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Rotating type II null fluids

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Abstract. A method of obtaining solutions of Einstein field equations, representing rotating type II null fluids is presented. One explicit solution is given and its details are discussed. The well-known deSitter metric is derived as a particular case.

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1. Introduction

Spherically symmetric solutions describing null fluids have been widely used to discuss gravitational collapse. It began with Vaidya [1–3] and Lindquist *et al* [4] describing outflowing radiation from collapsing spheres. Papapetrou [5] first showed that Vaidya [2] solution can give rise to naked singularities providing a counterexample to cosmic censorship conjecture (Penrose [6]). Bonnor and Vaidya [7] have obtained the charged version of Vaidya solution. Later this solution was used extensively to understand various aspects of black hole formation (see Israel [8] and Plebanski and Stachel [9]).

Recently Husain [10] further generalized this solution to include null fluid with an equation of state. Wang [11] has obtained a very general spherically symmetric null fluid solution which includes most of the known solutions.

In the present note we intend to indicate a method to study a rotating distribution of null fluid with an equation of state. We shall begin with Wang's expression for the energy-momentum tensor of null fluid and use it along with a metric explicitly exhibiting a non-zero rotation vector, to derive the field equations of our problem.

2. Field equations

Following Hawking and Ellis [12], Wang has explicitly written the type II energy momentum tensor for a null fluid distribution in the form

$$T_{ik} = \mu l_i l_k + (\rho + p)(l_i n_k + l_k n_i) - p g_{ik}, \tag{1}$$

l_i and n_i being two null vectors satisfying the relations $l_i n^i = 1, l_i l^i = n_i n^i = 0$.

To describe the geometry of a rotating null fluid distribution, we use the following general metric introduced earlier by Vaidya *et al* [13]:

$$\mathrm{d}s^2 = 2(\mathrm{d}u + g\sin\alpha\mathrm{d}\beta)\mathrm{d}t - 2L(\mathrm{d}u + g\sin\alpha\mathrm{d}\beta)^2 - M^2(\mathrm{d}\alpha^2 + \sin^2\alpha\mathrm{d}\beta^2) \quad (2)$$

with

$$L = L(u,\alpha,t),\ M = M(u,\alpha,t),\ g = g(\alpha).$$

Here t is the Galilean time and if r is the Galilean radial distance then u is "retarded" distance, $u = r + t$. The coordinates are $x^1 = u, x^2 = \alpha, x^3 = \beta$ and $x^4 = t$.

We introduce the tetrad

$$\theta^1 = \mathrm{d}u + g\sin\alpha\mathrm{d}\beta,\ \theta^2 = M\mathrm{d}\alpha,\ \theta^3 = M\sin\alpha\mathrm{d}\beta,\ \theta^4 = \mathrm{d}t - L\theta^1. \quad (3)$$

The metric (2) can now be expressed in the simple form

$$\mathrm{d}s^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\theta^a\theta^b.$$

From now on the bracketed indices denote tetrad components. We have already calculated the tetrad components $\mathrm{R}_{(ab)}$ of the Ricci tensor for the above metric [13]. We record them here as an appendix for ready reference.

The tetrad components of null vectors, l_i and n_i can be taken as $l_{(a)} = (1,0,0,0)$ and $n_{(a)} = (0,0,0,1)$. Therefore the relation (1) will now give the tetrad components $T_{(ab)}$ as $T_{(11)} = \mu, T_{(14)} = \rho, T_{(22)} = T_{(33)} = p, T_{(12)} = T_{(13)} = T_{(23)} = T_{(24)} = T_{(34)} = T_{(44)} = 0$.

So Einstein field equations

$$R_{(ab)} - \frac{1}{2}Rg_{(ab)} = -8\pi T_{(ab)},$$

will now lead to

$$R_{(11)} = -8\pi\mu, R_{(22)} = R_{(33)} = -8\pi\rho, R_{(14)} = -8\pi p \quad (4)$$

and

$$R_{(23)} = R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = R_{(44)} = 0, \quad (5)$$

where $R_{(ab)}$ are given by the expressions listed in the appendix.

Thus if the metric (2) is to describe the gravitational field of a null fluid distribution given by (1), then we must solve the differential equations (5) to get the metric coefficients L, M and g.

