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Unitary Modules with Non-Vanishing Relative Lie Algebra Cohomology

For a real semisimple Lie algebra $\mathfrak{g}_0$ it is of interest to find the class of unitarizable, irreducible Harish–Chandra modules X such that $H^i(\mathfrak{g}_0, k_0, X \otimes F)$ is non-zero for some integer i. Here, k_0 is a maximal compactly imbedded subalgebra of $\mathfrak{g}_0$, F is a finite dimensional irreducible module for $\mathfrak{g}_0$ and $H^i(\ldots)$ are the relative Lie algebra cohomology spaces. It is known ([8]) that the class of such modules is a subclass of the class of modules $A_{\mathfrak{q},\lambda}$ constructed in [5] and [9]. Here, $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ (dropping the subscript 0 means complexification) under a Cartan involution θ fixing k_0, and λ varies over a subset of t^* where t is a fundamental Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$. There is a conjecture, as yet unproved, that $A_{\mathfrak{q},\lambda}$ belong to the subclass above. The only difficulty in proving this conjecture is the inability to prove that the $A_{\mathfrak{q},\lambda}$ in question are unitarizable. This problem has been solved in some special cases — to mention a few due to Speh ([7]) in the case of $\mathfrak{sl}(n, \boldsymbol{R})$, by Enright ([3]) in the complex case, by Baldoni-Silva and Barbash ([1]) in the case of real rank one groups and by the present author ([6]) in the case of highest weight modules. The general problem is often vaguely referred to as the problem of unitarizability of $A_{\mathfrak{q},\lambda}$. A solution has now been obtained by Vogan and more recently, also by Wallach.

For all $\mathfrak{g}$, when $\mathfrak{q}$ is of quasi-abelian type (see the definition below), it has been proved recently by Enright, Parthasarathy, Wallach, and Wolf ([4]) that $A_{\mathfrak{q},\lambda}$ are unitarizable. (We say $\mathfrak{q}$ is quasi-abelian if $[u \cap k, u \cap p] = 0$ where u is the nilradical of $\mathfrak{q}$).

In this paper we will assume that rank of $\mathfrak{g}_0$ = rank of k_0 and discuss the unitarizability of $A_{\mathfrak{q},\lambda}$ whenever $\mathfrak{q}$ contains the Borel subalgebra r corresponding to a Borel–de Siebenthal chamber P (Definition: P contains a unique non-compact simple root β and its coefficient in the highest root is 1 or 2).

We assume that $\mathfrak{g}_0$ is irreducible. Recall that rank of k_0 = rank of $\mathfrak{g}_0$. Let β be the unique noncompact simple root for P and, if $(\mathfrak{g}_0, k_0)$ is not a hermitian symmetric pair, let γ be the unique simple root for P_k which is not a simple root for P. Let u be the nilradical of $\mathfrak{q}$. Write

$$u \cap k = u'_k + u''_k,$$

where the coefficient of β in the roots in u'_k is zero and the coefficient of β in the roots in u''_k is two.

By looking at the construction in [5] one can infer that, if $\mathfrak{q}' \cap p = \mathfrak{q} \cap p$, then we may work with $\mathfrak{q}'$ instead of $\mathfrak{q}$. Thus we can assume that $\mathfrak{q}$ is maximal among all $\mathfrak{q}'$, such that $\mathfrak{q} \cap p = \mathfrak{q}' \cap p$.

By a verification case by case one can check that the following is true:

(*) *Either* $\mathfrak{q}$ *is of quasi-abelian type or, if* $\mathfrak{q}_1$, *with the Levi decomposition* $m_1 + u_1$ *is the maximal parabolic subalgebra obtainted by deleting* β, *then* $u_1 \cap k = u''$.

We only have to deal with the latter case since for parabolics of quasi-abelian type the result is proved in [4].

From the construction in [5] recall that the modules $A_{\mathfrak{q},\lambda}$ are obtained by a chain of "completions" of a $\mathfrak{g}$-Verma module $V_{\mathfrak{g},P,\mu}$ with respect to a reduced expression for $w \in W_k$, where $P_k \cap -wP_k$ = the roots in $u \cap k$. Here, the word "completion" is used in the sense of [2]. Thus, we can make

(*Step* 1) completion with respect to a reduced expression for w' where $P_k \cap -w'P_k$ = the roots in u'_k,

followed by

(*Step* 2) completion with respect to a reduced expression for w'' where $P_k \cap -w''P_k$ = the roots in u''_k.

Since β has coefficient zero in the roots in u'_k, the simple reflection s_γ does not occur in the reduced expression for w'. Thus, after completion of step 1, the original Verma module becomes another Verma module $V_{\mathfrak{g},P,\mu'}$. In view of (*), step 2 is now just like producing a module of type $A_{\mathfrak{g}_1,*}$.

The parabolic $\mathfrak{q}_1$ is of quasi-abelian type. However, the parameter μ' is in general off the list of parameters for which unitarizability has been established in [4].

The crucial "Dirac operator inequality" associated with unitarizability, which is often an aid in proving unitarizability, can be seen to hold for the irreducible quotient of $V_{\mathfrak{g},P,\mu}$. This inequality is preserved during completions. Thus a large part of $V_{\mathfrak{g},P,\mu'}$ (considered as a k-module) which is adequate to analyse the module $A_{\mathfrak{q},\lambda}$ will satisfy the Dirac operator inequality. This circumstance enables one to employ the techniques familiar in the quasi-abelian case and leads to a proof of the unitarizability of $A_{\mathfrak{q},\lambda}$.

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