is obtained by deleting α_{q,ν_q} and all arrows to it from (Λ_q,Ω_q'). We assume that for each $p\,(1\leqslant p\leqslant n)$, α_{p1} is a sink of (Λ_p,Ω_p), α_{p2} is a sink of $(\Lambda_p,s_{p1}\Omega_p)\ldots$ etc.... $\alpha_{p\nu_p}$ is a sink of $(\Lambda_p,s_{p(\nu_p-1)}\cdots s_{p2}s_{p1}\Omega_p)$. We then set

$$(\Lambda_{q+1},\Omega_{q+1}')=(\Lambda_{q+1},s_{q+1(\nu_{q+1}-1)}\cdots s_{q+1,2}s_{q+1,1}\Omega_{q+1}).$$

(2.2) The natural t-structure in $D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$ {resp. $D^b(\operatorname{mod}(\Lambda_{q+1},\Omega_{q+1}))$} will be denoted by $(D_q'^{\leqslant 0},D_q'^{\geqslant 0})$ {resp. $D_{q+1}^{\leqslant 0},D_{q+1}^{\geqslant 0})$}. We will now apply the constructions of the previous section taking $(\Lambda,\Omega)=(\Lambda_q,\Omega_q')$ and $(\Lambda_U,\Omega_U)=(\Lambda_{q+1},\Omega_{q+1})$. More particularly, remark 1.17a implies the following proposition.

{The functors $i^*,i_*,i^!,j_!,j^*,j_*,\tilde{\imath}_*,\tilde{\imath}^!,\tilde{\jmath}^*,\tilde{\jmath}_*$ defined in § 1 are denoted by the same symbols in the situation $(\Lambda,\Omega)=(\Lambda_q,\Omega_q')$ and $(\Lambda_U,\Omega_U)=(\Lambda_{q+1},\Omega_{q+1})$}.

PROPOSITION 2.3

Let $(\tilde{D}_{q+1}^{\leqslant 0},\tilde{D}_{q+1}^{\geqslant 0})$ be any t-structure in $D^b(\operatorname{mod}(\Lambda_{q+1},\Omega_{q+1}))$ such that $\tilde{D}_{q+1}^{\leqslant 0}\subseteq D_{q+1}^{\leqslant 0}$. Define $\tilde{D}_q^{\leqslant 0}=\{X\in D_q'^{\leqslant 0}\,|\,j^*X\in\tilde{D}_{q+1}^{\leqslant 0}\}$ and $\tilde{D}_q^{\geqslant 0}=\tilde{D}_q^{\geqslant 1}[1]$ where $\tilde{D}_q^{\geqslant 1}=\{X\in D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))\,|\,\operatorname{Hom}(Y,X)=0,\ \forall Y\in\tilde{D}_q^{\leqslant 0}\}$. Then $(\tilde{D}_q^{\leqslant 0},\tilde{D}_q^{\geqslant 0})$ is a t-structure in $D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$, such that $\tilde{D}_q^{\leqslant 0}\subseteq D_q'^{\leqslant 0}$.

{Recall from [1, 1.3.4] that if $(D_{..}^{\leqslant 0},D_{..}^{\geqslant 0})$ is a t-structure then $D_{..}^{\geqslant 1}$ is the right orthogonal of $D_{..}^{\leqslant 0}$}.

Remark 2.4. In the context of Proposition 2.3, the functor $j_*:D^b(\operatorname{mod}(\Lambda_{q+1},\Omega_{q+1}))\to D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$ is exact, with respect to the t-structures $(\tilde{D}_{q+1}^{\leqslant 0},\tilde{D}_{q+1}^{\geqslant 0})$ in $D^b(\operatorname{mod}(\Lambda_{q+1},\Omega_{q+1}))$ and $(\tilde{D}_q^{\leqslant 0},\tilde{D}_q^{\geqslant 0})$ in $D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$. Indeed, $X\in\tilde{D}_{q+1}^{\leqslant 0}\Rightarrow X\in D_{q+1}^{\leqslant 0}$ (by assumption)$\Rightarrow j_*X\in D_q'^{\leqslant 0}$ (1.17a). But as $j^*j_*X\approx X$ (1.15b), Proposition 2.3$\Rightarrow j_*X\in\tilde{D}_q^{\leqslant 0}$. This shows j_* is '*right*' exact ($:=j_*(D_{..}^{\leqslant 0})\subseteq D_{..}^{\leqslant 0}$). But in the general context of 'recollement' j_* is always '*left*' exact ($:=D_{..}^{\geqslant 0}$ goes into $D_{..}^{\geqslant 0}$)—an easy fact [1, Prop. 1.4.16(i)]. Moreover, as already remarked (1.18a) in the context of Proposition 2.3, $i^!$ is exact. (Again, $i^!$ is only left exact in the general context of 'recollement' [1, Prop. 1.4.16(i)].)

(2.5) To apply Proposition 2.3 for the (descending) inductive construction of $(\tilde{D}_q^{\leqslant 0},\tilde{D}_q^{\geqslant 0})$ for $1\leqslant q\leqslant n$, we will transport $(\tilde{D}_q^{\leqslant 0},\tilde{D}_q^{\geqslant 0})$ to $D^b(\operatorname{mod}(\Lambda_q,\Omega_q))$ using an isomorphism (described below) of $D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$ with $D^b(\operatorname{mod}(\Lambda_q,\Omega_q))$. Recalling 2.1 and using 1.6, we now have

Lemma 2.5a. For $1\leqslant\mu<\nu_q$, $\alpha_{q\mu}$ is a source of $(\Lambda_q,s_{q\mu}s_{q\mu-1}\cdots s_{q1}\Omega_q)$. Thus, the functor $R_{q\mu}^-=R_{\alpha_{q\mu}}^-:D^b(\operatorname{mod}(\Lambda_q,s_{q\mu}\cdots s_{q1}\Omega_q))\to D^b(\operatorname{mod}(\Lambda_q,s_{q(\mu-1)}\cdots s_{q1}\Omega_q))$ is an equivalence of triangulated categories. Let $R_q'=R_{q1}^-\circ R_{q2}^-\circ\cdots\circ R_{q(\nu_q-1)}^-$. Then the functor R_q' is an equivalence of $D^b(\operatorname{mod}(\Lambda_q,\Omega_q'))$ with $D^b(\operatorname{mod}(\Lambda_q,\Omega_q))$.

We will recall briefly the notation in 1.5 where we defined R_α^- to show

(2.6) R_α^- is right exact with respect to the natural t-structures.

Proof. If V''_α is the cone on the morphism 1.5a, we have a distinguished triangle

$$V^{\cdot}_\alpha \to \bigoplus_{\overrightarrow{\alpha\beta} \in E} V^{\cdot}_\beta \to V''_\alpha \to V^{\cdot}_\alpha[1]$$

(with notation as in 1.5a). If $V^{\cdot} \in D^{\leqslant 0}_{..}$ for the natural t-structure then the above triangle implies that $V''_\alpha \in D^{\leqslant 0}_{..}$ for the natural t-structure of $D^b(\mathrm{mod}\, k)$. If we recall how R^-_α is defined (1.5), it now ensues that $R^-_\alpha(V^{\cdot}) \in D^{\leqslant 0}_{..}$ for the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda, s_\alpha\Omega))$.

We can now conclude that the functor R'_q in Lemma 2.5a is right exact with respect to the natural t-structures, i.e.,

$$R'_q(D'^{\leqslant 0}_q) \subseteq D^{\leqslant 0}_q. \tag{2.7}$$

In particular, if $\tilde{D}^{\leqslant 0}_q$ is given by Proposition 2.3, then

$$R'_q(\tilde{D}^{\leqslant 0}_q) \subseteq R'_q(D'^{\leqslant 0}_q) \subseteq D^{\leqslant 0}_q. \tag{2.7a}$$

By abuse of notation we write $\tilde{D}^{\leqslant 0}_q$ instead of $R'_q(\tilde{D}^{\leqslant 0}_q)$ and regard $(\tilde{D}^{\leqslant 0}_q, \tilde{D}^{\geqslant 0}_q)$ as a t-structure in $D^b(\mathrm{mod}\,(\Lambda_q, \Omega_q))$. Thus

$$\tilde{D}^{\leqslant 0}_q \subseteq D^{\leqslant 0}_q. \tag{2.7b}$$

Now we can apply Proposition 2.3 to produce a t-structure $(\tilde{D}^{\leqslant 0}_{q-1}, \tilde{D}^{\geqslant 0}_{q-1})$ in $D^b(\mathrm{mod}\,(\Lambda_{q-1}, \Omega'_{q-1}))$ and so on...

(2.8) We can now define for each $q (1 \leqslant q \leqslant n)$ a t-structure $(\tilde{D}^{\leqslant 0}_q, \tilde{D}^{\geqslant 0}_q)$ in $D^b(\mathrm{mod}\,(\Lambda_q, \Omega_q))$ as follows. For $(\tilde{D}^{\leqslant 0}_n, \tilde{D}^{\geqslant 0}_n)$ we take the image of the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda_n, \Omega'_n))$ under the equivalence R'_n (2.5a): $D^b(\mathrm{mod}\,(\Lambda_n, \Omega'_n)) \to D^b(\mathrm{mod}\,(\Lambda_n, \Omega_n))$. Then we inductively construct $(\tilde{D}^{\leqslant 0}_{n-1}, \tilde{D}^{\geqslant 0}_{n-1})$, $(\tilde{D}^{\leqslant 0}_{n-2}, \tilde{D}^{\geqslant 0}_{n-2}), \ldots, (\tilde{D}^{\leqslant 0}_1, \tilde{D}^{\geqslant 0}_1)$ respectively in $D^n(\mathrm{mod}\,(\Lambda_{n-1}, \Omega_{n-1}))$, $D^b(\mathrm{mod}\,(\Lambda_{n-2}, \Omega_{n-2})), \ldots, D^b(\mathrm{mod}\,(\Lambda, \Omega))$.

(2.9) Finally, the t-structure $(\tilde{D}^{\leqslant 0}_1, \tilde{D}^{\geqslant 0}_1)$ is what we refer to in the sequel as the 't-structure associated to the data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} | 1 \leqslant p \leqslant n\}$'.

(2.10a) Example. The natural t-structure of $D^b(\mathrm{mod}\,(\Lambda, \Omega))$ can be obtained as the t-structure associated to a data in the following way (roughly, spreading out an admissible enumeration (1.2) of (Λ, Ω)).

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an admissible enumeration of (Λ, Ω). Define a data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} | 1 \leqslant p \leqslant n\}$ by $v_p = 1 \forall p$ and $\alpha_{p1} = \alpha_p$. In this example $(\Lambda_q, \Omega'_q) = (\Lambda_q, \Omega_q)$ for all q and R'_q is the identity functor. The t-structure associated to this data (2.9) is simply the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda, \Omega))$.

(2.10b) Example. Let (Λ, Ω) be the quiver $\overset{1}{.} \to \overset{2}{.} \to \overset{3}{.}$ (this is a Dynkin quiver of type A_3). If α, β, γ are the simple roots corresponding to the vertices 1, 2, 3 respectively then the positive roots are $\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma$ and $\alpha + \beta + \gamma$. The indecomposable objects (up to isomorphism) of $\mathrm{mod}\,(\Lambda, \Omega)$ are in 1-1 correspondence with the positive roots (according to Gabriel's observation). We denote these indecomposable objects by V_α, $V_\beta, V_\gamma, V_{\alpha+\beta}, V_{\beta+\gamma}, V_{\alpha+\beta+\gamma}$ respectively. Then the indecomposable objects of $D^b(\mathrm{mod}\,(\Lambda, \Omega))$ (up to isomorphism) are (i) $\{V_\theta[i] | i \in \mathbb{Z}$ and $\theta \in (\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma)\}$ (see for

instance [5,4.1]). If one restricts to $i \geqslant 0$ (resp. $i \leqslant 0$) in (i) then one gets the indecomposables of $D^{\leqslant 0}$ (resp. $D^{\geqslant 0}$) for the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$. The t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$ associated to the data $\{(3), (2, 1), (2)\}$ can be described as follows. The indecomposables of $\tilde{D}_1^{\leqslant 0}$ are of the form $V_\theta[i]$ where (i) either $\theta = \alpha, \gamma, \alpha+\beta, \alpha+\beta+\gamma$ and $i \geqslant 0$ or (ii) $\theta = \beta, \beta+\gamma$ and $i > 0$. The indecomposables of $\tilde{D}_1^{\geqslant 0}$ are of the form $V_\theta[i]$ where (i) either $\theta = \alpha, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma$ and $i \leqslant 0$, or (ii) $\theta = \beta$ and $i \leqslant 1$.

(2.11) We will now define some t-structures intermediate to $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ and the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$. They play a role in the proof of Theorem 7.1.

Fix a q, $1 \leqslant q \leqslant n$ and v such that $1 \leqslant v < v_q$. Define

(2.11a) $R'_{q,(v)}: D^b(\mathrm{mod}(\Lambda_q, s_{qv} \cdots s_{q1}\Omega_q)) \to D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ by $R'_{q,(v)} = R^-_{q1} \circ R^-_{q2} \circ \cdots \circ R^-_{qv}$ (2.5a). Thus for example,

(2.11b) $R'_{q,v_q-1} = R'_q$ (2.5a) in our earlier notation. Define $(\tilde{D}^{\leqslant 0}_{q,(v)}, \tilde{D}^{\geqslant 0}_{q,(v)})$ in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ to be the image of the natural t-structure of $D^b(\mathrm{mod}(\Lambda_q, s_{qv} \cdots s_{q1}\Omega_q))$ in $D^b(\mathrm{mod}(\Lambda_q, \Omega_q))$ under the isomorphism $R'_{q,(v)}$ and $\tilde{C}_{q,(v)}$ its heart. Starting with this $(\tilde{D}^{\leqslant 0}_{q,(v)}, \tilde{D}^{\geqslant 0}_{q,(v)})$ apply the inductive construction of Proposition 2.3 to obtain new t-structures $(\tilde{D}^{\leqslant 0}_{q-1}, \tilde{D}^{\geqslant 0}_{q-1}), \ldots, (\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_{q-1}, \Omega_{q-1})), \ldots, D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$ respectively.

