On invariant measures of the Euclidean algorithm

S.G. Dani and Arnaldo Nogueira

Abstract. We study the ergodic properties of the additive Euclidean algorithm f defined in $\mathbb{R}_+^2$. A natural extension of f is obtained using the action of $SL(2,\mathbb{Z})$ on a subset of $SL(2,\mathbb{R})$. We prove that even though f is ergodic and has an infinite invariant measure equivalent to the Lebesgue measure, such a measure is not unique; (in fact there is a continuous family of such measures). While it is folklore that this could happen for a map which is not conservative, as is the case with f, there seems to be no recorded example in the literature to that effect, and f provides a natural example for which it is the case.

1. Introduction.

Let $\mathbb{R}_+^2 = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$. The *Euclidean algorithm* is the map defined by

$$f : (x_1, x_2) \in \mathbb{R}_+^2 \longmapsto \begin{cases} (x_1 - x_2, x_2), & \text{if } x_1 \geq x_2 \\ (x_1, x_2 - x_1), & \text{otherwise.} \end{cases} \tag{1.1}$$

When x_1 and x_2 are natural numbers the action of successive powers of f on (x_1, x_2) corresponds to the application of the Euclidean algorithm for finding the *greatest common divisor (g.c.d.)*, say d, of x_1 and x_2, and there exists a $k \in \mathbb{N}$ such that $f^k(x_1, x_2) = (d, 0)$ or $(0, d)$. That is the source of the name for the transformation as above.

Let $E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, the two 2×2 elementary matrices. They generate the group $SL(2,\mathbb{Z})$ consisting of all integral unimodular 2×2 matrices. The map f as above is then given by

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}_+^2 \longmapsto \begin{cases} E_1^{-1}x, & \text{if } x_1 \geq x_2 \\ E_2^{-1}x, & \text{otherwise,} \end{cases}$$

where the matrices act on x as linear operators.

Let l denote the Lebesgue measure on $\mathbb{R}^2_+$. It is easy to see that f is a noninvertible map which is nonsingular with respect to l; that is, if $l(E) = 0$ then $l(f^{-1}(E)) = 0$. By [N], f has the following property.

Theorem 1.1. For all $x \in \mathbb{R}^2_+$, the orbit of x under f equals the orbit of x under the linear action of $SL(2, \mathbb{Z})$ graphed in $\mathbb{R}^2_+$, that is,

$$\cup_{n=0}^{\infty} \cup_{k=0}^{\infty} f^{-k}(\{f^n(x)\}) = \mathbb{R}^2_+ \cap SL(2, \mathbb{Z})x.$$

By a well-known result of Hedlund the linear action of $SL(2, \mathbb{Z})$ on $\mathbb{R}^2$ is ergodic; (see [BM] for instance). Theorem 1.1 therefore implies the following.

Corollary 1.2. The map f is ergodic relative to the Lebesgue measure l.

This note was inspired by the question whether f admits an invariant measure absolutely continuous with respect to l, and if it exists, such a measure is unique up to scalar multiples; we note that f is not conservative and therefore, even though f is ergodic, existence of an invariant density does not ensure it being unique. We prove the following.

Theorem 1.3 There exists a family $\{\nu_t\}_{t \in \mathbb{R}}$ of measures on $\mathbb{R}^2_+$ such that each ν_t is f-invariant and absolutely continuous with respect to l, and for $s, t \in \mathbb{R}$, ν_s and ν_t are not scalar multiples of each other, unless $s = t$.

Though it is generally recognised that an ergodic nonsingular transformation which is not conservative may have more than one invariant measure in the given measure class, there seems to be no recorded example of this in the literature. It may be noted that in the light of Theorem 1.3 the Euclidean algorithm transformation as above furnishes a natural example for which this happens.

2. An invariant measure of f.

Consider the map F of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ defined by

$$F : (x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ \longmapsto \begin{cases} (f(x), E_2 y) = (E_1^{-1}, E_2 y), & \text{if } x_1 \geq x_2, \\ (f(x), E_1 y) = (E_2^{-1}, E_1 y), & \text{otherwise.} \end{cases} \tag{2.1}$$

We see in particular that the Lebesgue measure on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ (viewed canonically as $\mathbb{R}^4_+$) is invariant under the action of F. Using this we first describe a simple construction of an f-invariant measure absolutely continuous with respect to l. Let $\pi : \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be the canonical projection $\pi(x, y) = x$. Then $\pi(F(x, y)) = f(\pi(x, y))$ for all (x, y). The recipe to obtain an f-invariant measure is to consider a suitable subset of $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ invariant under F, and to integrate it along the fibers of π, (with respect to the other variable y); the set needs to be chosen so that the integrals along the fibers are finite.

