Scaling relation for determining the critical threshold for continuum percolation of overlapping discs of two sizes

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Abstract. We study continuum percolation of overlapping circular discs of two sizes. We propose a phenomenological scaling equation for the increase in the effective size of the larger discs due to the presence of the smaller discs. The critical percolation threshold as a function of the ratio of sizes of discs, for different values of the relative areal densities of two discs, can be described in terms of a scaling function of only one variable. The recent accurate Monte Carlo estimates of critical threshold by Quintanilla and Ziff [*Phys. Rev.* **E76**, 051115 (2007)] are in very good agreement with the proposed scaling relation.

Keywords. Continuum percolation; overlapping discs; critical percolation threshold.

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In recent years, there has been a lot of interest in studying continuum percolation, owing to its many applications. For a review on continuum percolation, see [1]. Continuum percolations of overlapping objects of various sizes and shapes, spheres and discs [2–6], ellipsoids [7], plates [8], sticks [9], oriented cubes [10] etc., have been studied. In applications like the modelling of porous media, one of the most important parameters is the distance from percolation threshold, and several approximation schemes have been proposed to determine the percolation threshold for different types of disorders.

In this paper, we discuss the case of continuum percolation of overlapping discs of two sizes in a plane. We propose a phenomenological equation for the increase in the effective size of the larger discs in the presence of the smaller discs. We check our theory against data on critical thresholds by Quintanilla and Ziff [11]. The agreement is found to be very good.

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We consider a percolation model of a mixture of circular discs of two sizes randomly placed in a plane. Consider a finite area S and randomly drop discs in S. The probability that a given small areal element dA contains the centre of a dropped disc is $n\,dA$, independent of other discs. Once a centre of the disc is chosen, it is assigned a radius R_1 with probability f and R_2 with probability $(1-f)$. We denote the ratio of radii R_1/R_2 by λ. The number density of discs with radius R_1 is then $n_1 = fn$, and that of radius R_2 is $n_2 = (1-f)n$. The total number density of discs, irrespective of radius, is $n = n_1 + n_2$. We propose an approximate formula for the critical percolation threshold in terms of λ and f. We express this function of two variables in terms of the function $\xi(A)$ which gives the correlation length ξ as a function of the areal density A of single-sized discs.

The earliest proposal of a general rule for determining the critical threshold is the constant covered fraction rule by Scher and Zallen [2]. Early simulations [12,13] noted that the total covered fractional area at critical threshold was nearly constant for a mixture of discs of different sizes, if the polydispersity of the mixture was small. However, if the polydispersity is large, and one takes discs with several different radii, the total covered fraction at critical threshold can be made as close to one as we wish [6]. The original heuristic arguments have been made rigorous later [14].

We start by summarizing the qualitative arguments of [6]. Let us assume, without any loss of generality, that $R_1 < R_2$. We consider the plane on which smaller discs of radius R_1 each have been thrown in randomly with n_1 discs per unit area. The areal density of these discs is then $A_1 = \pi R_1^2 n_1$. Note that A_1 is a dimensionless number giving the ratio of total area of discs thrown into the area of the plane. In the case of percolation of discs of equal radii, the areal density of the discs at the percolation threshold is independent of the size of the discs. Let this critical value of A be denoted by A^*. We assume that A_1 is below the critical threshold A^*, and the small discs by themselves do not percolate. From numerical simulations, the value of A^* is known quite accurately as $A^* \approx 1.128085$. The corresponding value of the covered area fraction is given by $\phi^* = 1 - \exp(-A^*) \approx 0.6763475(5)$ [11].

The two-point correlation function, $G(r)$, is defined as the probability that two points at a distance r from each other, chosen at random, belong to the same cluster when only the smaller discs have been dropped. Below criticality, this decays exponentially with distance, i.e., $G(r) \sim \exp(-r/\xi_1)$. And using simple scaling invariance of the problem $R_1 \to \alpha R_1$, we have

$$\xi_1(A_1) = R_1 g(A_1), \tag{1}$$

where the function $g(x)$ determines how the correlation length varies with areal density, and is independent of R_1.

Now we throw in a single disc of the larger radius R_2, and look at the cluster of discs that is connected to this single large disc. Then, each such cluster looks like a somewhat bigger fuzzy disc of size $R_2 + \Delta R_2$. Let us assume that the variation between different clusters may be neglected. This approximation is quite good if $R_2 \gg \xi_1$, but less valid if R_2/ξ_1 is not so large. The percolation problem can then be considered as a percolation of these larger effective discs. The number density n_2^* of these effective larger equal-sized discs of radius $R_2 + \Delta R_2$ that have to be dropped to reach criticality is given by
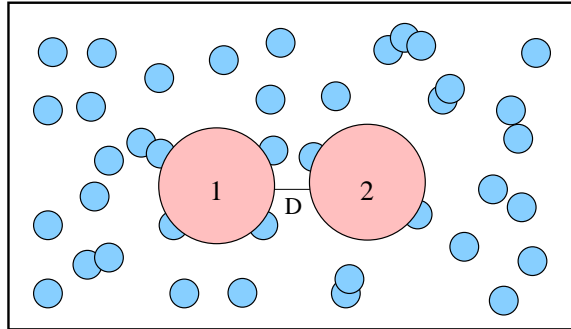
Figure 1. Two large discs of radius R_2 in a background of randomly dropped smaller discs. The least separation between the discs is D.

$$n_2^*\pi(R_2 + \Delta R_2)^2 = A^*. \tag{2}$$

We will consider this equation as the definition of ΔR_2.