(2.11c) We will use the notation $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ to denote the t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ obtained in this way. In this notation, the t-structure associated to the data $\{\alpha_{p1}, \ldots, \alpha_{pv_p} \mid 1 \leqslant p \leqslant n\}$ is $(\hat{D}^{\leqslant 0}_{n,(v_n-1)}, \hat{D}^{\geqslant 0}_{n,(v_n-1)})$. One has the obvious inclusion relations

$$\hat{D}^{\leqslant 0}_{q,(v)} \subseteq \hat{D}^{\leqslant 0}_{q,(v-1)}, \hat{D}^{\leqslant 0}_{q,(v)} \subseteq \hat{D}^{\leqslant 0}_{q-1,(\mu)} \tag{2.11d}$$

and also $\hat{D}^{\geqslant 0}_{q,(v)} \supseteq \hat{D}^{\geqslant 0}_{q,(v-1)}, \hat{D}^{\geqslant 0}_{q,(v)} \supseteq \hat{D}^{\geqslant 0}_{q-1,\mu}$.

Remark (2.12). Let P be the simple projective of $\mathrm{mod}(\Lambda_q, s_{q,v-1} \cdots s_{q1}\Omega_q)$ corresponding to the sink $\alpha_{q,v}$. Then, the indecomposables of $\tilde{D}^{\leqslant 0}_{q,(v)}$ are obtained by dropping the isomorphism class of $R'_{q,(v-1)}(P)$ from the indecomposables of $\tilde{D}^{\leqslant 0}_{q,(v-1)}$.

3. A necessary and sufficient condition

If $(D^{\leqslant 0}, D^{\geqslant 0})$ is any t-structure in $D^b(\mathrm{mod}(\Lambda, \Omega))$ denote by $\mathscr{G}$ the full subcategory $D^{\leqslant 0} \cap D^{\geqslant 0}$. It is an abelian category called the 'heart' of the t-structure [1, 1.3.1]. If $D^b_*(\mathscr{G})$ denotes the bounded derived category of $\mathscr{G}$, then a functor 'real' ('realization'): $D^b(\mathscr{G}) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ is defined in [1, 3.1.10]. In this article, we are concerned with the t-structures for which the functor real is an equivalence of triangulated categories.

(3.1) Recall the functors $j_!, j_*: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q))$ and $j_*: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, s_{qv_q}\Omega'_q))$ (2.2 and 1.8). We now define functors

$$\begin{aligned} &\rho_q: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega'_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q)) \text{ by} \\ &\rho_q = j_! R'_{q+1}, \text{ (2.5a).} \end{aligned} \tag{3.1a}$$

For each $p (1 \leqslant p \leqslant n)$ define $\mathscr{G}'(p) =$ the heart of the natural t-structure of

$D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$. For simplicity of notation $R'_p(\mathscr{G}'(p))$ will also be denoted by $\mathscr{G}'(p)$.

Let $\tilde{\mathscr{G}}_1$ denote the heart of $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ constructed in § 2 (2.9).

A necessary and sufficient condition (effaceability) for the functor real to be an isomorphism is given in [1, Prop. 3.1.16] (see 3.4 below). For the t-structure $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ (2.9) we will derive a consequence of this condition. In this form, though not very transparent, it is easy to verify in practice as will become clear with a few examples (5.1 and 5.2).

(3.2) Write $V(q)$ for the simple object of $\mathrm{mod}(\Lambda_q, \Omega'_q)$ corresponding to the vertex α_{q,v_q} (i.e. we have a one-dimensional vector space over the vertex α_{q,v_q} and null vector space over other vertices).

Theorem 3.3. *The functor real:* $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ *is an equivalence of triangulated categories if and only if*

For each p, q with $1 \leqslant p \leqslant q < n$ *either* $\rho_p \circ \rho_{p+1} \circ \cdots \circ \rho_q(V(q+1))$ *belongs to* $\mathscr{G}'(p)$ *or equals* $j_* \circ R'_{p+1} \circ (\rho_{p+1} \circ \rho_{p+2} \circ \cdots \circ \rho_q)(V(q+1))$. *{Here* j_* *is functor (1.8a, 2.2)* $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda_p, \Omega'_p))$*}.* (3.3a)

When this condition is satisfied up to isomorphism the objects $R'_1 \circ (\rho_1 \circ \cdots \circ \rho_{q-1})(V(q))$, $(1 \leqslant q \leqslant n)$ *are precisely all the indecomposable projectives of* $\tilde{\mathscr{G}}_1$.

Proof. By (1, Prop. 3.1.16).

(3.4) A necessary and sufficient condition for real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ to be an equivalence is the following:

Given $A, B \in \tilde{\mathscr{G}}_1$ and $f \in \mathrm{Hom}(A, B[n])$ where $n > 0$, $\exists$ a monomorphism (in $\tilde{\mathscr{G}}_1$) $B \to C$ such that f has image zero under the canonical map $\mathrm{Hom}(A, B[n]) \to \mathrm{Hom}(A, C[n])$.

Remark (3.5) Observe that condition 3.4 is equivalent to "the property 3.4 holds even if one only assumes $A \in \tilde{D}_1^{\geqslant 0}$."

Indeed, first if $A = A'[-m]$ where $m \geqslant 0$ then $\mathrm{Hom}(A, B[n]) \approx \mathrm{Hom}(A', B[m+n])$ and so the property 3.4 for A' implies the property for A. Next, suppose $A \in \tilde{D}_1^{\geqslant 0} \cap \tilde{D}_1^{\leqslant m+1}$. Then, one has a truncation triangle $\tau_{\leqslant m}(A) \to A \to \tau_{\geqslant m+1}(A)$ where $\tau_{\leqslant m}(A) \in D^{\geqslant 0} \cap D^{\leqslant m}$ and $\tau_{\geqslant m+1}(A) \in \tilde{\mathscr{G}}_1[-m-1]$. A suitable induction argument (see the proof of 3.18) completes the proof.

(3.6) Fix p, $(1 \leqslant p \leqslant n)$. The n-p sequences $\alpha_{q1}, \ldots, \alpha_{qn_q} (p+1 \leqslant q \leqslant n)$ give rise to a t-structure in $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ via the construction of § 2. Clearly, this coincides with what was denoted $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ in the inductive construction of § 2 (2.8). The composite of the functors $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \xrightarrow{j_*} D^b(\mathrm{mod}(\Lambda_p, \Omega'_p)) \xrightarrow{R'_p} D^b(\mathrm{mod}(\Lambda_p, \Omega_p)) \xrightarrow{j_*} D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1})) \xrightarrow{R'_{p-1}} D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega_{p-1})) \xrightarrow{j_*} \cdots \to D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$ is seen to be exact (using Remark 2.4) with respect to the t-structures $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ and $(\tilde{D}_1^{\leqslant 0}, \tilde{D}_1^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda, \Omega))$. {To avoid a possible confusion, we point out here that even though R'_p is only right exact with respect to the natural t-structures (by 2.7) it is exact with respect to $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$ in $D^b(\Lambda_p, \Omega'_p)$ and its transport by R'_p denoted by the same notation $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_p, \Omega_p))$}.

(3.6a) We denote this composite by $\prod_{p,*}$. Thus, $\prod_{p,*}:D^b(\mathrm{mod}(\Lambda_{p+1},\Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda,\Omega))$ is exact with respect to $(\tilde{D}^{\leqslant 0}_{p+1},\tilde{D}^{\geqslant 0}_{p+1})$ and $(\tilde{D}^{\leqslant 0}_1,\tilde{D}^{\geqslant 0}_1)$.

(3.7) We denote by $\tilde{\mathscr{G}}_p$ the heart of the t-structure $(\tilde{D}^{\leqslant 0}_p,\tilde{D}^{\geqslant 0}_p)$ in $D^b(\mathrm{mod}(\Lambda_p,\Omega'_p))$ and we use the same notation $\tilde{\mathscr{G}}_p$ for $R'_p(\tilde{\mathscr{G}}_p)$. By $\mathscr{G}(p)$ (resp. $\mathscr{G}'(p)$) we denote the heart of the natural t-structure of $D^b(\mathrm{mod}(\Lambda_p,\Omega_p))$ (resp. $D^b(\mathrm{mod}(\Lambda_p,\Omega'_p))$). Again $R'_p(\mathscr{G}'(p))$ will also be denoted by $\mathscr{G}'(p)$.

(3.8) By 1.10 (ii), $\mathrm{Hom}(j_*X,j_*Y)\approx\mathrm{Hom}(j^*j_*X,Y)\approx\mathrm{Hom}(X,Y)$. Thus, for $X,Y\in\tilde{\mathscr{G}}_{p+1}$, $\prod_{p,*}(X)$ and $\prod_{p,*}(Y)$ belong to $\tilde{\mathscr{G}}_1$ and $\mathrm{Hom}(\prod_{p,*}(X),\prod_{p,*}(Y))\approx\mathrm{Hom}(X,Y)$ and $\mathrm{Hom}(\prod_{p,*}(X),\prod_{p,*}(Y)[n])\approx\mathrm{Hom}(X,Y[n])$ for $n\in\mathbb{Z}$.

(3.9) Similarly the composite of the functors $D^b(\mathrm{mod}(\Lambda_1,\Omega_1))\approx D^b(\mathrm{mod}(\Lambda_1,\Omega'_1))\xrightarrow{j^*}D^b(\mathrm{mod}(\Lambda_2,\Omega_2))\approx D^b(\mathrm{mod}(\Lambda_2,\Omega'_2))\xrightarrow{j^*}\cdots\xrightarrow{j^*}D^b(\mathrm{mod}(\Lambda_{p+1},\Omega_{p+1}))$ is exact with respect to $(\tilde{D}^{\leqslant 0}_1,\tilde{D}^{\geqslant 0}_1)$ and $(\tilde{D}^{\leqslant 0}_{p+1},\tilde{D}^{\geqslant 0}_{p+1})$. Denote this composite functor by $\prod^*_p$. As it is exact, in particular, if $\Psi:A\to B$ is a monomorphism in $\tilde{\mathscr{G}}_1$, then $\prod^*_p\Psi:\prod^*_pA\to\prod^*_pB$ is a monomorphism in $\tilde{\mathscr{G}}_{p+1}$. We also observe that $\prod^*_p\circ\prod_{p,*}\approx$ identity.

Lemma 3.10. If real: $D^b(\tilde{\mathscr{G}}_1)\to D^b(\mathrm{mod}(\Lambda,\Omega))$ is an equivalence then real: $D^b(\tilde{\mathscr{G}}_{p+1})\to D^b(\mathrm{mod}(\Lambda_{p+1},\Omega_{p+1}))$ is also an equivalence.

Proof. Let $X,Y\in\tilde{\mathscr{G}}_{p+1}$ and $f\in\mathrm{Hom}(X,Y[n])$, $n>0$. Let $\varphi:\prod_pY\to M$ be a monomorphism in $\tilde{\mathscr{G}}_1$ such that $\prod_p(f)$ goes to zero under the canonical map $\mathrm{Hom}(\prod_pX,\prod_pY[n])\to\mathrm{Hom}(\prod_pX,M[n])$. Using Remark 3.9, we then see that $\prod^*_p(\varphi):\prod^*_p\prod_{p,*}Y(\approx Y)\to\prod^*_pM$ is a monomorphism and f goes to zero under $\mathrm{Hom}(X,Y[n])\to\mathrm{Hom}(X,\prod^*_pM[n])$. Hence by [1, Prop. 3.1.16] real: $D^b(\tilde{\mathscr{G}}_{p+1})\to D^b(\mathrm{mod}(\Lambda_{p+1},\Omega_{p+1}))$ is an equivalence. q.e.d.

Proof of the necessity part in Theorem. By suitable induction hypothesis, we can assume the validity of the assertion in the theorem for (Λ_2,Ω_2). Using induction hypothesis and Lemma 3.10, we conclude that for $2\leqslant p\leqslant q<n$ either $\rho_p\circ\rho_{p+1}\circ\cdots\circ\rho_q(V(q+1))$ belongs to $\mathscr{G}'(p)$ or equals $j_*R'_{p+1}(\rho_{p+1}\circ\cdots\circ\rho_qV(q+1))$ and furthermore $R'_2(V(2))$ and $R'_2(\rho_2\circ\cdots\circ\rho_{q-1})(V(q))$ $(2<q\leqslant n)$ are all the indecomposable projectives of $\tilde{\mathscr{G}}_2$ (upto isomorphism).

(3.11) Recall the functors $j_*,j_!:D^b(\mathrm{mod}(\Lambda_2,\Omega_2))\to D^b(\mathrm{mod}(\Lambda_1,\Omega'_1))$ and $D^b(\mathrm{mod}(\Lambda_1,\Omega'_1))\underset{i_*}{\overset{i^!}{\rightleftarrows}}D^b(\mathrm{mod}\,k)$ (see 1.8a, c, 1.11, 1.12 and 2.2). For any $K\in D^b(\mathrm{mod}(\Lambda_1,\Omega'_1))$, we have a distinguished triangle

$$i_*i^!K\to K\to j_*j^*K\xrightarrow{d}i_*i^!K[1].$$

Let P be an indecomposable projective of $\tilde{\mathscr{G}}_2$. Taking $K=j_!P$ and using the isomorphism $j^*j_!P\approx P$, we have a distinguished triangle

$$i_*i^!j_!P\to j_!P\to j_*P\xrightarrow{d}i_*i^!j_!P[1]. \tag{3.12}$$

Since $j_!$ is right exact [1, 1.4.16 i], i_* is exact (*loc cit*) and $i^!$ is exact (1.18a). We conclude $i_* i^!(j_! P) \in D_F^{\leqslant 0}$ (1.18).

(3.13) We claim that in fact $i_* i^!(j_! P) \in \mathscr{G}_F (= D_F^{\leqslant 0} \cap D_F^{\geqslant 0})$.

To see this note that $j_! P$ and $j_* P$ are both indecomposable objects of $D^b(\text{mod}(\Lambda, \Omega))$. This follows, for example, from the assumption that P is indecomposable and the fact $\text{Hom}(j_! P, j_! P) = \text{Hom}(P, j^* j_! P) = \text{Hom}(P, P)$ and $\text{Hom}(j_* P, j_* P) = \text{Hom}(j^* j_* P, P) = \text{Hom}(P, P)$.