For $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, let $< x, y >= x_1y_1 + x_2y_2$ be the canonical scalar product in $\mathbb{R}^2$. We have

$$\phi(x, y) =< x, y >=< E^{-1}x, E^t y >= \phi(F(x, y)),$$

where E is either E_1 or E_2 and E^t is the transpose o E. Thus ϕ is a nonconstant function invariant under F; (in particular, F is not ergodic). Let

$$\Omega = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ :< x, y > \leq 1\}. \tag{2.2}$$

Then Ω is F-invariant. Let $x = (x_1, x_2) \in \mathbb{R}^2_+$. Then for $y \in \mathbb{R}^2_+$ we see that $(x, y) \in \Omega$ if and only if y belongs to the set

$$\Omega(x) = \{z \in \mathbb{R}^2_+ :< x, z > \leq 1\}.$$

The latter is a right-angled triangle whose catets are $1/x_1$ and $1/x_2$. Integrating the restriction of the Lebesgue measure to Ω along the fibers of π, as indicated above, we conclude that.

Theorem 2.1. The measure $d\nu = \dfrac{1}{2x_1x_2}dx_1dx_2$ is invariant under f.

3. A natural extension of f.

Let F and Ω be as before; see (2.1) and (2.2). Let

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ :< x, y >= 1\}.$$

Then Ω_1 is F-invariant. Let F_1 be the restriction of F to Ω_1. Let $\rho : \Omega \to \Omega_1$ be the map defined by $(x, y) \in \Omega \longrightarrow (x, y/ < x, y >) \in \Omega_1$. Then it is easy to see that $F_1 \circ \rho = \rho \circ F$. We define a measure μ on Ω_1 by setting, for any Borel subset $B \subset \Omega_1$,

$$\mu(B) = \lambda(\rho^{-1}(B)).$$

It can be seen that μ is a σ-finite measure on Ω_1. Also the relation $F_1 \circ \rho = \rho \circ F$ shows that μ is invariant under F_1.

Clearly F_1 is an extension of f, with the projection π_1 from $(x, y) \in \Omega_1$ to the first coordinate x as the extension map. We note that it is in fact a natural extension (see Aaronson [A], p.90 ff), in the sense that it is a minimal invertible extension.

Theorem 3.1. The map F_1 is a natural extension of f.

Proof. Since F_1 is an extension, with an infinite invariant measure μ as above, to prove the theorem it suffices to show that the partition of Ω_1 into equivalence classes of the relation defined by $(x, y) \sim (x', y')$, for $(x, y), (x', y') \in \Omega_1$, if $\pi_1(F_1^{-i}((x, y))) = \pi_1(F_1^{-i}((x', y')))$ for all $i \geq 0$, is the trivial partition mod μ. Let $(x, y), (x', y') \in \Omega_1$ and let $F_1^{-i}((x, y)) = (x_i, y_i)$ and $F_1^{-i}((x', y')) = (x_i, y_i')$ for all $i \geq 0$. Then $f(x_i) = x_{i-1}$ for all $i \geq 1$. By the definition of f there exist $A_i \in \{E_1^{-1}, E_2^{-1}\}$ such that $f(x_i) = A_i x_i = x_{i-1}$ for all $i \geq 1$. Then $F_1((x_i, y_i)) = (A_i x_i, A_i^{t-1} y_i)$ for all $i \geq 1$. Hence $y_i = A_i^t y_{i-1}$ for all $i \geq 1$. Similarly $y_i' = A_i^t y_{i-1}'$ for all $i \geq 1$. But there exists a unique number y such that the sequence defined by $y_0 = y$ and $y_i = A_i^t y_{i-1}$ for all $i \geq 1$ consists only of nonnegative numbers. This shows that $y' = y$, and therefore the partition as above is the trivial partition.

$\square$

We now give another realisation of the natural extension. For this we identify the subset Ω_1 canonically with the subset of $SL(2, \mathbb{R})$ given by

$$\Omega^{(1)} = \left\{ \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix} \in SL(2, \mathbb{R}) : x_1, x_2, y_1, y_2 \geq 0 \right\}. \tag{3.1}$$

The map F_1 then corresponds to

$$F_1 : g \in \Omega^{(1)} \longmapsto E^{-1} g \in \Omega^{(1)}, \tag{3.2}$$

where if $g = \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix}$ then $E = E_1$ if $x_1 \geq x_2$, and $E = E_2$ otherwise.