In [6], the simple approximation

$$\Delta R_2 \approx c\xi_1 \tag{3}$$

was used, where c is some constant of order one. This gives the correct limiting behaviour that for any initial density A_1 of the smaller discs, the critical value of the areal density of larger discs $A_2^*(R_2)$ tends to A^* as R_2 tends to infinity, keeping A_1 fixed. Also, the other limit when we keep A_2 fixed at any value below A^*, and slowly increase A_1 till we reach critical percolation, then the critical value of $A_1^*(R_1)$ to reach criticality tends to A^* as R_1 tends to zero [14].

However, eq. (3) strongly underestimates the value of ΔR_2. Consider two discs of radius R_2 thrown in a sea of randomly dropped smaller discs of areal density A_1. Call these discs 1 and 2 and, let the minimum distance between these discs be denoted by D (figure 1). We denote by $\mathrm{Prob}_D(1 \rightsquigarrow 2)$ the probability that there is a path of overlapping smaller discs between the larger discs, and they belong to the same cluster. Thus, $\mathrm{Prob}_D(1 \rightsquigarrow 2)$ is a measure of the connectivity correlations in the problem of percolation of single-sized discs.

Clearly, $\mathrm{Prob}_D(1 \rightsquigarrow 2)$ is a decreasing function of the separation D, which will decrease exponentially from 1 to 0, as D varies from 0 to infinity. For large D, this decreases as $\exp(-D/\xi_1)$. The dependence of this on R_2 comes from the fact that the prefactor of the exponential would depend on R_2. Also, for D comparable to ξ_1, the D-dependence cannot be approximated well by a simple exponential. However, we can define an effective size $\Delta R_2^{\mathrm{eff}}$ by the requirement that this probability is a fixed value, say $1/2$, when $D = 2\Delta R_2^{\mathrm{eff}}$. Then, a better estimate of ΔR_2 than eq. (3) is given by

$$\Delta R_2 \approx \Delta R_2^{\mathrm{eff}}. \tag{4}$$

$\Delta R_2^{\mathrm{eff}}$ is a function of R_1, R_2 and n_1 (or, equivalently ξ_1). For D comparable to ξ_1, we cannot use the large D exponential decay of $\mathrm{Prob}_D(1 \rightsquigarrow 2)$ to estimate $\Delta R_2^{\mathrm{eff}}$. However, if $R_1 \ll \xi_1$, then we can assume that the leading dependence is
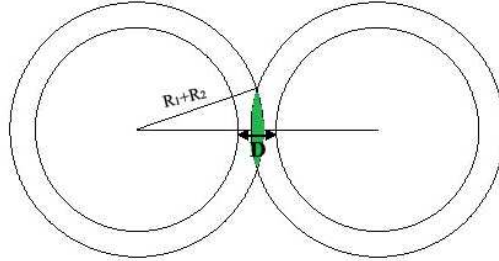
Figure 2. Two large discs of radius R_2 with separation between the discs D. A larger circle of radius $R_1 + R_2$ is drawn surrounding each of the discs. If the centre of any small disc falls in the intersection region (shown shaded) of the larger circles, it forms a connecting path betwen them.

from ξ_1, and correction terms involving powers of R_1/ξ_1 can be neglected. Then, ΔR_2^{eff}, to leading order, is only a function of R_2 and ξ_1. Using the fact that the probabilities are invariant if all distances are scaled by same factor, we get

$$\Delta R_2^{\text{eff}} = \xi_1 h(R_2/\xi_1), \tag{5}$$

where $h(x)$ is some, as yet unspecified, scaling function of its argument x. Now, clearly, $\text{Prob}_D(1 \rightsquigarrow 2)$ is a monotonically increasing function of R_2, which tends to 1 as R_2 tends to infinity, keeping D fixed, as then the problem is that of percolation in a very long strip, and somewhere or other, there will be a connection of smaller discs. This implies that ΔR_2^{eff} must tend to infinity if R_2 tends to infinity. Also, in the case $R_1 \ll R_2 \ll \xi_1$, it must tend to infinity as ξ_1 tends to infinity. The simplest form of $h(x)$ that is consistent with these requirements is a simple power-law form, which gives

$$\Delta R_2^{\text{eff}} = k\xi_1^a R_2^{1-a}. \tag{6}$$

Here k is some constant of order 1. The main improvement in this form over eq. (3) is the inclusion of dependence on R_2.