For simplicity let us write N for $i_* i^! j_! P$. Thus $N \in D_F^{\leqslant 0}$ and $N = H^\circ(N) \oplus \tau_{\leqslant -1}(N)$. The map $d: j_* P \to N[1]$ is a sum of two maps $d': j_* P \to H^\circ(N)[1]$ and $d'': j_* P \to (\tau_{\leqslant -1}(N)[1])$. If we show $d'' = 0$, then the distinguished triangle $H^\circ(N) \oplus \tau_{\leqslant -1}(N) \to j_! P \to j_* P \xrightarrow{d' + d''} H^\circ(N)[1] \oplus (\tau_{\leqslant -1}(N))[1]$ would imply $j_! P = \tau_{\leqslant -1}(N) \oplus N'$ where N' occurs in a distinguished triangle $H^\circ(N) \to N' \to j_* P \xrightarrow{d'} H^\circ(N)[1]$ (Note: $N' \neq 0$. For, applying j^* to the last triangle $0 \to j^* N' \to P \to 0$ is a distinguished triangle). Since $j_! P$ is indecomposable, it would follow $\tau_{\leqslant -1} N = 0$ and $N \approx H^\circ(N) \in \mathscr{G}_F$ as desired.

(3.13a) It remains to show $d'' = 0$. Indeed we will show that $\text{Hom}(j_* P, Y) = 0$ for any $Y \in i_* D_F^{\leqslant -2}$. If not $\exists n \geqslant 2$, such that $\text{Hom}(j_* P, V[n]) \neq 0$, where V is the unique simple object of $i_* \mathscr{G}_F (\approx \text{mod}\, k)$. Let f be a nonzero element of $\text{Hom}(j_* P, V[n])$. Since by assumption real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\text{mod}(\Lambda, \Omega))$ is an equivalence and since $j_* P$ and $V \in \tilde{\mathscr{G}}_1$, using [1, 3.1.16] $\exists$ a monomorphism $V \hookrightarrow M$ in $\tilde{\mathscr{G}}_1$ such that f goes to zero under the map $\text{Hom}(j_* P, V[n]) \to \text{Hom}(j_* P, M[n])$. We have a distinguished triangle

$$i_* i^! M \to M \to j_* j^* M, \tag{3.14}$$

which is in fact a short exact sequence $0 \to i_* i^! M \to M \to j_* j^* M \to 0$ in $\tilde{\mathscr{G}}_1$ as $i^!$ and j_* (and of course i_* and j^* also) are exact. (Here, we use [1, 1.2.2.1].) The monomorphism $V \hookrightarrow M$ factorizes as $V \hookrightarrow i_* i^! M \to M$; this follows from the observation $V \in i_* \mathscr{G}_F$ and $\text{Hom}(i_*, \ldots, j_*, \ldots) = 0$. Applying $\text{Hom}(j_* P, \ldots)$ to the distinguished triangle 3.14 we have a long exact sequence

$$\begin{aligned}
&\text{Hom}(j_* P, i_* i^! M) \to \text{Hom}(j_* P, M) \to \text{Hom}(j_* P, j_* j^* M) \\
&\quad \to \text{Hom}(j_* P, (i_* i^! M)[1]) \to \text{Hom}(j_* P, M[1]) \\
&\quad \to \text{Hom}(j_* P, (j_* j^* M)[1]) \to \text{Hom}(j_* P, (i_* i^! M)[2]) \\
&\quad \to \text{Hom}(j_* P, M[2]) \to \text{Hom}(j_* P, (j_* j^* M)[2]).
\end{aligned} \tag{3.14a}$$

But, $\text{Hom}(j_* P, (j_* j^* M)[l]) \approx \text{Hom}(P, j^* M[l]) = 0$ for $l \geqslant 1$ as P is projective in $\tilde{\mathscr{G}}_2$. The long exact sequence implies $\text{Hom}(j_* P, (i_* i^! M)[l]) \to \text{Hom}(j_* P, M[l])$ is an isomorphism for $l \geqslant 2$. Thus f goes to zero under the map $\text{Hom}(j_* P, V[n]) \to \text{Hom}(j_* P, (i_* i^! M)[n])$. But this is impossible since the inclusion $V \hookrightarrow i_* i^! M$ actually splits ($V \hookrightarrow i_* i^! M$ is an inclusion in $\mathscr{G}_F \approx \text{mod}\, k$).

We have now shown $d'' = 0$ and 3.13 is proved.

For simplicity let us identify $D^b(\text{mod}(\Lambda_1, \Omega_1'))$ with $D^b(\text{mod}(\Lambda_1, \Omega_1))$ using the equivalence R_1'.

We have now a short exact sequence $0 \to N \to j_! P \to j_* P \to 0$ in $\mathscr{G}_1$, where $N = i_* i^! j_! P \in i_* \mathscr{G}_F$.

If $N = 0$ then $j_! P \approx j_* P$ which is as asserted in the theorem. If $N \neq 0$, N is isomorphic to a finite sum of copies of V, the simple object corresponding to the vertex α_{1k_1} of (Λ_1, Ω_1'). Since α_{1k_1} is a sink of (Λ_1, Ω_1') V is a projective object of mod (Λ_1, Ω_1'). Here we identify mod (Λ_1, Ω_1') to $\mathscr{G}'(1)$, the heart of the natural t-structure of $D^b(\text{mod}(\Lambda_1, \Omega_1'))$.

(3.15) It will be shown below that if $X \in D^b(\text{mod}(\Lambda_1, \Omega_1'))$ is indecomposable and $\text{Hom}(V, X) \neq 0$ then $X \in \mathscr{G}'(1)$.

Thus, when $N \neq 0$, $j_! P \in \mathscr{G}'(1)$ which is as asserted in the theorem.

Except for 3.15 this completes the proof of the necessity part of the theorem. That $j_! P$ is projective in $\tilde{\mathscr{G}}_1$ follows from below as we assumed real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\text{mod}(\Lambda, \Omega))$ is an equivalence. It remains to show 3.15.

(3.15a) It is known that for $A, B \in \text{mod}(\Lambda, \Omega)$, (where, as usual, we assume Ω has no oriented cycles and it is a quiver without relations) $\text{Ext}^l(A, B) = 0$ for $l > 1$. This leads to the fact that if $X \in D^b(\text{mod}(\Lambda, \Omega))$, $X \approx \oplus(H^l(X))[-l]$. This can also be deduced from [5, Lemma 4.1]. This immediately implies that if $P \in \text{mod}(\Lambda, \Omega))$ is projective and $X \in D^b(\text{mod}(\Lambda, \Omega))$ is indecomposable then $\text{Hom}(P, X) \neq 0 \Rightarrow X \in \text{mod}(\Lambda, \Omega)$. This implies 3.15.

Sufficiency part in the theorem. We begin the proof of the sufficiency part by first showing

(3.16) Assume that the condition 3.3a in Theorem 3.3 is satisfied. Let $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ be the object of $D^b(\text{mod}(\Lambda_p, \Omega_p'))$ as in Theorem 3.3. Then $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ belongs to $\tilde{\mathscr{G}}_p$ (3.7). Furthermore $\text{Hom}(\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1)), X) = 0$, $\forall X \in \tilde{D}_p^{\leqslant -1}$.

(3.16a) Later, after we show real: $D^b(\tilde{\mathscr{G}}_p) \to D^b(\text{mod}(\Lambda_p, \Omega_p'))$ is an equivalence, the above property can be rephrased as saying that $\rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$ is projective in $\tilde{\mathscr{G}}_p$.

(As usual, we identify $D^b(\text{mod}(\Lambda_p, \Omega_p'))$ with $D^b(\text{mod}(\Lambda_p, \Omega_p))$ using the isomorphism R_p'.)

The following assertion should be considered for the case $p = q + 1$ of 3.16.

(3.16b) $R_p'(V(p))$ belongs to $\tilde{\mathscr{G}}_p$. Furthermore, $\text{Hom}(R_p'(V(p)), X) = 0$, $\forall X \in \tilde{D}_p^{\leqslant -1}$.

$V(p) \in \text{mod}(\Lambda_p, \Omega_p') \subseteq D_p'^{\leqslant 0}$ (2.2). By Proposition 2.3, $\tilde{D}_p^{\leqslant 0} = \{X \in D_p'^{\leqslant 0} \mid j^* X \in \tilde{D}_{p+1}^{\leqslant 0}\}$. As $V(p) \in D_p'^{\leqslant 0}$ and $j^* V(p) = 0$, we conclude $V(p) \in \tilde{D}_p^{\leqslant 0}$. But also $V(p) \in D_p'^{\geqslant 0} \subseteq \tilde{D}_p^{\geqslant 0}$. Hence $V(p) \in \tilde{D}_p^{\leqslant 0} \cap \tilde{D}_p^{\geqslant 0} = \tilde{\mathscr{G}}_p$. Thus $R_p'(V(p)) \in R_p'(\tilde{\mathscr{G}}_p)$, which, by our convention is also written $\tilde{\mathscr{G}}_p$. As α_{p,v_p} is a sink of (Λ_p, Ω_p') and $V(p)$ is the simple object of mod (Λ_p, Ω_p') corresponding to that vertex, $V(p)$ is a projective object of mod (Λ_p, Ω_p'). Thus, $\text{Hom}(V(p), X) = 0$, for $X \in D_p'^{\leqslant -1}$. Since $\tilde{D}_p^{\leqslant -1} \subseteq D_p'^{\leqslant -1}$, 3.16b follows.

We will prove 3.16 by descending induction on p. Our induction hypothesis is

(3.17) The assertion 3.16 is true for $p + 1$.

$$\text{Write } P \text{ for } R_{p+1}' \cdot \rho_{p+1} \cdots \rho_q(V(q+1)).$$

Recall the functors

$$D^b(\text{mod}(\Lambda_{p+1}, \Omega_{p+1})) \underset{j_*, j_!}{\overset{j^*}{\rightleftarrows}} D^b(\text{mod}(\Lambda_p, \Omega_p')) \underset{i_*}{\overset{i^*, i^!}{\rightleftarrows}} D^b(\text{mod}\, k).$$

By assumption 3.3a is satisfied. So either $j_!P = j_*P$, or $j_!P \in \mathscr{G}'(p)$. In the first case $j_!P \in \tilde{\mathscr{G}}_p$ since $P \in \tilde{\mathscr{G}}_{p+1}$ and j_* is exact (2.4) with respect to $(\tilde{D}_{p+1}^{\leqslant 0}, \tilde{D}_{p+1}^{\geqslant 0})$ and $(\tilde{D}_p^{\leqslant 0}, \tilde{D}_p^{\geqslant 0})$. In the second case $j_!P \in \mathscr{G}'(p) \subseteq \tilde{D}_p^{\geqslant 0}$. But also $j_!P \in \tilde{D}_p^{\leqslant 0}$ as $j_!$ is right exact [1, 1.4.16(i)]. Thus $j_!P \in \tilde{\mathscr{G}}_p$.

For any $X \in \tilde{D}_p^{\leqslant -1}$, in the distinguished triangle $i_* i^! X \to X \to j_* j^* X$ all the objects are in $\tilde{D}_p^{\leqslant -1}$ as $i^!$ and j_* are exact. From the long exact sequence obtained by applying $\mathrm{Hom}(j_!P, \ldots)$ to the above distinguished triangle and from the fact $\mathrm{Hom}(j_!, \ldots, i_*, \ldots) = 0$, we conclude that $\mathrm{Hom}(j_!P, X) \approx \mathrm{Hom}(j_!P, j_* j^* X)$. But, $\mathrm{Hom}(j_!P, j_* j^* X) = \mathrm{Hom}(j^* j_! P, j^* X) = \mathrm{Hom}(P, j^* X) = 0$ using induction hypothesis as $j^* X \in \tilde{D}_{p+1}^{\leqslant -1}$. Thus, $\mathrm{Hom}(j_!P, X) = 0$. But $j_!P = \rho_p \cdot \rho_{p+1} \cdots \rho_q(V(q+1))$. So 3.16 is proved.

Lemma 3.18. Suppose $0 \to A_1 \to A_2 \to A_3 \to 0$ *is a short exact sequence in* $\tilde{\mathscr{G}}_1$, *or, more generally, suppose* $A_1 \to A_2 \to A_3$ *is any distinguished triangle in* $D^b(\mathrm{mod}(\Lambda, \Omega))$. *Let m be any integer* $\geqslant 2$. *Let* $i = 1$ *or* 3. *Suppose that for any* $X \in \tilde{\mathscr{G}}_1$ *and any* $f \in \mathrm{Hom}(A_i, X[m])\ \exists$ *a monomorphism* $X \to M$ *in* $\tilde{\mathscr{G}}_1$ *such that f goes to zero under the natural map* $\mathrm{Hom}(A_i, X[m]) \to \mathrm{Hom}(A_i, M[m])$. *Then the same property also holds for* $i = 2$.

Proof. Consider the exact sequence

$$\mathrm{Hom}(A_3, X[m]) \xrightarrow{\varphi_X} \mathrm{Hom}(A_2, X[m]) \xrightarrow{\psi_X} \mathrm{Hom}(A_1, X[m]).$$

Let $f \in \mathrm{Hom}(A_2, X[m])$. Let $X \to M'$ be a monomorphism such that $\varphi_X f$ goes to zero under $\mathrm{Hom}(A_1, X[m]) \to \mathrm{Hom}(A_1, M'[m])$. Consider the commutative diagram (where the rows are exact)

$$\begin{array}{ccccc}
\mathrm{Hom}(A_3, X[m]) & \xrightarrow{\psi_X} & \mathrm{Hom}(A_2, X[m]) & \xrightarrow{\varphi_X} & \mathrm{Hom}(A_1, X[m]) \\
{\scriptstyle \eta_3}\downarrow & & {\scriptstyle \eta_2}\downarrow & & {\scriptstyle \eta_1}\downarrow \\
\mathrm{Hom}(A_3, M'[m]) & \xrightarrow{\psi_{M'}} & \mathrm{Hom}(A_2, M'[m]) & \xrightarrow{\varphi_{M'}} & \mathrm{Hom}(A_1, M'[m]).
\end{array}$$

Since $\varphi_{M'} \eta_2(f) = 0$, $\eta_2(f)$ comes from $\tilde{f} \in \mathrm{Hom}(A_3, M'[m])$. Let $M' \to M''$ be a monomorphism such that $\tilde{f}$ goes to zero under the map $\mathrm{Hom}(A_3, M'[m]) \to \mathrm{Hom}(A_3, M''[m])$. Then the monomorphism $X \to M''$ does the job for f. q.e.d.

In view of the lemma, it suffices to prove the sufficiency part for each simple object A. (One can easily show that in the abelian category $\tilde{\mathscr{G}}_1$, all objects have finite length.)