Using Theorem 1.1 and $(3.1 - 3.2)$, we deduce the following.

Theorem 3.2. Let $g = \begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix} \in \Omega^{(1)}$. Then the orbit of g under F_1 equals the orbit of g under the action of $SL(2, \mathbb{Z})$ on $SL(2, \mathbb{R})$ by translations on the left, graphed in $\Omega^{(1)}$, that is,

$$\cup_{n=-\infty}^{\infty} F_1^n(\{g\}) = \Omega^{(1)} \cap SL(2, \mathbb{Z}) g.$$

Proof. Let $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. If x_1/x_2 is rational then the assertion follows easily from the Euclidean algorithm for pairs of natural numbers. Now suppose that x_1/x_2 is irrational. Let $A \in SL(2,\mathbb{Z})$ be such that $Ag \in \Omega^{(1)}$. In order to prove our claim it suffices to show that there exists $k \in \mathbb{Z}$ such that $Ag = F_1^k(g)$. As $Ag \in \Omega^{(1)}$, $Av \in \mathbb{R}^2_+$. By Theorem 1.1, there exist $m, n \geq 1$ such that $f^m(Av) = f^n(v)$. By the definition of f, there exist γ, γ' in the semigroup generated by E_1^{-1} and E_2^{-1}, such that $f^n(v) = \gamma v$ and $f^m(Av) = \gamma' Av$. Thus we get $\gamma' Av = \gamma v$, which means that $\gamma^{-1}\gamma' A$ fixes v. Since x_1/x_2 is irrational this implies that $\gamma^{-1}\gamma' A$ is the identity matrix, and so $\gamma' A = \gamma$. Since $f^n(v) = \gamma v$ and $f^m(Av) = \gamma' Av$, comparing the definitions of F_1 and f we see that $F_1^m(Ag) = \gamma' Ag = \gamma g = F_1^n(g)$. Therefore $Ag = F_1^{n-m}(g)$. This proves our claim. □

4. Invariant densities for f.

Using the model for the natural extension as described in the last section we now provide a construction of a large class of measures on $\mathbb{R}^2_+$ which are absolutely continuous with respect to the Lebesgue measure l, and f-invariant.

For simplicity of notation let $G = SL(2,\mathbb{R})$ and $\Gamma = SL(2,\mathbb{Z})$. The quotient space $\Gamma\backslash G$ can be realised canonically as the space of unimodular lattices in $\mathbb{R}^2$, associating to each (right) coset Γg, $g \in G$, the lattice in $\mathbb{R}^2$ generated by the rows of g (it being independent of the representative g in Γg). The space $\Gamma\backslash G$ carries a unique probability measure, say m, invariant under the action of G on $\Gamma\backslash G$ on the right (see [BM]).

For each $s \in \mathbb{R}$ let $g_s = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}$ and for every $t \in \mathbb{R}$ let $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Then $\{g_s\}$ and $\{h_t\}$ are one-parameter subgroups of G. Their actions on $\Gamma\backslash G$ induced by the G-action correspond, respectively, to the geodesic and horocycle flows associated with the modular surface, and these are both ergodic with respect to the measure m.

Let $L^\infty(\Gamma\backslash G)^+$ denote the space of all nonnegative bounded measurable functions on $\Gamma\backslash G$. We now associate to each $\varphi \in L^\infty(\Gamma\backslash G)^+$ a f-invariant absolutely continuous measure on $\mathbb{R}^2_+$. Let λ denote the Haar measure on G. For any $\varphi \in L^\infty(\Gamma\backslash G)^+$ let μ_φ be the measure on $\Omega^{(1)}$ defined by

$$\mu_\varphi(A) = \int_A \varphi(\Gamma g)\, d\lambda(g),$$

for all Borel subsets A of $\Omega^{(1)}$. We claim that μ_φ is a F_1-invariant measure on $\Omega^{(1)}$. In view of Theorem 3.1 every Borel subset A of $\Omega^{(1)}$ can be decomposed as a countable disjoint union $A = \cup A_i$ such that on each A_i the action of

F_1 coincides with the action of an element γ_i of Γ. Therefore to prove the claim it suffices to see that $\mu_\varphi(\gamma A) = \mu_\varphi(A)$ for all Borel subsets A of $\Omega^{(1)}$ and $\gamma \in \Gamma$ such that γA is contained in $\Omega^{(1)}$; this is clear from the definition of μ_φ, and proves the claim.

Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2_+$. The action of $G = SL(2,\mathbb{R})$ on $\mathbb{R}^2 - (0)$ is transitive, and thus $\mathbb{R}^2 - (0)$ may be realised as a homogeneous space G/U, with $U = \{h_t : t \in \mathbb{R}\}$, the stabiliser of e_1. By the Fubini theorem for homogeneous spaces, for any $\psi \in L^\infty(\Gamma \backslash G)$ we have

$$\int \psi(g) d\lambda(g) = \int_{\mathbb{R}^2 - (0)} \left(\int_{\mathbb{R}} \psi(gh_t) dt \right) dl,$$

where l is the Lebesgue measure on $\mathbb{R}^2$, and the expression in parenthesis is viewed as a function on $\mathbb{R}^2$ with $\int_U \psi(gh_t)dt$ as the value at the point ge_1, namely at $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ if $g = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. Hence for any bounded measurable function ψ vanishing outside $\Omega^{(1)}$ we have

$$\int \psi(g) d\mu_\varphi(g) = \int \psi(g)\varphi(\Gamma g) d\lambda(g) = \int_{\mathbb{R}^2 - (0)} \left(\int_{\mathbb{R}} \varphi(\Gamma g h_t)\psi(gh_t) dt \right) dl,$$

where the parenthetical integral is the value of the outer integrand at the point ge_1. This implies that the image of μ_φ on $\mathbb{R}^2_+$ is a σ-finite measure, say ν_φ, which is absolutely continuous with respect to the Lebesgue measure l, and $d\nu_\varphi = \bar{\varphi}(x_1, x_2) dx_1 dx_2$, with

$$\bar{\varphi}(x_1, x_2) = \int_{\mathbb{R}} \varphi(\Gamma g h_t)\chi(gh_t) dt, \tag{4.1}$$

where χ denotes the characteristic function of $\Omega^{(1)}$ in G, and $g \in G$ is any element such that $ge_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Thus we have for each $\varphi \in L^\infty(\Gamma \backslash G)^+$ a measure ν_φ on $\mathbb{R}^2_+$ which is absolutely continuous with respect to l. Also, since μ_φ is F_1-invariant it follows that ν_φ is f-invariant.

Now let $(x_1, x_2) \in \mathbb{R}^2_+$ and choose $g = \begin{pmatrix} x_1 & -x_2^{-1} \\ x_2 & 0 \end{pmatrix}$. Then $gh_t = \begin{pmatrix} x_1 & x_1 t - x_2^{-1} \\ x_2 & x_2 t \end{pmatrix} \in \Omega^{(1)}$ if and only if $x_1 t - x_2^{-1} < 0$ and $x_2 t > 0$, or equivalently if and only if $0 < t < 1/x_1 x_2$. Therefore we get that

$$\bar{\varphi}(x_1, x_2) = \int_0^{1/x_1 x_2} \varphi(\Gamma \begin{pmatrix} x_1 & -x_2^{-1} \\ x_2 & 0 \end{pmatrix} h_t) dt$$

$$= \int_0^{1/x_1x_2} \varphi(\Gamma \begin{pmatrix} x_1 & x_1t - x_2^{-1} \\ x_2 & x_2t \end{pmatrix})dt. \tag{4.2}$$

We note that when φ is chosen to be the constant function 1 the measure $\nu_1 = \nu_\varphi$ is $\dfrac{1}{x_1x_2}dx_1dx_2$, the invariant measure as in Theorem 2.1. It may also be noted that for any $\varphi \in L^\infty(\Gamma\backslash G)^+$ the measure ν_φ is absolutely continuous with respect to ν_1 and the Radon-Nikodym derivative $\frac{d\nu_\varphi}{d\nu_1}$ is bounded by the essential supremum of φ.

Proposition 4.1 Let φ be the characteristic function of a nonempty open subset Θ of $\Gamma\backslash G$. Then $\frac{d\nu_\varphi}{d\nu_1}$ takes the value 1 on a set of positive Lebesgue measure.