The power-law dependence of R_2^{eff} on R_2 is seen most easily by considering a perturbation expansion of $\text{Prob}_D(1 \rightsquigarrow 2)$ in powers of n_1. Let

$$\text{Prob}_D(1 \rightsquigarrow 2) = n_1 F_1(D, R_2) + n_1^2 F_2(D, R_2) + \cdots. \tag{7}$$

In the first order in n_1, the configurations that contribute to $\text{Prob}_D(1 \rightsquigarrow 2)$ are those where a single small disc overlaps with both the bigger discs. This is possible only if $D < 2R_1$, and in that case, if there is at least one small disc in the region which is within a distance $R_1 + R_2$ from the centres of the discs 1 and 2 (see figure 2). For small n_1, the probability of this event is proportional to the area of the shaded region in figure 2. Using elementary geometry, it is easily seen that for $R_2 \gg R_1$, the area is proportional to $R_2^{1/2}(2R_1 - D)^{3/2}$. Thus we get

$$F_1(D, R_2) \sim R_2^{1/2}(2R_1 - D)^{3/2}, \quad \text{for } 0 \leq D \leq 2R_1. \tag{8}$$
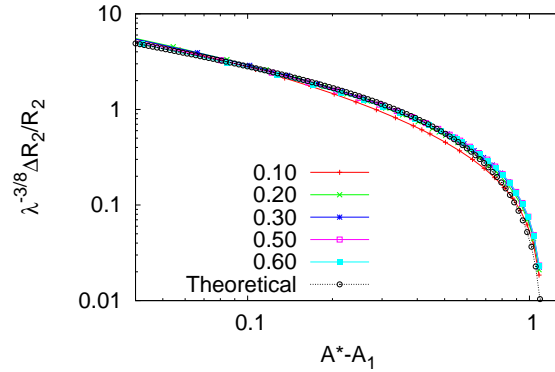
Figure 3. Scaling collapse of the Monte Carlo data of [11]. Re-scaled data of ΔR_2 is plotted vs. $A^* - A_1$, the deficit in the areal density A_1 of smaller discs from the critical value for monodisperse discs A^*, for different values of the ratios of radii λ.

Thus, we see that F_1, and by extension $\text{Prob}_D(1 \rightsquigarrow 2)$ has a strong dependence on R_2. Of course, for $\xi_1 \gg R_1$, higher-order terms in n_1 make significant contribution, and they would change the precise form of the functional dependence on R_2.

Again, we assume that the larger discs act as discs of radius $R_2 + \Delta R_2$, with $\Delta R_2 \simeq \Delta R_2^{\text{eff}}$, given by eq. (6). Expressing n_2 in terms of A_2, the areal density of the larger discs, the criticality condition may be written as

$$\frac{\Delta R_2}{R_2} = \sqrt{A^*/A_2} - 1 \approx k[\lambda g(A_1)]^a. \tag{9}$$

The above equation is clearly invariant under scaling of all lengths by the same factor. We can determine the value of a, in the limit $\xi_1 \gg R_2$. Then, assume $A_1 = A^*(1 - \epsilon)$. Then, $\epsilon \ll 1$ implies that $\xi_1 \gg R_1$.

Clearly, the number density of additional discs of radius R_2 required to reach criticality would be less than with discs of size R_1. Hence, in terms of areal densities, this bound becomes $A_2 < A^* \epsilon \lambda^{-2}$. Also, as discussed in [6], the total areal density of discs at criticality is greater than A^* when all discs are not of the same size, $A_2 \geq \epsilon A^*$. Thus, $A_2 \sim \epsilon$. Then, $\Delta R_2 \sim \epsilon^{-1/2}$. Since it is known that $g(x) \sim (A^* - x)^{-\nu}$ for x near A^*, with $\nu = 4/3$. Thus, comparing powers of ϵ we see that $a = 3/8$.

Our proposed approximation can be directly checked against numerical data. Quintanilla and Ziff have given a very extensive table of data giving different values of A_1, A_2 for different values of R_1/R_2, that define critical surface [15]. Using eq. (9), if we plot $Y = \lambda^{-a}[\sqrt{A^*/A_2} - 1]$ vs. $X = A^* - A_1$, all points should fall on a single curve $Y = g(A^* - X)^a$. The result is shown in figure 3, where we have plotted data corresponding to five different values of $\lambda = 0.10, 0.20, 0.30$ and 0.50. We get a very good collapse. We do not show other values, in order not to clutter up the figure, but have checked that the collapse is as good with them as well. Note that no free parameters have been used to generate the scaling collapse.

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Define $\Phi(x) = [g(x)]^a$. The function $\Phi(x)$, which gives the equation of the curve is in principle calculable if we can solve the problem of percolation probability with single sized discs. As of now, we only know the behaviour of Φ in certain regimes. For small x, $\Phi(x) \sim x$ and for x near A^*, $\Phi(x)$ varies as $(1 - x/A^*)^{-1/2}$. Hence we parametrize the curve as

$$\Phi(x) \sim kx(1 + cx)(1 - x/A^*)^{-1/2}. \tag{10}$$

The values $k = 0.25$ and $c = 2.20$, give a fairly good fit. The curve using these fitting parameters is also shown in figure 2.

Acknowledgements

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