Lemma 3.19. The simple objects (up to isomorphism) of $\tilde{\mathscr{G}}_1$, *are* $S_1, S_2, \ldots, S_n$ *where* $S_p = R'_1 \cdot j_* \cdot R'_2 \cdot j_* \ldots \cdot R'_{p-1} \cdot j_* \cdot R'_p V(p)$, *where* $V(p)$ *is the simple object of* $\mathrm{mod}(\Lambda_p, \Omega'_p)$ *corresponding to the vertex* α_{p, v_p}.

Proof. Let A be any object of $\tilde{\mathscr{G}}_1$. Consider the triangle $i_* i^! A \to A \to j_* j^* A$. Here $i_*, j_*, \ldots$ etc. refer to functors between $D^b(\mathrm{mod}(\Lambda_2, \Omega_2))$, $D^b(\mathrm{mod}(\Lambda_1, \Omega'_1))$ $D^b(\mathrm{mod}\, k)$. As we have seen before this actually comes from a short exact sequence $0 \to i_* i^! A \to A \to j_* j^* A \to 0$ in $\tilde{\mathscr{G}}_1$. Suppose A is a simple object then either $A \approx j_* j^* A$ and $j^* A$ is a simple object of $\tilde{\mathscr{G}}_2$ (j_* is exact!) or $A \approx i_* i^! A$. In the latter case, $A \approx V(1)$ (or, rather, $A \approx R'_1 V(1)$). By iterating we see the validity of the assertion in the lemma. q.e.d.

Now let $i_*, j_*, \ldots$ etc. refer to the functors between $D^b(\mathrm{mod}(\Lambda_p, \Omega_p))$ $D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1}))$ and $D^b(\mathrm{mod}\, k)$. Then we have a short exact sequence (in $\widetilde{\mathscr{G}}_{p-1} \subseteq D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1}))$ $0 \to i_* i^! j_! R'_p(V(p)) \to j_! R'_p(V(p)) \to j_* R'_p V(p) \to 0$ (3.12). Hence, we have a short exact sequence (in $\widetilde{\mathscr{G}}_{p-1} \subseteq D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega_{p-1}))$.

$$0 \to R'_{p-1} i_* i^! j_! R'_p V(p) \to R'_{p-1} j_! R'_p(V(p)) \to R'_{p-1} j_* R'_p(V(p)) \to 0\,. \tag{3.20}$$

Now apply $j_*: D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega_{p-1})) \to D^b(\mathrm{mod}(\Lambda_{p-2}, \Omega'_{p-2}))$ (which is exact with respect to the t-structures $(\widetilde{D}^{\leqslant 0}_{p-1}, \widetilde{D}^{\geqslant 0}_{p-1})$ and $(\widetilde{D}^{\leqslant 0}_{p-2}, \widetilde{D}^{\geqslant 0}_{p-2})$. Thus, we have a surjection

$$j_* R'_{p-1} j_! R'_p V(p) \twoheadrightarrow j_* R'_{p-1} j_* R'_p V(p). \tag{3.21}$$

But we also have a short exact sequence (3.12)

$$0 \to i_* i^! j_! K \to j_! K \to j_* K \to 0,$$

where $K = R'_{p-1} j_! R'_p V(p)$, which gives a surjection

$$j_! R'_{p-1} j_! R'_p V(p) \twoheadrightarrow j_* R'_{p-1} j_! R'_p V(p). \tag{3.22}$$

Composing 3.22 and 3.21 we get a surjection

$$j_! R'_{p-1} j_! R'_p(V(p)) \twoheadrightarrow j_* R'_{p-1} j_* R'_p V(p)$$

and also

$$R'_{p-2} j_! R'_{p-1} j_! R'_p(V(p)) \twoheadrightarrow R'_{p-2} j_* R'_{p-1} j_* R'_p V(p).$$

Iterating we get a surjection

$$R'_1 j_! R'_2 j_! \cdots R'_{p-1} j_! R'_p(V(p)) \to R'_1 j_* R'_2 j_* \cdots R'_{p-1} j_* R'_p(V(p)).$$

Recalling our notation (3.1a, 3.19) this is the same as a surjection

$$R'_1(\rho_1 \circ \rho_2 \cdots \circ \rho_{p-1})(V(p)) \to S_p. \tag{3.23}$$

If K is the kernel of 3.23, then we have an exact sequence

$$0 \to K \to R'_1(\rho_1 \cdot \rho_2 \cdots \rho_{p-1})(V(p)) \to S_p \to 0.$$

(3.23a) Write P_p for the middle term. Thus we have a short exact sequence

$$0 \to K \to P_p \to S_p \to 0.$$

Now we will show the property

(3.24) "If k is an integer $\geqslant 1$, $X \in \widetilde{\mathscr{G}}_1$ and $f \in \mathrm{Hom}(S_p, X[k])$ $(1 \leqslant p \leqslant n)$ then there exists a monomorphism $X \hookrightarrow M$ in $\widetilde{\mathscr{G}}_1$ such that f goes to zero under $\mathrm{Hom}(S_p, X[k]) \to \mathrm{Hom}(S_p, M[k])$" by induction on k. For $k = 1$, the condition is satisfied (without any hypothesis). The morphism f occurs in a distinguished triangle

$$X \to M \to S_p \xrightarrow{f} X[1].$$

Since $X, S_p \in \tilde{\mathscr{G}}_1$, we conclude that $M \in \tilde{\mathscr{G}}_1$ and $0 \to X \to M \to S_p \to 0$ is a short exact sequence. The inclusion $X \to M$ does the job (the composite of adjacent morphisms in a triangle is zero; $S_p \to X[1] \to M[1]$ is a triangle).

(3.25) Now assume 3.24 is true for $1 \leqslant k \leqslant l$. Then by Lemma 3.18, 3.24 is true with S_p replaced by any arbitrary $K \in \tilde{\mathscr{G}}_1$.

Applying Hom$(\ldots, X[l+1])$ to the triangle $K \to P_p \to S_p \to K[1] \to P_p[1]$, we get an exact sequence

$$\begin{aligned} &\operatorname{Hom}(P_p[1], X[l+1]) \to \operatorname{Hom}(K[1], X[l+1]) \\ &\quad \to \operatorname{Hom}(S_p, X[l+1]) \to \operatorname{Hom}(P_p, X[l+1]). \end{aligned}$$

But $\operatorname{Hom}(P_p[1], X[l+1]) \approx \operatorname{Hom}(P_p, X[l]) = 0$ by 3.16 (see notation in 3.23a) and $\operatorname{Hom}(P_p, X(l+1]) = 0$ also for the same reason. Thus, we have an isomorphism $\operatorname{Hom}(K[1], X[l+1]) \to \operatorname{Hom}(S_p, X[l+1])$. Let $f \in \operatorname{Hom}(S_p, X[l+1])$. Choose $\tilde{f} \in \operatorname{Hom}(K[1], X[l+1])$ lying over f. As $\operatorname{Hom}(K[1], X[l+1]) \approx \operatorname{Hom}(K, X[l])$, by induction hypothesis 3.25 $\exists$ monomorphism $X \to M$ which annihilates $\tilde{f}$. But we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(K[1], X[l+1]) & \xrightarrow{\sim} & \operatorname{Hom}(S_p, X[l+1]) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(K[1], M[l+1]) & \xrightarrow{\sim} & \operatorname{Hom}(S_p, M[l+1]) \end{array}$$

and clearly

$$\begin{array}{ccc} \tilde{f} & \mapsto & f \\ \downarrow & & \downarrow \\ 0 & \mapsto & 0 \end{array}$$

This completes the proof of the sufficiency part of Theorem 3.3.

In particular, as remarked in 3.16a, the property 3.16 now implies that $R'_1(\rho_1 \cdot \rho_2 \cdots \rho_{p-1})(V(p))$ $(1 \leqslant p \leqslant n)$ are projective objects in $\tilde{\mathscr{G}}_1$. By Lemma 3.19 and 3.23, these indecomposable projective objects are enough to cover all the simple objects of $\tilde{\mathscr{G}}_1$. Hence they are all the indecomposable projectives.

This finishes the proof of Theorem 3.3. We illustrate with some examples in § 5.

4. A property of intermediate t-structures

The purpose of this section is to show that for a given data 2.1 if real: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\operatorname{mod}(\Lambda, \Omega))$ is an equivalence, then it is an equivalence also for the intermediate t-structures defined in 2.11. The proof is mostly a repetition of what we saw in the necessity part of Theorem 3.3.

Fix a q and v as in 2.11. We freely use the notation introduced there.

(4.1) We remark that the t-structure $(\hat{D}^{\leqslant 0}_{q,(v)}, \hat{D}^{\geqslant 0}_{q,(v)})$ in $D^b(\operatorname{mod}(\Lambda, \Omega))$ (cf. 2.11c) can actually be obtained as the one associated (2.9) to a data $\{\beta_{p1},\dots,\beta_{pv'_p}|1\leqslant p\leqslant n\}$ defined as follows: Choose any admissible enumeration (1.2) $\gamma_1,\dots,\gamma_{n-q+1}$ of $(\Lambda_q, s_{qv}\cdots s_{q1}\Omega_q)$ with $\gamma_1=\alpha_{q,v+1}$. We set $v'_p=v_p$ for $1\leqslant p<q$ and $\beta_{p\mu}=\alpha_{p\mu}$ for $1\leqslant p<q$ and $1\leqslant\mu\leqslant v_p$. Put $v'_q=v+1$, $\beta_{q\mu}=\alpha_{q\mu}$ for $1\leqslant\mu\leqslant v+1$. Put $v'_p=1$ for $q<p\leqslant n$ and $\beta_{p1}=\gamma_{p-q+1}$.

When we try to do the construction of § 2 with the above β-data, the admissible enumeration $\gamma_1,\dots,\gamma_{n-q+1}$ ensures (because of 2.10a) that at level q of the inductive construction of § 2, we get the natural t-structure of $D^b(\mathrm{mod}(\Lambda_q, s_{qv}\cdots s_{q1}\Omega_q))$. The remaining steps iterate on this using Proposition 2.3. These are exactly the steps which produced $(\hat{D}^{\leqslant 0}_{q,(v)},\hat{D}^{\geqslant 0}_{q,(v)})$ in 2.11.

PROPOSITION 4.2

Suppose real: $D^b(\tilde{\mathscr{G}}_1)\to D^b(\mathrm{mod}(\Lambda,\Omega))$ *is an equivalence. Then property* 3.3a *holds for the β-data in* 4.1.

Proof. The case $q=1$ is trivial. So let $q>1$. Using suitable induction hypothesis we can assume the validity of the proposition for the data $\{\beta_{p1},\dots,\beta_{pv'_p}|2\leqslant p\leqslant n\}$ in (Λ_2,Ω_2). Let $\tilde{\mathscr{G}}_{\beta 2}$ denote the heart of the t-structure in $D^b(\mathrm{mod}(\Lambda_2,\Omega_2))$ associated to this data and let P be an indecomposable projective in $\tilde{\mathscr{G}}_{\beta 2}$. We have a distinguished triangle (3.12)

$$i_*i^!(j_!P)\to j_!P\to j_*P\to i_*i^!(j_!P)[1],$$

where the functors are as in 3.11. (Incidentally, the quivers (Λ_2,Ω_2), (Λ_1,Ω'_1) the functors $j_!$, j^*, j_*, i^*, i_*, $i^!$ etc., appearing in the constructions with the β-data coincide with the corresponding objects for the α-data. This is so since $q>1$.) For the same reasons as explained following 3.12, $i_*i^!j_!(P)\in D_F^{\leqslant 0}$ and similar to 3.13 we wish to conclude

$$i_*i^!j_!(P)\in\mathscr{G}_F. \tag{4.3}$$

The proof of this part, as detailed below, is similar to that of 3.13, if we use 3.5. Again, we reduce to showing $\mathrm{Hom}(j_*P, Y)=0$ for $Y\in i_*D_F^{\leqslant -2}$ (by the same arguments as preceding 3.13a). If $\mathrm{Hom}(j_*P,Y)\neq 0$ for some $Y\in i_*D_F^{\leqslant -2}$, $\exists n\geqslant 2$ such that $\mathrm{Hom}(j_*P, V[n])\neq 0$ where V is the unique simple object of $i_*\mathscr{G}_F(\approx \mathrm{mod}\,k)$. Let f be a nonzero element of $\mathrm{Hom}(j_*P, V[n])$. As $j_*P\in\hat{D}^{\geqslant 0}_{q,(v)}$ (using 2.4 for the β-data) and as $\hat{D}^{\geqslant 0}_{q,(v)}\subseteq\tilde{D}_1^{\geqslant 0}$ (2.11d), it follows $j_*P\in\tilde{D}_1^{\geqslant 0}$. Using 3.5, $\exists$ a monomorphism $V\hookrightarrow M$ in $\tilde{\mathscr{G}}_1$ such that f goes to zero under the map $\mathrm{Hom}(j_*P, V[n])\to\mathrm{Hom}(j_*P, M[n])$. We then have the long exact sequence 3.14a and we still have $\mathrm{Hom}(j_*P, (j_*j^*M)[l])=\mathrm{Hom}(P, j^*M[l])=0$ as P is projective in $\tilde{\mathscr{G}}_{\beta 2}$ and $j^*M\in\tilde{D}_2^{\leqslant 0}\subseteq\tilde{D}_{\beta 2}^{\leqslant 0}$ (2.11d). After this, the same arguments which follow 3.14 show $i_*i^!j_!P\in\mathscr{G}_F$ and finally that $j_!P=j_*P$ or $j_!P\in\mathscr{G}'(1)$. This completes the proof of 4.2.

COROLLARY 4.4

If real: $D^b(\tilde{\mathscr{G}}_1)\to D^b(\mathrm{mod}(\Lambda,\Omega))$ *is an equivalence and* $\hat{\mathscr{G}}_{q,(v)}$ *denotes the heart of the t-structure* $(\hat{D}^{\leqslant 0}_{q,(v)},\hat{D}^{\geqslant 0}_{q,(v)})$ (2.11c) *then real*: $D^b(\hat{\mathscr{G}}_{q,(v)})\to D^b(\mathrm{mod}(\Lambda,\Omega))$ *is also an equivalence.*

Proof. This follows from 4.1, 4.2 and the sufficiency part of Theorem 3.3.