Proof. We note that $\Gamma\Omega^{(1)} = G$; this may be deduced by observing that every lattice of row vectors has a basis of the form $\{(a, -b), (c, d)\}$, with $a, b, c, d > 0$. Let Ψ be a compact subset of $\Omega^{(1)}$ with nonempty interior, such that $\Gamma\backslash\Gamma\Psi$ is contained in Θ and $\Psi e_1 = \{ge_1 : g \in \Psi\}$ is contained in $\{(x_1, x_2) \in \mathbb{R} : x_1, x_2 > 0\}$. Then there exists a $\delta > 0$ such that $\Gamma g h_t \in \Theta$ for all $g \in \Psi$ and $t \in (-\delta, \delta)$. Also there exists a $C > 0$ such that if $g \in \Psi$ and $ge_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, then $\frac{1}{x_1x_2} \leq C$. Now consider any element $g' = gg_s$, with $g \in \Psi$ and $s > \log \frac{1}{2}(C/\delta)$, such that $\Gamma g' \in \Gamma\Psi$. Then we have, for all $t \in \mathbb{R}$,

$$\varphi(\Gamma g'h_t)\chi(g'h_t) = \varphi(\Gamma g'h_t)\chi(gg_sh_t) = \varphi(\Gamma g'h_t)\chi(gg_sh_tg_{-s}),$$

where (as before) χ is the characteristic function of $\Omega^{(1)}$, and the last equality holds since $\Omega^{(1)}$ is invariant under the right translations by $\{g_s\}$. As $g_sh_tg_{-s} = h_{e^{2s}t}$, and $\mid e^{2s}t \mid > C$ if $\mid t \mid \geq \delta$, it follows that $\chi(gg_sh_tg_{-s}) = 0$ for all t such that $\mid t \mid \geq \delta$. On the other hand for $t \in (-\delta, \delta)$, $\varphi(\Gamma g'h_t) = 1$, since $\Gamma g' \in \Gamma\Psi \subset \Theta$. Thus we see that $\varphi(\Gamma g'h_t)\chi(g'h_t) = \chi(g'h_t)$ for all $t \in \mathbb{R}$. By (4.1) this shows that $\frac{d\nu_\varphi}{d\nu_1}(g'e_1) = 1$.

Since the flow induced by $\{g_s\}$, namely the geodesic flow, is ergodic, the set of elements g' in $\Omega^{(1)}$ for which the condition as above is satisfied is a set of positive (Haar) measure, and hence its image in $\mathbb{R}^2_+$ is a set of positive Lebesgue measure. This proves the proposition. □

Proof of Theorem 1.3. Let S be a smooth open surface in $\Gamma\backslash G$, transversal to the horocycle flow, namely the action of $\{h_t\}$ on the right, such that $(\sigma, t) \mapsto \sigma h_t$ is a diffeomorphism of $S \times (-1, 1)$ onto an open subset, say B, of $\Gamma\backslash G$. For each $r \in (-1,1)$ let B_r be the image of $S\times(-r,r)$ in $\Gamma\backslash G$. Then each B_r is an open subset of $\Gamma\backslash G$; let φ_r be the characteristic function of B_r. To prove the theorem it suffices to show that no two of the f-invariant measures $\{\nu_{\varphi_r}\}_{r\in(-1,1)}$ are scalar multiples of each other. Since by Proposition 4.1 their essential suprema are 1, they can be scalar multiples of each other only if they are equal. We shall show that they are in fact distinct.

Now let $a,b \in (-1,1)$, say $-1 < a < b < 1$, and consider ν_{φ_a} and ν_{φ_b}. Let $g \in \Omega^{(1)}$ be arbitrary. Then the set, say T, of t in $\mathbb{R}$ for which $gh_t \in \Omega^{(1)}$ is an interval of length $1/x_1x_2$, where x_1, x_2 are such that $ge_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let $T_1 = \{t \in T : \Gamma gh_t \in B\}$, $T_a = \{t \in T : \Gamma gh_t \in B_a\}$ and $T_b = \{t \in T : \Gamma gh_t \in B_b\}$. We note that T_1 is a string of disjoint intervals, say k of them, and the lengths of all except possibly the first and the last one are 2. Suppose $k \geq 3$. Then each of the middle intervals intersects T_a and T_b in intervals of lengths $2a$ and $2b$ respectively; the end intervals could be smaller, but the length of the intersection with T_b is at least as much as the length of the intersection with T_a. Thus the Lebesgue measures of T_a and T_b are $2(k-2)a + c$ and $2(k-2)b + d$ respectively, where c and d are the total lengths of the segments in the first and the last intervals, and we have $0 \leq c \leq d \leq 4$. Hence

$$\frac{d\nu_{\varphi_a}}{d\nu_1}(ge_1) = \int_{\mathbb{R}} \varphi_a(\Gamma gh_t)\chi(gh_t)dt < \int_{\mathbb{R}} \varphi_b(\Gamma gh_t)\chi(gh_t)dt = \frac{d\nu_{\varphi_b}}{d\nu_1}(ge_1),$$

provided k as above is at least 3.