3. A solution of the field equations

We wish to solve the differential equations (5). $R_{(23)}$ is identically zero. Next we fix our attention the four equations

$$R_{(24)} = R_{(34)} = R_{(12)} = R_{(13)} = 0.$$

Using the expressions for $R_{(ab)}$ given in the appendix, it is easy to show that the above four equations (i.e. the vanishing of $R_{(12)}, R_{(13)}, R_{(24)}, R_{(34)}$) will lead us to the conclusion that (i) $R_{(44)}$ does not depend on u and α and so is a function of t only; (ii) so also $R_{(14)} - LR_{(44)}$ is a function of t only. Thus it is natural to begin with the four equations

$$R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = 0.$$

A solution of these four equations is

$$M^2 = \left(\frac{f}{Y\phi_t}\right)(X^2+Y^2), X = u - \phi(t), Y = -y, 2f = g_\alpha + g\cot\alpha \quad (6)$$

and

$$2L = 2A(t) - \frac{4B(t)X}{X^2+Y^2} \quad (7)$$

with

$$B_t + A\phi_t = \frac{1}{2}. \quad (8)$$

Here ϕ, A and B are functions of t alone, a suffix denotes a derivative (i.e. $B_t = (\partial B/\partial t), g_\alpha = (\partial g/\partial\alpha)$ etc.) and y is defined by $g\mathrm{d}\alpha = \mathrm{d}y$.

Using (6) the expression for $R_{(44)}$ becomes

$$R_{(44)} = -\frac{1}{2}\left[2\left(\frac{\phi_{tt}}{\phi_t}\right)_t - \left(\frac{\phi_{tt}}{\phi_t}\right)^2\right]. \quad (9)$$

But our field equations demand $R_{(44)} = 0$. One obvious solution of this equation is $\phi(t) = t$ which was considered by Vaidya *et al* [13]. Another interesting solution of $R_{(44)} = 0$ is

$$\phi(t) = -\frac{b}{t}, \quad (10)$$

where b is an arbitrary constant. Therefore a general solution of the field equations (5) is given by (6), (7), (8) and (10). Here it should be noted that the metric potential $g(\alpha)$ remains undetermined.

Kerr [14] has derived a stationary generalization of Schwarzschild exterior solution known as Kerr solution. For Kerr metric we have

$$g(\alpha) = k\sin\alpha \text{ i.e. } y = -k\cos\alpha, \quad (11)$$

where k is an arbitrary constant related with the angular momentum of the rotating system.

As we are interested in rotating null fluid distributions, we take $g(\alpha)$ as given in (11).

If we take $B(t) = 0$, then (8) gives $A = t^2/2b$. Let us redefine the time coordinate T by $T = -b/t$. Then the metric (2) with $B(t) = 0$ takes the form

$$\mathrm{d}s^2 = \frac{b}{T^2}[2(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta)\mathrm{d}T - (r^2 + k^2\cos^2\alpha)(\mathrm{d}\alpha^2 + \sin^2\alpha\mathrm{d}\beta)^2$$
$$-(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta^2)] \quad (12)$$

where $r = u - T$. It can be verified that the metric (12) satisfies the field equations $R_{ik} = \Lambda g_{ik}, \Lambda = 3/b$. Here Λ is the cosmological constant. Therefore the background metric (12) represents the expanding de Sitter universe in rotating coordinates.

This shows that even when $B(t) \neq 0$ it is more convenient to use T as the time coordinate instead of t. So we write the metric of our solution in the form

$$\mathrm{d}s^2 = \frac{b}{T^2}\Bigg[2(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta)\mathrm{d}T - (r^2 + k^2\cos^2\alpha)(\mathrm{d}\alpha^2 + \sin^2\alpha\mathrm{d}\beta^2) - \left(a - \frac{2mr}{r^2 + k^2\cos^2\alpha}\right)(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta)^2\Bigg]. \tag{13}$$

Here a and m are functions of T defined by

$$a = 2AT^2/b, \; m = 2BT^2/b. \tag{14}$$

The relation (8) now becomes

$$m_T = 1 - a + \frac{2m}{T}. \tag{15}$$

From (4) we can determine the physical parameters p, ρ and μ. They are given by

$$8\pi bp = 2Ta_T - \frac{1}{2}T^2 a_{TT} - 3a \tag{16}$$

$$8\pi b\rho = 3a - Ta_T + \frac{1}{(r^2 + k^2\cos^2\alpha)}\left[T^2\left(1 - a - \frac{2m}{T}\right) - rT\left(2 - 2a + \frac{2m}{T} + Ta_T\right)\right] \tag{17}$$

and

$$8\pi\mu = \frac{1}{(r^2 + k^2\cos^2\alpha)}\left[ra_T - 2\left(1 - a + \frac{m}{T}\right)\right]. \tag{18}$$

Clearly the pressure p of the null fluid is a function of time only as was to be expected. The radiation densities ρ and μ depend upon all the three variables u, T and α.