5. Some examples

(5.1) Let (Λ, Ω) be the quiver $1 \to 2 \to 3 \to 4$. Consider the data $\{(\alpha_{11}), (\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}), (\alpha_{31}, \alpha_{32}), (\alpha_{41})\} = \{(4), (3,2,3,1), (2,3), (2)\}$. Let α, β, γ, δ denote the simple roots of the Dynkin diagram $1 \to 2 \to 3 \to 4$ corresponding to the vertices 1, 2, 3, 4. We identify indecomposables of $\mathrm{mod}(\Lambda, \Omega)$ with the corresponding positive roots (via Gabriel's theorem). Identify $\mathrm{mod}(\Lambda_p, \Omega'_p)$, $\mathrm{mod}(\Lambda_p, \Omega_p)$ etc. with the image in $D^b(\mathrm{mod}(\Lambda, \Omega))$ under composite functors: $D^b(\mathrm{mod}(\Lambda_p, \Omega'_p)) \xrightarrow{R'_p} D^b(\mathrm{mod}(\Lambda_p, \Omega_p)) \xrightarrow{j_*} D^b(\mathrm{mod}(\Lambda_{p-1}, \Omega'_{p-1})) \xrightarrow{R'_{p-1}} \cdots \xrightarrow{j_*} \cdots \cdots \xrightarrow{R'_1} D^b(\mathrm{mod}(\Lambda_1, \Omega_1))$. We will indicate the quivers (Λ_p, Ω'_p) {resp. (Λ_p, Ω_p)} by writing the simple objects of $\mathrm{mod}(\Lambda_p, \Omega'_p)$ {resp. (Λ_p, Ω_p)} (for the above imbedding into $D^b(\mathrm{mod}(\Lambda, \Omega))$ in the place of the corresponding vertices.

$(\Lambda_1, \Omega'_1) = (\Lambda_1, \Omega_1) = (\Lambda, \Omega)$, indicated by	$\alpha \to \beta \to \gamma \to \delta$,
(Λ_2, Ω_2)	$\alpha \to \beta \to \gamma$
$(\Lambda_2, s_{21}\Omega_2)$	$\alpha \to \beta + \gamma \leftarrow \gamma[1]$,
$(\Lambda_2, s_{22}s_{21}\Omega_2)$	$\alpha + \beta + \gamma \leftarrow (\beta + \gamma)[1] \to \beta$,
$(\Lambda_2, s_{23}s_{22}s_{21}\Omega_2) = (\Lambda_2, \Omega'_2)$	$\alpha + \beta + \gamma \leftarrow \gamma[1] \leftarrow \beta[1]$,
(Λ_3, Ω_3)	$\gamma[1] \leftarrow \beta[1]$,
$(\Lambda_3, s_{31}\Omega_3) = (\Lambda_3, \Omega'_3)$	$\gamma[2] \to (\beta + \gamma)[1]$,
$(\Lambda_4, \Omega_4) = (\Lambda_4, \Omega'_4)$	$\gamma[2]$.

Next, we describe $(\rho_p \rho_{p+1} \cdots \rho_q)(V(q+1))$ (cf. 3.3a). Recall (3.1a) $\rho_q = j_! R'_{q+1}$ where $j_!: D^b(\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})) \to D^b(\mathrm{mod}(\Lambda_q, \Omega'_q))$ (1.8c). The functor $j_!$ is in general easy to write down. Identify $\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$ and $\mathrm{mod}(\Lambda_q, \Omega'_q)$ with subcategories of $D^b(\mathrm{mod}(\Lambda, \Omega))$ as indicated previously. Let $X \in \mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$. Then $Y = j_! X$ is the extension of X obtained as follows: Let $\theta_1, \ldots, \theta_k$ be the vertices of $(\Lambda_{q+1}, \Omega_{q+1})$ adjacent to $\alpha_{q\nu_q}$. Then $Y(\alpha_{q\nu_q}) = X(\theta_1) \oplus X(\theta_2) \oplus \cdots \oplus X(\theta_k)$. The morphism $Y(\theta_i \alpha_{q\nu_q})$ is the inclusion $X(\theta_i) \to Y(\alpha_{q\nu_q})$. In particular if $X(\theta_i) = 0$ for $i = 1, \ldots, k$ then $Y(\alpha_{q\nu_q}) = 0$ and $j_! X = j_* X$.

For example, suppose in our example $q = 2$. Identify $\mathrm{mod}(\Lambda_{q+1}, \Omega_{q+1})$ and $\mathrm{mod}(\Lambda_q, \Omega'_q)$ with subcategories of $D^b(\mathrm{mod}(\Lambda, \Omega))$ as indicated previously. Then $j_!(\gamma[1]) = \alpha + \beta$, $j_!(\beta[1]) = \beta[1]$ and $j_!((\beta + \gamma)[1]) = \alpha$.

Coming back to the description of $\rho_p \rho_{p+1} \cdots \rho_q(V(q+1))$, we have

$$V(4) = \gamma[2]$$

$$\{V(3), \rho_3(V(4))\} = \{(\beta + \gamma)[1], \beta[1]\}$$

$\{V(2), \rho_2(\beta+\gamma)[1], \rho_2\beta[1]\} = \{\alpha+\beta+\gamma, \alpha, \beta[1]$

$\{V(1), \rho_1(\alpha+\beta+\gamma), \rho_1(\alpha), \rho_1\beta[1]\} = \{\delta, \alpha+\beta+\gamma+\delta, \alpha, \beta[1]\}.$

The conditions 3.3a are clearly satisfied; for instance $\rho_1\rho_2\rho_3(V(4)) = \beta[1]$ which does not belong to $\mathscr{G}'(1)$ $(= \operatorname{mod}(\Lambda_1, \Omega_1'))$ but $\rho_1\rho_2\rho_3(V(4)) = j_*\rho_2\rho_3(V(4))$.

(5.2) Let (ρ, Ω) be the quiver $1 \to 2 \to 3 \to 4$. Consider the data $\{(4), (3, 2, 1, 3), (2), (1)\}$. We follow the same conventions as in 5.1.

$(\Lambda_1, \Omega_1) = (\Lambda_1, \Omega_1') = (\Lambda, \Omega)$	$\alpha \to \beta \to \gamma \to \delta,$
(Λ_2, Ω_2)	$\alpha \to \beta \to \gamma,$
s_{21}	$\alpha \to \beta+\gamma \leftarrow \gamma[1],$
s_{22}	$\alpha+\beta+\gamma \leftarrow (\beta+\gamma)[1] \to \beta$
$(\Lambda_2, s_{23}\cdot s_{22}\cdot s_{21}\Omega_2)$ $= (\Lambda_2, \Omega_2')$	$(\alpha+\beta+\gamma)[1] \to \alpha \to \beta,$
$(\Lambda_3, \Omega_2) = (\Lambda_3, \Omega_3')$	$(\alpha+\beta+\gamma)[1] \to \alpha,$
$(\Lambda_4, \Omega_4) = (\Lambda_4, \Omega_4')$	$(\alpha+\beta+\gamma)[1],$

$\rho_p\rho_{p+1}\cdots\rho_q(V(q+1))$ are as follows.

$V(4) = (\alpha+\beta+\gamma)[1],$

$\{V(3), \rho_3(V(4))\} = \{\alpha, (\beta+\gamma)[1]\},$

$\{V(2), \rho_2\alpha, \rho_2(\beta+\gamma)[1]\} = \{\beta, \alpha+\beta, \gamma[1]\},$

$\{V(1), \rho_1\beta, \rho_1(\alpha+\beta), \rho_1\gamma[1]\} = \{\delta, \beta, \alpha+\beta, (\gamma+\delta)[1]\}.$

The condition 3.3a is not satisfied; $\rho_1\rho_2\rho_3 V(4) = \gamma + \delta[1] \notin \mathscr{G}'(1)$ and $j_*R_2'\rho_2\rho_3 V(4) = j_*R_2'\gamma[1] = \gamma[1] \neq \rho_1\rho_2\rho_3 V(4)$.

6. Theorem 7.1 with hypothesis

Let A and B be finite dimensional k-algebras and M an $A-B$ bimodule. Following $[B]$, we say that (A, M, B) is a tilting triple if

(i) $\operatorname{Ext}^1(M, M) = 0.$ (6.1)
(ii) $\operatorname{Ext}^i(M, N) = 0$ for $i \geqslant 2$ and $N \in \operatorname{mod}(A)$.
(iii) $\exists$ an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ where T_1 and T_2 are direct summands of finite direct sums of M.
(iv) $B^{\text{opp}} = \operatorname{End}_A(M)$.

One important fact is that $D^b(\text{mod } B) \approx D^b(\text{mod } A)$ when A is of finite cohomological dimension, [5, § 1]. We review this in a convenient form.

(6.2) Let (A, M, B) be a tilting triple. Let $(D_B^{\leqslant 0}, D_B^{\geqslant 0})$ {resp. $(D_A^{\leqslant 0}, D_A^{\geqslant 0})$} be the natural

t-structure of $D^b(\mathrm{mod}\, B)$ {resp. $D^b(\mathrm{mod}\, A)$}. Let $\tilde{D}_B^{\leq 0} = \{K \in D_A^{\leq 0} \mid \mathrm{Hom}(M, K[i]) = 0 \forall i > 0\}$. Define $\tilde{D}_B^{\leq -1} = \tilde{D}_B^{\leq 0}[1]$ and $\tilde{D}_B^{\geq 0} = \{L \in D^b(\mathrm{mod}\, A) \mid \mathrm{Hom}(K, L) = 0, \forall K \in \tilde{D}_B^{\leq -1}\}$. Then $(\tilde{D}_B^{\leq 0}, \tilde{D}_B^{\geq 0})$ is a t-structure in $D^b(\mathrm{mod}\, A)$. We have $D_A^{\leq -1} \subset \tilde{D}_B^{\leq 0} \subset D_A^{\leq 0}$. There exists an equivalence $\varphi: D^b(\mathrm{mod}\, B) \to D^b(\mathrm{mod}\, A)$ such that $\varphi(D_B^{\leq 0}) \subset \tilde{D}_B^{\leq 0}$ and $\varphi(D_B^{\geq 0}) \subset \tilde{D}_B^{\geq 0}$. Furthermore if B is regarded as a module over itself in the canonical way then $\varphi(B) \approx M$. {Throughout we will assume that when M is decomposed into indecomposables (in mod A) the multiplicities are ≤ 1.}

(6.3) Example. In the context of Lemma 2.5a fix a q and assume

$$D_q^{\leq -1} \subset R_q' D_q'^{\leq 0}.$$

Let $B = A_q'$ be the quiver algebra of (Λ_q, Ω_q') and $A = A_q$ the quiver algebra of (Λ_q, Ω_q). Let $\varphi = R_q': D^b(\mathrm{mod}\, A_q') \to D^b(\mathrm{mod}\, A_q)$. Let $M_q = \varphi(A_q')$. Suppose

$$M_q \in \mathrm{mod}\,(A_q).$$

Then (A_q, M_q, A_q') is a tilting triple.

To see this, write $\tilde{D}_B^{\leq 0} = \varphi(D_q'^{\leq 0})$ and $\tilde{\mathscr{G}}_B = \varphi(\mathrm{mod}\,(\Lambda_q, \Omega_q')$. Then

$$D_q^{\leq -1} \subset \tilde{D}_B^{\leq 0} \subset D_q^{\leq 0} \subset \tilde{D}_B^{\leq 1}.$$

Thus, by 3.15a any object X of mod $A = D_q^{\leq 0} \cap D_q^{\geq 0}$ can be written as $X' \oplus X''$ where $X' \in \tilde{\mathscr{G}}_B[-1]$ and $X'' \in \tilde{\mathscr{G}}_B$. Write $A = A' \oplus A''$ in this fashion. As $A'[1] \in \tilde{\mathscr{G}}_B \approx \mathrm{mod}\, B (B = A_q')$, we have a projective resolution $0 \to P_1' \to P_2' \to A'[1] \to 0$ in mod B and hence a distinguished triangle $A' \to P_1' \to P_2'$. Also as A is projective in mod A, $\mathrm{Hom}(A, Y) = 0 \forall Y \in D_q^{\leq -1}$. As A'' is a direct summand of A and $\tilde{D}_B^{\leq -1} \subset D_q^{\leq -1}$, $\mathrm{Hom}(A'', Y) = 0 \forall Y \in \tilde{D}_B^{\leq -1}$. From this we conclude that A'' is projective in mod B. Thus, we have a distinguished triangle.

$$A' \oplus A'' \to P_1 \oplus A'' \to P_2, \text{ i.e., } A \to P_1 \oplus A'' \to P_2. \tag{6.3a}$$

But the projectives of $\tilde{\mathscr{G}}_B(\approx \mathrm{mod}\, B = \mathrm{mod}\,(\Lambda_q, \Omega_q') = \mathrm{mod}\,(A_q'))$ are direct summands of direct sums of $M \approx \varphi(B)$. Thus 6.3a gives rise to an exact sequence $0 \to A \to T_1 \to T_2 \to 0$ of the type required in 6.1 (iii). The other conditions in 6.1 are easy to verify.

(6.4) For $1 \leq p \leq q \leq n$, define by descending induction on p a t-structure $(\tilde{D}_{q,p}^{\leq 0}, \tilde{D}_{q,p}^{\geq 0})$ in $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p'))$ and $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))$ by

$$\tilde{D}_{q,p}^{\leq 0} = \{X \in D_p'^{\leq 0} \mid j^* X \in \tilde{D}_{q,p+1}^{\leq 0}\}.$$

$\tilde{D}_{q,p}^{\leq 0}$ in $D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p))$ is obtained by pushing the above $\tilde{D}_{q,p}^{\leq 0}$ by the isomorphism R_p'. Here $j^*: D^b(\mathrm{mod}\,(\Lambda_p, \Omega_p')) \to D^b(\mathrm{mod}\,(\Lambda_{p+1}, \Omega_{p+1}))$ is the functor 1.9. Also to start induction, we define $(\tilde{D}_{q,q}^{\leq 0}, \tilde{D}_{q,q}^{\geq 0})$ to be $(D_q'^{\leq 0}, D_q'^{\geq 0})$, the natural t-structure of $D^b(\mathrm{mod}\,(\Lambda_q, \Omega_q'))$ and its pushforward by R_q' in $D^b(\mathrm{mod}\,(\Lambda_q, \Omega_q))$.

