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Quasi-static Axisymmetric Deformation of a Poroelastic Half-space with Anisotropic Permeability and Compressible Constituents by Surface Loads

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ABSTRACT: A quasi-static axisymmetric solution is obtained of the fully coupled diffusion-deformation system of equations for a poroelastic half-space possessing anisotropic permeability and compressible solid and fluid constituents. This solution is used to study analytically the deformation of a half-space by surface loads. The problem of normal disc loading is discussed in detail. The effects of the compressibility of the solid and fluid constituents and the anisotropy in permeability are studied numerically.

1 Introduction

Deformation of a poroelastic half-space by surface loads has been studied extensively (see, e.g., Wang, 2000). However, in most of the investigations, the permeability is assumed to be isotropic. On account of the sedimentation process producing horizontal stratification planes, the permeability in the horizontal and vertical directions may differ. Therefore, it is useful to investigate the effect of the anisotropy in permeability on the quasi-static deformation of a half-space by surface loads. Recent studies on the deformation of a poroelastic half-space with anisotropic permeability include Chen (2004, 2005) and Singh *et al.* (2007). While Chen (2004) assumed the solid constituents of the poroelastic medium to be incompressible, Chen (2005) assumed both the fluid and solid constituents as incompressible. Singh *et al.* (2007) discussed the consolidation of a poroelastic half-space with anisotropic permeability and compressible constituents by two-dimensional surface loads.

The purpose of the present paper is to study the quasi-static deformation of a poroelastic half-space by axisymmetric surface loading. The permeability in the vertical direction may be different from the permeability in the horizontal direction. The fluid and solid constituents are assumed to be compressible. The problem of normal disc loading is discussed in detail. An explicit analytical solution in the Laplace-Hankel transform domain is obtained. Schapery's formula (Schapery, 1962) is used for the Laplace transform inversion and the extended Simpson's rule for the Hankel transform inversion. Detailed numerical computations are performed to study the effects of the anisotropy in permeability and the compressibility of the fluid and solid constituents.

2 Governing equations

Let (r, θ, z) denote the cylindrical polar coordinates and (u_r, u_θ, u_z) the corresponding displacement components. For axial symmetry, $\partial / \partial \theta \equiv 0$. A poroelastic material with compressible fluid and solid constituents can be characterized by four constitutive constants. Let these constants be: the shear modulus (G), the drained Poisson's ratio (ν), the undrained Poisson's ratio (ν_u) and the Biot-Willis coefficient (α). For axial symmetry we have the following governing equations in which σ denotes the total stress tensor and p the pore pressure.

2.1 Equilibrium equations

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{\partial^2 u_r}{\partial z^2}+\frac{1}{1-2n}\frac{\partial \in}{\partial r}-\frac{a}{G}\frac{\partial p}{\partial r}=0 \tag{1}$$

$$\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{1-2n}\frac{\partial \in}{\partial z}-\frac{a}{G}\frac{\partial p}{\partial z}=0 \tag{2}$$

$$h\tilde{N}^2 \mathrm{p} = \mathrm{G}\ \tilde{N}^2\ \hat{I} \tag{3}$$

where

$$\hat{I} = \mathrm{div}\ \mathbf{u},\ h=\frac{1-2n}{2(1-n)}a\ . \tag{4}$$

2.2 Constitutive equations

$$s_{rr} = 2G\left(\frac{\partial u_r}{\partial r}+\frac{n}{1-2n}\in\right)-ap$$

$$s_{qq} = 2G\left(\frac{u_r}{r}+\frac{n}{1-2n}\in\right)-ap$$

$$s_{zz} = 2G\left(\frac{\partial u_z}{\partial z}+\frac{n}{1-2n}\in\right)-ap$$

$$s_{rz} = G\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right), s_{rq} = s_{zq} = 0\,. \tag{5}$$

2.3 Darcy's law

$$\mathrm{q_r} = -c_{\mathrm{r}}\ ¶\,\mathrm{p}\ /\ ¶\,\mathrm{r},\ \mathrm{q}_q = 0,\ \mathrm{q_z} = -c_{\mathrm{z}}\,¶\,\mathrm{p}\ /\ ¶\,\mathrm{z} \tag{6}$$

where **q** is the fluid flux and (c_{r}, c_{z}) is the Darcy conductivity in the (r, z) direction.