The functions m and a are connected by only one relation (15). Therefore the explicit solution of the field equations can be derived by choosing one more restriction on the behaviour of these two functions.

One such interesting case is

$$m(T) = lT^n, \tag{19}$$

where l and n are arbitrary constants. In this case we have

$$a(T) = 1 - l(n-2)T^{n-1}. \tag{20}$$

The physical parameters p, ρ and μ for this case can be obtained from (16), (17) and (18). They are given by

$$8\pi bp = -3 + \frac{l}{2}(n-2)(n-3)(n-4)T^{n-1}, \tag{21}$$

$$8\pi b\rho = 3 + l(n-2)(n-4)T^{n-1} + \frac{l(n-4)T^n}{(r^2 + k^2\cos^2\alpha)}[T - r(1-n)], \tag{22}$$

and

$$8\pi\mu = \frac{-l(n-1)T^{n-2}}{(r^2 + k^2\cos^2\alpha)}[2T + (n-2)r]. \tag{23}$$

One particular case is noteworthy. Let us assume $n = 4$. In this case we have

$$8\pi p = -8\pi\rho = -\frac{3}{b}, \text{ i.e. } p + \rho = 0, \tag{24}$$

and

$$8\pi\mu = -\frac{6lT^2(T+r)}{(r^2 + k^2\cos^2\alpha)}. \tag{25}$$

The explicit form of the line element for the case $n = 4$ is

$$\mathrm{d}s^2 = \frac{b}{T^2}\Bigg[2(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta)\mathrm{d}T - (r^2 + k^2\cos^2\alpha)\mathrm{d}\alpha^2 + \sin^2\alpha\mathrm{d}\beta^2) - \left\{1 - 2lT^3\left(1 + \frac{rT}{r^2 + k^2\cos^2\alpha}\right)\right\}(\mathrm{d}u + k\sin^2\alpha\mathrm{d}\beta)^2\Bigg]. \tag{26}$$

We have verified that the metric (26) satisfies the field equations

$$R_{ik} = \Lambda g_{ik} - 8\pi\mu l_i l_k,\ l_i l^i = 0, \tag{27}$$

with $\Lambda = 3/b$. Thus the metric (26) represents the expanding de Sitter universe pervaded by null fluid.

One may also note that when $n = 1$, we get $\mu = 0$. In this case we obtain a solution corresponding to ρ and p only.

Many other interesting choices for the functions $m(T)$ and $a(T)$ satisfying the relation (15) are possible. We have also chosen the metric potential $g(\alpha)$ in a particular form. But the field equations do not put any restriction on $g(\alpha)$. So we can choose $g(\alpha)$ in many ways. These remarks indicate that there can be many explicit solutions of Einstein's equations representing fields of rotating null fluids with energy momentum tensor of type II. For the sake of brevity we shall not enter into these details here. Thus we have seen that the Kerr-NUT metric (2) is compatible with rotating type II null fluid distribution.

Appendix

$R_{(23)} = 0$

$$R_{(44)} = (2/M)[M_{tt} - f^2/M^3]$$

$$R_{(24)} = (g/M)[(M_t/M)_y - (f/M^2)_u]$$

$$R_{(34)} = -(g/M)[(M_t/M)_u + (f/M^2)_y]$$

$$R_{(14)} = L_{tt} + (2/M)[M_{tu} + (LM_t)_t + (Lf^2/M^3)]$$

$$R_{(12)} = LR_{(24)} + (g/M)[(L_t + M_u/M)_y + (2fL/M^2)_u]$$

$$R_{(13)} = LR_{(34)} + (g/M)[(2fL/M^2)_y - (L_t + M_u/M)_u]$$

$$R_{(22)} = R_{(33)} = (1/M^2)[g^2(M_u/M)_u + g^2(M_y/M)_y$$
$$+2f(M_y/M) + 4(f^2L/M^2) - 1 - (M^2)_{ut} - \{L(M^2)_t\}_t]$$

$$R_{(11)} = L^2R_{(44)} + (1/M^2)[g^2(L_{uu} + L_{yy}) + 2fL_y + 2L_uMM_t$$
$$+4LMM_{ut} - 2L_tMM_u + 2MM_{uu}].$$

Here a suffix denotes partial derivative (e.g. $M_u = \partial M/\partial u, L_y = \partial L/\partial y$ etc.) and the variable y replaces the variable α in differentiation the defining relation being

$$g\mathrm{d}\alpha = \mathrm{d}y,$$

and $2f$ stands for $g_\alpha + g \cot \alpha$.

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