(6.4a) We denote the heart of $(\tilde{D}_{q,p}^{\leq 0}, \tilde{D}_{q,p}^{\geq 0})$ by $\tilde{\mathscr{G}}_{q,p}$. Observe the following relation between the t-structures 2.11c and the above:

$$(\hat{D}_{q,(v_q-1)}^{\leq 0}, \hat{D}_{q,(v_q-1)}^{\geq 0}) = (\tilde{D}_{q,1}^{\leq 0}, \tilde{D}_{q,1}^{\geq 0}).$$

As a first step in proving Theorem 7.1, we will show

PROPOSITION 6.5

Given a data 2.1, assume that

(i) *real*: $D^b(\tilde{\mathscr{G}}_1) \to D^b(\mathrm{mod}(\Lambda, \Omega))$ *is an isomorphism*

(ii) (ii) *the assumption in 6.3 is satisfied for all q,* $1 < q \leqslant n$, *i.e.,* $D_q^{\leqslant -1} \subset R_q' D_q'^{\leqslant 0}$ *and* $R_q'(A_q') \in \mathrm{mod}(\Lambda_q, \Omega_q)$ *(cf. 6.3), in other words* $R_q'(P') \in \mathrm{mod}(\Lambda_q, \Omega_q)$ *for each indecomposable projective* P' *of* $\mathrm{mod}(\Lambda_q, \Omega_q')$.

Then for $1 \leqslant q \leqslant n\ \exists$ *a finite dimensional k-algebra* $\tilde{A}_{q,1}$ *such that* $\mathrm{mod}\,\tilde{A}_{q,1} \approx \tilde{\mathscr{G}}_{q,1}$ ($\tilde{\mathscr{G}}_{q,p}$ *is the heart of the t-structure* $(\tilde{D}_{q,p}^{\leqslant 0}, \tilde{D}_{q,p}^{\geqslant 0})$.) *Furthermore, there exists an* $\tilde{A}_{q,1} - \tilde{A}_{q+1,1}$ *bimodule* $\tilde{M}_{q+1,1}$ *such that* $(\tilde{A}_{q,1}, \tilde{M}_{q+1,1}, \tilde{A}_{q+1,1})$ *is a tilting triple.*

As a technical step required in the inductive proof of 6.5, we will in fact show

PROPOSITION 6.6

Under the same hypotheses as in 6.5, $\exists$ *finite dimensional k-algebras* $\tilde{A}_{q,p}$ *such that* $\mathrm{mod}(\tilde{A}_{q,p}) \approx \tilde{\mathscr{G}}_{q,p}$. *There exist* $\tilde{A}_{q,p} - \tilde{A}_{q+1,p}$ *bimodules* $\tilde{M}_{q+1,p}$ *such that* $(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ *is a tilting triple.*

Proof. Let M_{p+1} be the tilting module of $\mathrm{mod}(A_{p+1})$ described in 6.3 giving rise to the tilting triple $(A_{p+1}, M_{p+1}, A_{p+1}')$. Thus if we identify $\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}')$ as a subcategory of $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}))$ using the isomorphism R_{p+1}', then M_{p+1} is the direct sum of the projective indecomposables of $\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1}')$ (one copy each). Let $V(p)$ be the simple object of $\mathrm{mod}(\Lambda_p, \Omega_p')$ corresponding to the sink α_{p,v_p}. If $j_!$ is the usual functor $D^b(\mathrm{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\mathrm{mod}(\Lambda_p, \Omega_p'))$, set $\tilde{M}_{p+1,p} = j_!(M_{p+1}) \oplus V(p)$.

(6.7) We claim that $\tilde{M}_{p+1,p}$ is a tilting module in $\mathrm{mod}(A_p')$ ($\mathrm{mod}\,A_p' \approx \tilde{\mathscr{G}}_{p,p}$).
More generally, define

$$\begin{aligned} \tilde{M}_{q+1,p} = {} & \rho_p \cdot \rho_{p+1} \cdots \rho_q(A_{q+1}') \\ & \oplus \rho_p \cdot \rho_{p+1} \cdots \rho_{q-1}(V(q)) \\ & \oplus \rho_p \cdots \rho_{q-2}(V(q-1)) \\ & \oplus \cdots \\ & \oplus V(p) \end{aligned} \tag{6.8}$$

and $\tilde{A}_{q,p} = \mathrm{End}(\tilde{M}_{q,p})^{\mathrm{opp}}$.

(6.9) Assume inductively on q that $\mathrm{mod}\,\tilde{A}_{l,p} \approx \tilde{\mathscr{G}}_{l,p}$ the heart of $(\tilde{D}_{l,p}^{\leqslant 0}, \tilde{D}_{l,p}^{\geqslant 0})$ for $1 \leqslant l \leqslant q$ (and all p such that $1 \leqslant p \leqslant l$). Assume further that $(\tilde{A}_{l,p}, \tilde{M}_{l+1,p}, \tilde{A}_{l+1,p})$ is a tilting triple for $1 \leqslant l < q$ and $1 \leqslant p \leqslant l$ and also that

$$\tilde{D}_{l+1,p}^{\leqslant 0} = \{K \in \tilde{D}_{l,p}^{\leqslant 0} \mid \mathrm{Hom}(\tilde{M}_{l+1,p}, K[i]) = 0, \forall i > 0\} \tag{6.9a}$$

(For motivation compare the above hypotheses with 6.2)

(6.10) We wish to assert the validity of 6.9 and 6.9a replacing q by $q+1$.
For this we have to show that

$$\operatorname{mod} \tilde{A}_{q+1,p} \approx \tilde{\mathscr{G}}_{q+1,p} \quad \text{for} \quad 1 \leqslant p \leqslant q+1; \tag{6.11}$$

$(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ is a tilting triple and

$$\tilde{D}^{\leqslant 0}_{q+1,p} = \{K \in \tilde{D}^{\leqslant 0}_{q,p} \mid \operatorname{Hom}(\tilde{M}_{q+1,p}, K[i]) = 0, \forall i > 0\}. \tag{6.11a}$$

First we show $\tilde{M}_{q+1,p} \in \tilde{\mathscr{G}}_{q,p}$ ($\approx \operatorname{mod} \tilde{A}_{q,p}$).
Let Y be a direct summand of $\tilde{M}_{q+1,p}$. Then

(6.12)

(a) either $Y = V(p)$ or
(b) $Y = j_! R'_{p+1}(Z)$ where Z is a direct summand of $\tilde{M}_{q+1,p+1}$ and $j_!: D^b(\operatorname{mod}(\Lambda_{p+1}, \Omega_{p+1})) \to D^b(\operatorname{mod}(\Lambda_p, \Omega'_p))$ is the usual functor.

(This follows from 6.8). In the case 6.12(a) trivially $Y \in \tilde{\mathscr{G}}_{q,p}$.

(6.13) Consider the case 6.12b; i.e., $Y = j_! R'_{p+1}(Z)$. The data 2.1 gives rise to a data $\{\alpha_{t1}, \ldots, \alpha_{tv_t} \mid p \leqslant t \leqslant n\}$ in (Λ_p, Ω_p), for which $(\tilde{D}^{\leqslant 0}_{l,p}, \tilde{D}^{\geqslant 0}_{l,p})$ is one of the intermediate t-structures 2.11c (cf. 6.4a). By 3.10 and 4.4 we conclude

$$\text{real}: D^b(\tilde{\mathscr{G}}_{l,p}) \to D^b(\operatorname{mod}(\Lambda_p, \Omega_p) \text{ is an equivalence.} \tag{6.13a}$$

As a consequence Theorem 3.3 can be applied. The result of this application, one checks, is that

(i) either $j_! R'_{p+1}(Z) \in \mathscr{G}'(p)$ or (6.14)

(ii) $j_! R'_{p+1}(Z) = j_* R'_{p+1}(Z)$.

In the latter case, an obvious descending induction on p (namely, $\tilde{M}_{q+1,p+1} \in \tilde{\mathscr{G}}_{q,p+1}$) and the fact that j_* is exact with respect to the t-structures $(\tilde{D}^{\leqslant 0}_{q,p+1}, \tilde{D}^{\geqslant 0}_{q,p+1})$ and $(\tilde{D}^{\leqslant 0}_{q,p}, \tilde{D}^{\geqslant 0}_{q,p})$ show that $Y \in \tilde{\mathscr{G}}_{q,p}$. Starting the induction is taken care of by the assumption 6.5 (ii).

In the case 6.14 (i), we have $j_! R'_{p+1}(Z) \in \mathscr{G}'(p) \subset D'^{\geqslant 0}_p \subset \tilde{D}^{\geqslant 0}_{q,p}$. But also $j_! R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p}$ as $j_!$ is right exact [1, 1.4.16] and $R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p+1}$. Thus $j_! R'_{p+1}(Z) \in \tilde{D}^{\leqslant 0}_{q,p} \cap \tilde{D}^{\geqslant 0}_{q,p} = \tilde{\mathscr{G}}_{q,p}$.

This completes the proof that $\tilde{M}_{q+1,p} \in \tilde{\mathscr{G}}_{q,p}$. Next, we show that

$$\operatorname{Ext}^i_{\tilde{A}_{q,p}}(\tilde{M}_{q+1,p}, X) = 0 \quad \text{for} \quad i \geqslant 2 \text{ and any } X \in \operatorname{mod}(\tilde{A}_{q,p}). \tag{6.15}$$

But $\tilde{M}_{q+1,p} = V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})$ (6.8). As real: $D^b(\tilde{\mathscr{G}}_{q,p}) \to D^b(\operatorname{mod}(\Lambda_p, \Omega_p)$ is an equivalence (6.13a), $\operatorname{Ext}^i(\tilde{M}_{q+1,p}, X) = \operatorname{Hom}(\tilde{M}_{q+1,p}, X[i])$ in the derived category. Since $\tilde{\mathscr{G}}_{q,p} \subseteq \tilde{D}^{\leqslant 0}_{q,p} \subseteq D'^{\leqslant 0}_p$ and as $V(p)$ is a simple projective object of $\mathscr{G}'(p)$, $\operatorname{Hom}(V(p), X[i]) = 0$ for $X \in \tilde{\mathscr{G}}_{q,p}$ and $i > 0$. $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), X[i]) = \operatorname{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* X[i])$. But $j^* X \in \tilde{\mathscr{G}}_{q,p+1}$. This allows us to conclude $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), X[i]) = 0$, for $i > 1$ by using the induction hypothesis $\operatorname{Hom}(\tilde{M}_{q+1,p+1}, Y[i]) = 0$ for $i > 1$ and $Y \in \tilde{\mathscr{G}}_{q,p+1}$. Starting the induction is taken care of by the assumption 6.5 (ii): $\operatorname{Hom}(R'_q(A'_q), Z) = 0$ for $Z \in R'_q(D'^{\leqslant -1}_q)$, hence for $Z \in D^{\leqslant -2}_q$ as $D^{\leqslant -2}_q \subset R'_q D'^{\leqslant -1}_q$ (6.5 ii).

This completes the proof of 6.15. Next we will show

$$\operatorname{Ext}^1_{\tilde{A}_{q,p}}(\tilde{M}_{q+1,p}, \tilde{M}_{q+1,p}) = 0. \tag{6.16}$$

Again, this ext group is the same as $\operatorname{Hom}(\tilde{M}_{q+1,p}, \tilde{M}_{q+1,p}[1]) = \operatorname{Hom}(V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1}),\ V(p)[1] \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})[1])$. As already seen during the proof 6.15, $\operatorname{Hom}(V(p), \tilde{M}_{q+1,p}[1]) = 0$. Also, if we assume inductively $\operatorname{Hom}(\tilde{M}_{q+1,p+1}, \tilde{M}_{q+1,p+1}[1]) = 0$, then $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), j_! R'_{p+1}(\tilde{M}_{q+1,p+1})[1]) = 0$. Thus we are reduced to showing $\operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), V(p)[1]) = 0$. The latter equals $\operatorname{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* V(p)[1]) = 0$, as $j^* V(p) = 0$.

This proves 6.16.

The expression 6.8 allows us to count the number of indecomposable direct summands of $\tilde{M}_{q+1,p}$. By a result of Bongartz [2], we can replace the condition 6.1 (iii) in the definition of a tilting module by

(6.17) The number of nonisomorphic indecomposable summands in M as an A-module equals the number of distinct simple A-modules.

Thanks to this result we can now conclude $(\tilde{A}_{q,p}, \tilde{M}_{q+1,p}, \tilde{A}_{q+1,p})$ is a tilting triple. Only 6.11a remains to be shown. The assertion $\operatorname{mod} \tilde{A}_{q+1,p} \approx \tilde{\mathscr{G}}_{q+1,p}$ in 6.11 is then deduced from the isomorphism φ in 6.2.

Proof of 6.11a. Let $K \in$ right side of 6.11a. Then in particular

$$\operatorname{Hom}(j_! R'_{p+1} \tilde{M}_{q+1,p+1}, K[i]) = 0, \quad \forall i > 0. \tag{6.18}$$

But $\operatorname{Hom}(j_! R'_{p+1} \tilde{M}_{q+1,p+1}, K[i]) \approx \operatorname{Hom}(R'_{p+1} \tilde{M}_{q+1,p+1}, j^* K[i])$. So

$$\operatorname{Hom}(R'_{p+1} \tilde{M}_{q+1,p+1}, j^* K[i]) = 0 \quad \forall i > 0. \tag{6.19}$$

Make a (descending) induction hypothesis

$$\tilde{D}^{\leqslant 0}_{q+1,p+1} = \{K' \in \tilde{D}^{\leqslant 0}_{q,p+1} \mid \operatorname{Hom}(\tilde{M}_{q+1,p+1}, K'[i]) = 0, \forall i > 0\}. \tag{6.20}$$

By 6.19, $j^* K \in R'_{p+1}(\tilde{D}^{\leqslant 0}_{q+1,p+1})$. Now $6.4 \Rightarrow j_! j^* K \in \tilde{D}^{\leqslant 0}_{q+1,p} \subset D'^{\leqslant 0}_p$. Hence (6.21)

$$\operatorname{Hom}(V(p), j_! j^* K[i]) = 0 \quad \text{for } i > 0. \tag{6.22}$$

Applying $\operatorname{Hom}(V(p), *)$ to the triangle $j_! j^*(K[i]) \to K[i] \to i_* i^*(K[i]) \to j_! j^*(K[i+1])$, $6.22 \Rightarrow \operatorname{Hom}(V(p), K[i]) \approx \operatorname{Hom}(V(p), i_* i^* K[i])$. By choice of K ($\in$ right side of 6.11a), $\operatorname{Hom}(V(p), K[i]) = 0$ for $i > 0$. So, $\operatorname{Hom}(V(p), i_* i^* K[i]) = 0$ for $i > 0$. Therefore (a trivial property about the natural t-structure $(D^{\leqslant 0}_F, D^{\geqslant 0}_F)$ in $D^b(\operatorname{mod} k)$)

$$i^* K \in D^{\leqslant 0}_F \quad \text{and} \quad i_* i^* K \in \tilde{D}^{\leqslant 0}_{q+1,p}. \tag{6.23}$$

Now in the triangle $j_! j^* K \to K \to i_* i^* K$ both $j_! j^* K$ and $i_* i^* K \in \tilde{D}^{\leqslant 0}_{q+1,p}$. So $K \in \tilde{D}^{\leqslant 0}_{q+1,p}$. This completes the proof of one implication in 6.11a. For the other implication

Suppose $K \in \tilde{D}^{\leqslant 0}_{q+1,p}$, so that $j^* K \in \tilde{D}^{\leqslant 0}_{q+1,p+1}$ (cf. 6.4). We have to show that

$\operatorname{Hom}(\tilde{M}_{q+1,p}, K[i]) = 0, \forall i > 0$. As $\tilde{D}^{\leq 0}_{q+1,p} \subset D'^{\leq 0}_p$ $\operatorname{Hom}(V(p), K[i]) = 0, \forall i > 0$. Recall (6.8) $\quad \tilde{M}_{q+1,p} = V(p) \oplus j_! R'_{p+1}(\tilde{M}_{q+1,p+1})$. $\quad \operatorname{Hom}(j_! R'_{p+1}(\tilde{M}_{q+1,p+1}), K[i]) = \operatorname{Hom}(R'_{p+1}(\tilde{M}_{q+1,p+1}), j^* K[i] = 0$ for $i > 0$ as $j^* K \in \tilde{D}^{\leq 0}_{q+1,p+1}$ and we have 6.20. This completes the proof of 6.6 and 6.5.