2.4 Fluid diffusion equation

$$c_r\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right)+c_z\frac{\partial^2 p}{\partial z^2} = \frac{\partial}{\partial t}\left(a\in+\frac{1}{M}p\right) \tag{7}$$

where

$$M = \frac{2G(n_u-n)}{a^2(1-2n)(1-2n_u)}\,. \tag{8}$$

3 Solution

A solution of the governing equations can be obtained by following a modified version of the procedure used by Rajapakse and Senjuntichai (1993) and elaborated by Wang (2000, Sect. 9.6). We define the Laplace transform $\tilde{\mathrm{f}}(\mathrm{r,z,s})$ of a function $\mathrm{f}(\mathrm{r,z,t})$ by the relation

$$\tilde{f}(r,z,s)=\int_0^\infty f(r,z,t)e^{-st}dt\,. \tag{9}$$

Omitting the details, the Laplace transform of the solution of the governing equations which is bounded for z→¥ can be expressed in the form

$$\tilde{p}(r,z,s) = \int_0^\infty \left(A_1 e^{-mz} + A_2 e^{-kz}\right) J_0(kr) k dk \tag{10}$$

$$\tilde{u}_r(r,z,s) = \int_0^\infty \left[\frac{-hk}{G(m^2 - k^2)} A_1 e^{-mz} + A_6 z e^{-kz} + A_4 e^{-kz} \right] J_1(kr) k dk \tag{11}$$

$$\tilde{u}_z(r,z,s) = \int_0^\infty \left[\frac{-hm}{G(m^2 - k^2)} A_1 e^{-mz} + A_6 z e^{-kz} + A_5 e^{-kz} \right] J_0(kr) k dk \tag{12}$$

$$\tilde{s}_{rz}(r,z,s) = 2G \int_0^\infty \left[\frac{hmk}{G(m^2 - k^2)} A_1 e^{-mz} - A_6 k z e^{-kz} + \frac{1}{2}(A_3 - 2kA_4) e^{-kz} \right] J_1(kr) k dk \tag{13}$$

$$\tilde{s}_{zz}(r,z,s) = 2G \int_0^\infty \left[\frac{hk^2}{G(m^2 - k^2)} A_1 e^{-mz} - A_6 (1 + kz) e^{-kz} + \frac{1}{2}(A_3 - 2kA_4) e^{-kz} \right] J_0(kr) k dk \tag{14}$$

where

$$A_3 = \frac{h}{G}\left[1 - (1 - 2n)\frac{(1 - n_u)}{(n_u - n)}\frac{s_a}{s}\right] A_2, \quad A_6 = \frac{h}{2G}\left[1 + \frac{(1 - n_u) s_a}{(n_u - n) s}\right] A_2$$

$$A_5 = A_4 + \frac{h}{2Gk}\left[-1 + (3 - 4n)\frac{(1 - n_u) s_a}{(n_u - n) s}\right] A_2$$

$$m^2 = \left(\frac{c_r}{c_z}\right) k^2 + \frac{s}{c_z}, \quad s_a = s + (c_r - c_z) k^2$$

$$(c_r, c_z) = \frac{2G(1 - n)(n_u - n)}{a^2 (1 - 2n)^2 (1 - n_u)} (c_r, c_z). \tag{15}$$

Three arbitrary constants, A_1, A_2 and A_4, appear in the solution. These constants can be determined from the boundary conditions.