To complete the proof it suffices therefore to show that the set of g in $\Omega^{(1)}$ for which $k = k(g)$ as above is at least 3 has positive measure. We shall show that for any given $k_0 \in \mathbb{N}$ the set of g in $\Omega^{(1)}$ for which $k(g) \geq k_0$ is a set of positive measure.

Let $k_0 \in \mathbb{N}$ be given. Let φ be the characteristic function of B, and let θ be the function on $\Gamma\backslash G$ defined by $\theta(\Gamma g) = \int_0^1 \varphi(\Gamma gh_t)dt$. The action of $h := h_1$ on $\Gamma\backslash G$ is ergodic (see [BM]), and hence

$$\frac{1}{n}\Sigma_{i=0}^{n-1}\theta(\Gamma gh^i) \longrightarrow \int_{\Gamma\backslash G} \theta dm = m(B), \text{a.e.}.$$

Hence there exist $n \geq 4k_0/m(B)$ and a Borel subset E of $\Gamma\backslash G$ such that $m(E) > 0$ and

$$\frac{1}{n}\Sigma_{i=0}^{n-1}\theta(\Gamma gh^i) \geq m(B)/2$$

for all g in G such that $\Gamma g \in E$. Then for any g with $\Gamma g \in E$ we have

$$\int_0^n \varphi(\Gamma gh_t)dt = \Sigma_{i=0}^{n-1}\theta(\Gamma gh^i) \geq \frac{n}{2}m(B) \geq 2k_0.$$

Now let Ψ be a compact subset of $\Omega^{(1)}$ with nonempty interior, contained in the interior of $\Omega^{(1)}$, and such that $\Gamma\backslash\Gamma\Psi$ is contained in B. Then there exists a $\delta > 0$ such that Ψh_t is contained in $\Omega^{(1)}$ for all $t \in (-\delta, \delta)$. Since the geodesic flow is ergodic there exists $s > \frac{1}{2}\log n/\delta$ such that $m(E \cap (\Gamma\backslash\Gamma\Psi g_{-s})) > 0$. Now consider any g in G such that $\Gamma g \in E$ and $g = g' g_{-s}$ for some $g' \in \Psi$. We note that $gh_t = g' g_{-s} h_t \in \Omega^{(1)}$ if and only if $g' h_{e^{-2s}t} = g' g_{-s} h_t g_s \in \Omega^{(1)}$, and since $g' \in \Psi$ the set of t for which this holds contains the interval $(-\delta e^{2s}, \delta e^{2s})$, which in turn contains $(0, n)$. Also, since $\Gamma g \in E$, we have $\int_0^n \chi_B(\Gamma g h_t) dt \geq 2k_0$. Thus for such a g the set T as in the preceding argument contains the interval $(0, n)$, and the subset $T_1 \cap (0, n)$ has Lebesgue measure at least k_0. Since T_1 is a union of intervals of length at most 2, we get that T_1 contains at least k_0 intervals. This completes the argument and the proof of the theorem. $\square$

5. Interval exchange transformations.

We conclude with some remarks setting Theorem 1.3 in a broader context. Let $\lambda_1, \lambda_2 > 0$. Set $I = [0, \lambda_1 + \lambda_2), I_1 = [0, \lambda_1)$ and $I_2 = [\lambda_1, \lambda_1 + \lambda_2)$. The map $T : I \to I$ defined by $Tx = x + \lambda_2$, if $x \in I_1$, and $Tx = x - \lambda_1$, if $x \in I_2$, is an interval exchange of the intervals I_1 and I_2. The map f as in (1.1) corresponds to the of *Rauzy induction* defined for interval exchange transformations (see [V1]) in the special case with two intervals. In the light of our proof of Theorem 1.3 and the natural extension $\mathcal{U}$ defined by Veech (see [V1], p. 219), it may be seen that a result analogous to Theorem 1.3 would hold for the Rauzy induction $\mathcal{I}$ (see [V2], p. 1390).

Acknowledgement: The first named author would like to thank the Institute de Mathématiques de Luminy for its hospitality while this work was done.

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Addresses.

S.G. Dani
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005, India
E-mail: dani@math.tifr.res.in

Arnaldo Nogueira
Institut de Mathématiques de Luminy
163, avenue de Luminy, Case 907
13288 Marseille Cedex 9, France
E-mail: nogueira@iml.univ-mrs.fr