7. Removing hypothesis of section 6

(7.1) **Theorem.** *Suppose given a data* (2.1) *with associated t-structure* (2.9) $(\tilde{D}^{\leq 0}_1, \tilde{D}^{\geq 0}_1)$. *Assume real:* $D^b(\tilde{\mathscr{G}}_1) \to D^b(\operatorname{mod}(\Lambda, \Omega))$ *is an equivalence. Then the data* $\{\alpha_{p1}, \ldots, \alpha_{p\nu_p} | 1 \leq p \leq n\}$ *can be modified to another data* $\{\beta_{p1}, \ldots, \beta_{p\mu_p} | 1 \leq p \leq n\}$ *such that*

(a) *both data have the same associated t-structure*

$$(\tilde{D}^{\leq 0}_1, \tilde{D}^{\geq 0}_1).$$

(b) *the data* $\{\beta_{p1}, \ldots, \beta_{p\mu_p} | 1 \leq p \leq n\}$ *satisfies the conditions assumed in Proposition 6.5.*

In particular there exist finite dimensional k-algebras $A(1), \ldots, A(n)$ *such that* (i) $A(j-1)$ *is obtained by tilting* $A(j)$, (ii) $\operatorname{mod}(A_1) \approx \tilde{\mathscr{G}}_1$ *and* (iii) $A(n)$ *is the quiver algebra of* $(\Lambda, \bar{\Omega})$ *(a quiver without relations) where* $(\Lambda, \bar{\Omega})$ *and* (Λ, Ω) *have the same underlying graph but possibly with different orientations.*

Proof. Fix p, $1 < p \leq n$. Suppose that the given data $\{\alpha_{q1}, \ldots, \alpha_{q\nu_q} | 1 \leq q \leq n\}$ has been modified to another data $\{\alpha'_{q1}, \ldots, \alpha'_{q\nu'_q} | 1 \leq q \leq n\}$ such that

(7.2)

(i) both data have the same associated t-structure (2.9).

(ii) the condition (ii) in 6.5 is satisfied for the $\alpha'_{..}$ data for $p+1 \leq q \leq n$, i.e., $D^{\leq -1}_q \subseteq R'_q D'^{\leq 0}_q$ and $R'_q(A'_q) \in \mathscr{G}(q)$ for $p+1 \leq q \leq n$ where the notations refer to objects associated to the $\alpha'_{..}$ data.

We will then show that it is possible to modify $\{\alpha'_{q1}, \ldots, \alpha'_{q\nu'_q} | 1 \leq q \leq n\}$ to a data $\{\alpha''_{q1}, \ldots, \alpha''_{q\nu''_q} | 1 \leq q \leq n\}$ such that

(7.3)

(i) both data have the same associated t-structure.

(ii) the condition (ii) in 6.5 is satisfied for the α'' data for $p \leq q \leq n$.

Obviously, we can take the data $\alpha''_{..}$ = data $\alpha'_{..}$ if for the $\alpha'_{..}$ data

(7.4)

(a) $D^{\leq -1}_p \subseteq D'^{\leq 0}_p$ and

(b) any projective of $\mathscr{G}'(p)$ belongs to $\mathscr{G}(p)$

(in other words, if the condition 7.2 iii is satisfied for $q = p$). Until the $\alpha''_{..}$ data is constructed, in the following notation the objects are associated to $\alpha'_{..}$ data.

If 7.4(a) is not satisfied consider the smallest ν, $1 \leq \nu \leq \nu'_p$ such that $D^{\leq -1}_p \subset \tilde{D}^{\leq 0}_{p,(\nu)}$ and $D^{\leq -1}_p \not\subseteq \tilde{D}^{\leq 0}_{p,(\nu+1)}$ where $\tilde{D}^{\leq 0}_{..}$ are defined following 2.11(b). The sink $\alpha'_{p,\nu+1}$ of

$(\Lambda_p, s_{p,\nu}\cdots s_{p,1}\Omega_p)$ gives rise to a simple projective P of $\tilde{\mathscr{G}}_{p,(\nu)}$. By the minimality of ν and 2.12, one infers that

$$P[-1]\in\mathscr{G}(p), \quad \text{i.e.,} \quad P\in\mathscr{G}(p)[1]. \tag{7.5}$$

Let $j_! j_*: D^b(\text{mod}(\Lambda_p, \Omega_p)) \to D^b(\text{mod}(\Lambda_{p-1}, \Omega_{p-1}))$ be the usual functors. By 4.2, either $j_!(P) = j_*(P)$ or $j_!(P)\in\mathscr{G}'(p-1)$. The latter is impossible as it would imply $P(\approx j^* j_!(P))\in\mathscr{G}(p)$ contradicting 7.5. Thus

$$j_! P = j_* P. \tag{7.6}$$

The relation $D_p^{\leqslant -1} \subset \tilde{D}^{\leqslant 0}_{p,(\nu)}$ and the fact P is projective in $\tilde{\mathscr{G}}_{p,(\nu)} \Rightarrow \text{Hom}(P, X) = 0$ for $X\in D_p^{\leqslant -2}$. Hence $\text{Hom}(P[-1], Y) = 0$ for $Y\in D_p^{\leqslant -1}$. Together with 7.5, we now conclude that

$$P[-1] \text{ is projective in } \mathscr{G}(p). \tag{7.7}$$

View an object of $\mathscr{G}(p)$ ($\approx$ mod (Λ_p, Ω_p)) as a collection of vector spaces over vertices of Λ_p and linear maps corresponding to arrows. Like any nonzero projective of $\mathscr{G}(p)$, the vector space corresponding to $P[-1]$ is nonzero for at least one sink γ of (Λ_p, Ω_p). The relation 7.6 implies that there is no arrow from γ to $\alpha'_{p-1,\nu'_{p-1}}$ in $(\Lambda_{p-1}, \Omega'_{p-1})$ (e.g., use description of $j_!$ in 5.1). Thus, γ is a sink of $(\Lambda_{p-1}, \Omega'_{p-1})$ different from $\alpha'_{p-1,\nu'_{p-1}}$.

(7.8) Let V_γ be the simple object of $\mathscr{G}(p)$ corresponding to the vertex γ of (Λ_p, Ω_p). Since $\text{Hom}(V_\gamma, P[-1]) \neq 0$ and $P\in\text{mod}(\Lambda_p, s_{p\nu}\cdots s_{p1}\Omega_p)$

$$V_\gamma\in\text{mod}(\Lambda_p, s_{p\nu}\cdots s_{p1}\Omega_p)[l] \text{ where } l \leqslant -1. \tag{7.9}$$

(This follows using property 1.16a (i) in the definition of a t-structure.)

(7.10) Claim. For $1 \leqslant \mu < \nu'$, V_γ is a simple projective of $\tilde{\mathscr{G}}_{p,(\mu)}$. $\text{Hom}(V_\gamma, X) = 0$ for $X\in D_p^{\leqslant -1}$ as V_γ is projective in $\text{mod}(\Lambda_p, \Omega_p)$. Also, $\text{Hom}(X, V_\gamma) = 0$ for $X\in D_p^{\leqslant -1}$ (1.16a (i)). Since $\tilde{D}^{\leqslant 0}_{p,(\mu)} \subset D_p^{\leqslant 0}$, we have for $X\in\tilde{D}^{\leqslant -1}_{p,(\mu)}$

$$\text{Hom}(V_\gamma, X) = 0 \quad \text{and} \quad \text{Hom}(X, V_\gamma) = 0. \tag{7.10a}$$

The last fact $\Rightarrow V_\gamma\in\tilde{D}^{\leqslant 0}_{p,(\mu)}$. Hence $V_\gamma\in\tilde{\mathscr{G}}_{p,(\mu)}$. Then 7.10(a) $\Rightarrow V_\gamma$ is projective in $\tilde{\mathscr{G}}_{p,(\mu)}$. It remains to show that V_γ is a simple object of $\tilde{\mathscr{G}}_{p,(\mu)}$. Since V_γ is a simple projective of $\mathscr{G}(p)$, if $Y\in D_p^{\leqslant 0}$ and $f: Y \to V_\gamma$ is a nonzero morphism then f splits. As $\tilde{\mathscr{G}}_{p,(\mu)} \subset D_p^{\leqslant 0}$, the above remark implies that if $Y\in\tilde{\mathscr{G}}_{p,(\mu)}$ and Y is a nonzero subobject of V_γ in $\tilde{\mathscr{G}}_{p,(\mu)}$ then $Y = V_\gamma$. This completes the proof of 7.10.

It is now easy to conclude that $\gamma \neq \alpha'_{p,\mu}$ $(1 \leqslant \mu < \nu')$ and $\gamma = \alpha'_{p,\nu'}$. This property together with the fact that γ is a sink of $(\Lambda_{p-1}, \Omega_{p-1})$ and $\gamma \neq \alpha'_{p-1,\nu'_{p-1}}$ is all we need to construct a new data $\alpha''_{..}$ without changing the associated t-structure 2.9.

Define $\nu''_l = \nu'_l$ for $1 \leqslant l \leqslant n$, $l \neq p-1, p$, $\nu''_{p-1} = \nu'_{p-1} + 1$ and $\nu''_p = \nu'_p - 1$. $\alpha''_{l,\nu} = \alpha'_{l,\nu}$ for $l \neq p-1, p$ and $1 \leqslant \nu \leqslant \nu'_l$. Define the $\alpha''_{p-1,*}$ sequence by adjoining γ just before $\alpha'_{p-1,\nu'_{p-1}}$ in the $\alpha'_{p-1,*}$ sequence. Define the $\alpha''_{p,*}$ sequence by dropping $\alpha'_{p,\nu'}(=\gamma)$ from the $\alpha'_{p,*}$ sequence. In other words, $\alpha''_{p-1,l} = \alpha'_{p-1,l}$ for $1 \leqslant l < \nu'_{p-1}$, $\alpha''_{p-1,\nu'_{p-1}} = \gamma$ and $\alpha''_{p-1,1+\nu'_{p-1}} = \alpha'_{p-1,\nu'_{p-1}}$ while, $\alpha''_{p,l} = \alpha'_{p,l}$ for $1 \leqslant l < \nu'$, $\alpha''_{p,l} = \alpha'_{p,1+l}$ for $\nu' \leqslant l \leqslant \nu'_p - 1$.

(Λ_q, Ω_q) is the same for both data for $1 \leqslant q \leqslant n$, $q \neq p$. (Λ_q, Ω'_q) is the same for both data for

$1 \leqslant q \leqslant n, q \neq p-1$. (Λ_p, Ω_p) for the $\alpha''_{..}$ data is obtained by reflecting with respect to the sink γ of (Λ_p, Ω_p) for the $\alpha'_{..}$ data. We identify $D^b(\text{mod}\,(\Lambda_p, \Omega_p))$ for the $\alpha''_{..}$ data with $D^b(\text{mod}\,(\Lambda_p, \Omega_p))$ for the $\alpha'_{..}$ data using the reflection functor at γ. Similar remarks apply to $(\Lambda_{p-1}, \Omega'_{p-1})$ and $D^b(\text{mod}\,(\Lambda_{p-1}, \Omega'_{p-1}))$. With these identifications the t-structures $(\tilde{D}_q^{\leqslant 0}, \tilde{D}_q^{\geqslant 0})$ in the inductive construction of Proposition 2.3 coincide for both data for all q, $1 \leqslant q \leqslant n$.

Recall that we are trying to modify the $\alpha'_{..}$ data to make it satisfy 7.4a (forgetting 7.4b for a while) in addition to 7.2 (i) and (ii).

Note that the $\alpha''_{..}$ data constructed above satisfies 7.2 (i) and (ii). If it still does not satisfy 7.4(a) we repeat all the steps following 7.4 starting with this $\alpha''_{..}$ data in place of the $\alpha'_{..}$ data. At each stage of this iteration (as long as 7.4a is not satisfied) the length of the α''_{p*} sequence falls by one. So eventually after a finite number of steps 7.4a and 7.2 (i), (ii) are all satisfied.

Without loss of generality (or by changing notation) we can now assume that the $\alpha'_{..}$ data satisfies 7.4a and 7.2 (i), (ii).

Now suppose 7.4(b) is not satisfied. Let P be an indecomposable projective of $\mathscr{G}'(p)$ and suppose $P \notin \mathscr{G}(p)$ so that $P \in D_p^{\leqslant -1}$. Because of 7.4a, $\text{Hom}\,(P, X) = 0$ if $X \in D_p^{\leqslant -2}$. Thus $P[-1] \in \mathscr{G}(p)$ and is a projective of $\mathscr{G}(p)$. Repeating the steps after 7.5, we produce another data which satisfies 7.2 (i), (ii) and for which the length of the $\alpha''_{p,*}$ sequence is one less than the length of the $\alpha'_{p,*}$ sequence. (This new $\alpha''_{..}$ data also satisfies 7.4a, but we don't have to prove it; if it doesn't satisfy 7.4a, iterate the steps following 7.4a, b.)

Clearly, by iterating these steps, eventually we arrive at a data satisfying 7.3 (i) and (ii). This completes the proof of Theorem 7.1. q.e.d.

8. Completeness in the case of a Dynkin quiver

In this section, we show that all non-degenerate t-structures (see definition below) arise as the t-structure associated to a data if (Λ, Ω) is a Dynkin quiver. Theorem 2 then enables us to reprove a result of Happel.

We need some preparation, for which the assumption that (Λ, Ω) is a Dynkin quiver is not needed. Recall the functors $i_*, j_*, \ldots$ etc. Also recall that $(D_F^{\leqslant 0}, D_F^{\geqslant 0})$ is the natural t-structure of $D^b(\text{mod}\,k)$. A t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is said to be *non-degenerate* if $\bigcap_m D^{\leqslant m}$ and $\bigcap_m D^{\geqslant m}$. (cf. [1, 1.3.7]) both consist of the zero objects.

Lemma 8.1. Let α be a sink of (Λ, Ω). Define (Λ_U, Ω_U) as in 1.8. Let $(D^{\leqslant 0}, D^{\geqslant 0})$ be the natural t-structure of $D^b(\text{mod}\,(\Lambda, \Omega))$. *{Thus, in particular, $i_*(D_F^{\leqslant 0}) \subset D^{\leqslant 0}$ and $i_*(D_F^{\geqslant 0}) \subset D^{\geqslant 0}$.} Now, let $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ be any t-structure of $D^b(\text{mod}\,(\Lambda, \Omega))$ such that* (i) *$\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$ and* (ii) *$i_*(D_F^{\leqslant 0}) \subset \tilde{D}^{\leqslant 0}$. Then there is a unique t-structure $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ in $D^b(\text{mod}\,(\Lambda_U, \Omega_U))$ such that $\tilde{D}_U^{\leqslant 0} \subset D_U^{\leqslant 0}$ {$(D_U^{\leqslant 0}, D_U^{\geqslant 0})$ is the natural t-structure of $D^b(\text{mod}\,(\Lambda_U, \Omega_U))$} and $\tilde{D}^{\leqslant 0} = \{K \in D^{\leqslant 0} \mid j^* K \in \tilde{D}_U^{\leqslant 0}\}$. $\tilde{D}_U^{\leqslant 0}$ is given by $\tilde{D}_U^{\leqslant 0} = \{X \in D^b(\text{mod}\,(\Lambda_U, \Omega_U)) \mid j_* X \in \tilde{D}^{\leqslant 0}\}$.*

Proof. Let $\tilde{\tau}_{\leqslant 0}(K) \to K \to \tilde{\tau}_{\geqslant 1}(K)$ denote the truncation triangle of any $K \in D^b(\text{mod}\,(\Lambda, \Omega))$ with respect to the t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$. $(\tilde{\tau}_{\leqslant 0}(K) \in \tilde{D}^{\leqslant 0}$ and $\tilde{\tau}_{\geqslant 1}(K) \in \tilde{D}^{\geqslant 1}$.

(8.2) Let $X \in D^b(\text{mod}\,(\Lambda_U, \Omega_U))$. Let $K = j_* X$. Then $\tilde{\tau}_{\leqslant 0}(K) \approx j_* j^*(\tilde{\tau}_{\leqslant 0}(K))$ and $\tilde{\tau}_{\geqslant 1}(K) \approx j_* j^*(\tilde{\tau}_{\geqslant 1}(K))$.

The morphism $\tilde{\tau}_{\leqslant 0}(K) \to K$ induces isomorphism $\mathrm{Hom}(L, \tilde{\tau}_{\leqslant 0}(K)) \to \mathrm{Hom}(L, K)$ for $L \in \tilde{D}^{\leqslant 0}$. This follows from the triangle $\tilde{\tau}_{\geqslant 1}(K)[-1] \to \tilde{\tau}_{\leqslant 0}(K) \to K \to \tilde{\tau}_{\geqslant 1}(K)$, since $\mathrm{Hom}(L, *) = 0$ for the two extreme objects (1.16a (i)). $\mathrm{Hom}(i_*(..), K) = \mathrm{Hom}(i_*(.), j_*X) = \mathrm{Hom}(, i^! j_* X) = 0$. Thus,

$$\mathrm{Hom}(L, \tilde{\tau}_{\leqslant 0}(K)) = 0 \quad \text{if } L \in i_*(D_F^{\leqslant 0}). \tag{8.2a}$$

Since $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$, $i^!(\tilde{\tau}_{\leqslant 0}(K)) \in D_F^{\leqslant 0}$. By 8.2a, therefore,

$$\mathrm{Hom}(i_* i^!(\tilde{\tau}_{\leqslant 0}(K)), \tilde{\tau}_{\leqslant 0}(K)) = 0. \tag{8.2b}$$

In particular, in the triangle $i_* i^!(\tilde{\tau}_{\leqslant 0}(K)) \to \tilde{\tau}_{\leqslant 0}(K) \to j_* j^*(\tilde{\tau}_{\leqslant 0}(K))$ the first morphism is zero. The last fact actually implies that $i_* i^!(\tilde{\tau}_{\leqslant 0}(K)) = 0$: indeed, $\mathrm{Hom}(i_* i^!(\tilde{\tau}_{\leqslant 0}(K)), i_* i^!(\tilde{\tau}_{\leqslant 0}(K)) \to \mathrm{Hom}(i_* i^!(\tilde{\tau}_{\leqslant 0}(K)), \tilde{\tau}_{\leqslant 0}(K))$ is the zero map; but it also has to be an isomorphism as $\mathrm{Hom}(i_*(..), j_*(..)) = 0$.

As $i_* i^!(\tilde{\tau}_{\leqslant 0}(K)) = 0$, it follows $\tilde{\tau}_{\leqslant 0}(K) \approx j_* j^*(\tilde{\tau}_{\leqslant 0}(K))$. As $\tilde{\tau}_{\geqslant 1}(K)$ is the cone on $\tilde{\tau}_{\leqslant 0}(K) \to K$, it follows also that $j_* j^*(\tilde{\tau}_{\geqslant 1}(K)) \approx \tilde{\tau}_{\geqslant 1}(K)$. This completes the proof of 8.2.

Since j_* is a fully faithful functor 8.2 implies that we can define a t-structure $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ in $D^b(\mathrm{mod}(\Lambda_U, \Omega_U))$ by $\tilde{D}_U^{\leqslant 0} = \{X \in D^b(\mathrm{mod}(\Lambda_U, \Omega_U)) | j_* X \in \tilde{D}^{\leqslant 0}\}$ and $\tilde{D}_U^{\leqslant 0} = \{X | j_* X \in \tilde{D}^{\geqslant 0}\}$. The assumption that $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$ then implies $\tilde{D}_U^{\leqslant 0} \subset D_U^{\leqslant 0}$.

Let $K \in D^{\leqslant 0}$. In the triangle $i_* i^! K \to K \to j_* j^* K \to i_* i^! K[1]$ both extremes belong to $\tilde{D}^{\leqslant 0}$ (since $i_*(D_F^{\leqslant 0}) \subset \tilde{D}^{\leqslant 0}$). Thus, $K \in \tilde{D}^{\leqslant 0} \Leftrightarrow j_* j^* K \in \tilde{D}^{\leqslant 0}$. Thus, $\tilde{D}^{\leqslant 0} = \{K \in D^{\leqslant 0} | j_* j^* K \in \tilde{D}^{\leqslant 0}\} = \{K \in D^{\leqslant 0} | j^* K \in \tilde{D}_U^{\leqslant 0}\}$.

To complete the proof of Lemma 8.1, we leave the uniqueness part to the reader.

For the remaining part we assume that (Λ, Ω) is a Dynkin quiver.

Let $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ be any non-degenerate t-structure in $D^b(\mathrm{mod}(\Lambda, \Omega))$. Replacing if necessary by $(\tilde{D}^{\leqslant m}, \tilde{D}^{\geqslant m})$ for some $m \in \mathbb{Z}$, we can assume without loss that $\tilde{D}^{\leqslant 0} \subset D^{\leqslant 0}$.

(8.3) The indecomposables of $\tilde{D}^{\leqslant 0}$ are obtained by dropping a finite number, say l, of indecomposables of $D^{\leqslant 0}$.

Mod (Λ, Ω) has only finitely many indecomposables (as (Λ, Ω) is Dynkin) and any indecomposable of $D^b(\mathrm{mod}(\Lambda, \Omega))$ is of the form $K[i]$ where $K \in \mathrm{mod}(\Lambda, \Omega)$ and $i \in \mathbb{Z}$. Since $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is non-degenerate, $\exists n_0$ such that $m \geqslant n_0 \Rightarrow K[m] \notin \tilde{D}^{\geqslant 0}$ for any indecomposable of $\mathrm{mod}(\Lambda, \Omega)$. Thus, $\tilde{D}^{\geqslant 0} \subset D^{\geqslant -n_0}$; hence $D^{\leqslant -n_0} \subset \tilde{D}^{\leqslant 0}$. Thus the indecomposables of $D^{\leqslant 0}$ missing in $\tilde{D}^{\leqslant 0}$ are all to be found in $\mathscr{G} \cup \mathscr{G}[1] \cup \cdots \cup \mathscr{G}[n_0 - 1]$. This yields 8.3.

Let $\alpha_1, \ldots, \alpha_l, \alpha_{l+1}$ be any sequence of vertices of (Λ, Ω) such that α_1 is a sink of (Λ, Ω) and for $1 \leqslant i \leqslant l$, α_{i+1} is a sink of $(\Lambda, s_i \cdots s_1 \Omega)$. {Here, $s_j = s_{\alpha_j}$.} We now use a construction already encountered in 2.11. For $1 \leqslant \mu \leqslant l+1$, define $(\tilde{D}_{1,(\mu)}^{\leqslant 0}, \tilde{D}_{1,(\mu)}^{\geqslant 0})$ to be the image of the natural t-structure of $D^b(\mathrm{mod}(\Lambda, s_\mu \cdots s_1 \Omega))$ under the composite isomorphism $R_1^- \circ \cdots \circ R_\mu^-$ where $R_j^- : D^b(\mathrm{mod}(\Lambda, s_j \cdots s_1 \Omega)) \to D^b(\mathrm{mod}(\Lambda, s_{j-1} \cdots s_1 \Omega))$ is the isomorphism given by Proposition 1.6. Also, we set $(\tilde{D}_{1,(0)}^{\leqslant 0}, \tilde{D}_{1,(0)}^{\geqslant 0}) = (D^{\leqslant 0}, D^{\geqslant 0})$ the natural t-structure of $D^b(\mathrm{mod}(\Lambda, \Omega))$.

Choose v_1, $1 \leqslant v_1 \leqslant l+1$, such that $\tilde{D}^{\leqslant 0} \subset \tilde{D}_{1,(v_1-1)}^{\leqslant 0}$ and $\tilde{D}^{\leqslant 0} \not\subset \tilde{D}_{1,(v_1)}^{\leqslant 0}$. Such a v_1 exists (cf. 8.3 and 2.12). The data we are looking for (having as associated t-structure $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ shall have $\alpha_{1,\mu} = \alpha_\mu$ for $1 \leqslant \mu \leqslant v_1$. Apply Lemma 8.1 taking $(\Lambda, s_{v_1-1} \cdots s_1 \Omega)$ instead of (Λ, Ω) and α_{1,v_1} instead of α. As $(\tilde{D}^{\leqslant 0}, \tilde{D}^{\geqslant 0})$ is non-degenerate it is easy to see that $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ given by Lemma 8.1 is non-degenerate. We can use induction on $\#\Lambda$ to assume that $(\tilde{D}_U^{\leqslant 0}, \tilde{D}_U^{\geqslant 0})$ is given by a data

$\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|2\leqslant p\leqslant n\}$ in (Λ_U,Ω_U). Then it follows from Proposition 2.3 and Lemma 8.1 that $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ is the t-structure associated to the data $\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|1\leqslant p\leqslant n\}$.

Thus we have proved the following

PROPOSITION 8.4

Let (Λ,Ω) be a Dynkin quiver. Let $(D^{\leqslant 0},D^{\geqslant 0})$ be the natural t-structure in $D^b(\mathrm{mod}\,(\Lambda,\Omega))$ and $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ any non-degenerate t-structure. Assume, as we may, that $\tilde{D}^{\leqslant 0}\subset D^{\leqslant 0}$. Then $\exists$ a data $\{\alpha_{p,1},\ldots,\alpha_{p,v_p}|1\leqslant p\leqslant n\}$ in (Λ,Ω) for which the associated data (2.9) coincides with $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$.

Using Theorem 7.1, we have now the following result.

Theorem 8.5. *Let (Λ,Ω) be a Dynkin quiver. Let $(\tilde{D}^{\leqslant 0},\tilde{D}^{\geqslant 0})$ be a non-degenerate t-structure in $D^b(\mathrm{mod}\,(\Lambda,\Omega))$, with heart $\tilde{\mathscr{G}}$. Assume real: $D^b(\tilde{\mathscr{G}})\to D^b(\mathrm{mod}\,(\Lambda,\Omega))$ is an equivalence. Then there exist finite dimensional k-algebras $A(1),\ldots,A(n)(n=\#\Lambda)$ such that* (i) *$A(j-1)$ is obtained by tilting $A(j)$* (ii) *$\mathrm{mod}\,(A_1)\approx\tilde{\mathscr{G}}$ and* (iii) *$A(n)$ is the quiver algebra of $(\Lambda,\bar{\Omega})$ (a quiver without relations) where $(\Lambda,\bar{\Omega})$ has the same underlying Dynkin graph as (Λ,Ω) but possibly with different orientations.*

In [5, §5] similar results have been enunciated by D Happel, who proves the following: suppose (Λ,Ω) is a Dynkin quiver. Let A be an algebra such that $D^b(\mathrm{mod}\,A)\approx D^b(\mathrm{mod}\,k\,[\Lambda,\Omega])$. Then A is isomorphic to an iterated tilted algebra of Dynkin type. In fact one can choose a sequence of APR tilts.

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