4 Normal disc loading

Consider a poroelastic half-space z ³ 0 with z-axis vertically downwards. Suppose a total normal force Q_0 is uniformly applied over a circular surface area (z = 0, r £ a). If the surface is permeable and the load is applied in the positive z-direction, the boundary conditions yield

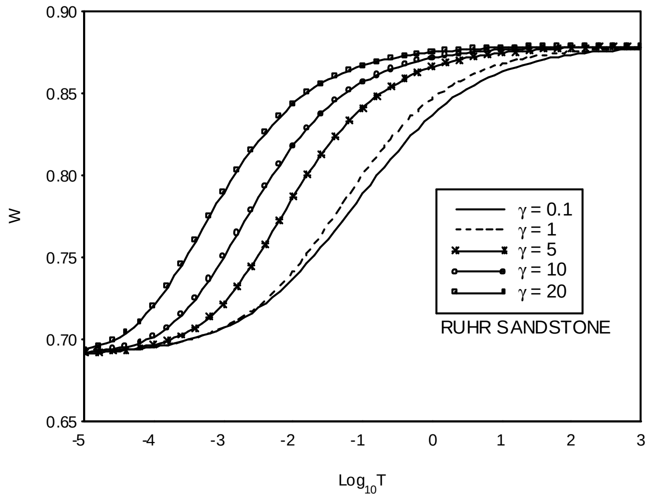

Figure 1. Effect of the permeability anisotropy on the time-settlement.

$$\mathrm{p} = 0,\ s_{rz} = 0,\ s_{zz} = \begin{cases} -Q_0/\left(pa^2\right) & , \quad r \le a \\ 0 & , \quad r > a \end{cases} \tag{16}$$

at z = 0. Using the boundary conditions, we obtain

$$A_1 = -A_2 = (n - n_u)(m + k)\frac{s}{hW}N_0$$

$$A_4 = -\left[s(n_u - n)\left(k^2 - m^2 + 2mk\right) + s_a(1 - n_u)(1 - 2n)\left(m^2 - k^2\right)\right]\frac{N_0}{2Gk(m - k)W} \tag{17}$$

where

$$N_0 = -\frac{Q_0}{p}\left[\frac{J_1(ak)}{ak}\right]$$

$$W = s(n_u - n)(k - m) - s_a(1 - n_u)(k + m)\,.$$

Explicit expressions for the pore pressure and the vertical displacement are:

$$\tilde{p}(r, z, s) = \frac{(n_u - n)sQ_0}{pah}\int_0^\infty \left(e^{-mz} - e^{-kz}\right)J_0(kr)J_1(ka)\frac{(m + k)}{W}dk \tag{18}$$

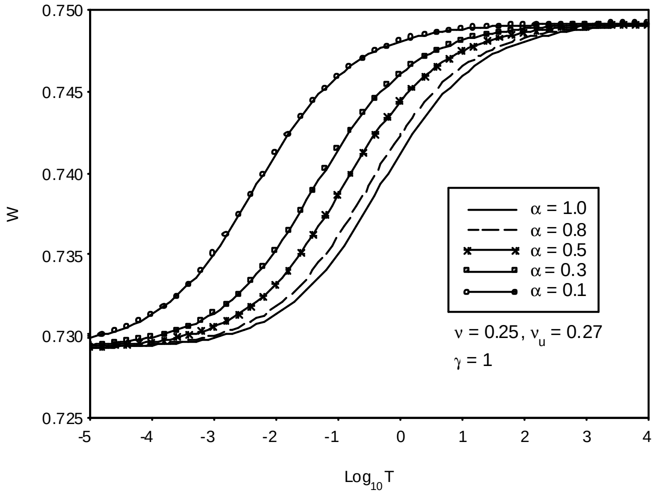

Figure 2. Effect of the compressibility of the solid constituents on the time-settlement.

$$\tilde{u}_z(r,z,s) = \frac{Q_0}{paG}\int_0^\infty \left[\frac{(n_u - n)ms}{k-m}\left(e^{-mz} - e^{-kz}\right) - \frac{1}{2}\left\{s(n_u - n) + s_a(1-n_u)\right\}(m+k)ze^{-kz} \right.$$

$$\left. -(1-n)(1-n_u)s_a\left(\frac{m+k}{k}\right)e^{-kz}\right] J_0(kr)J_1(ka)\frac{1}{W}dk \,. \qquad (19)$$

5 Numerical results

The solution obtained is in the Laplace-Hankel transforms domain. We have used Schapery's approximate formula (Schapery, 1962) for the Laplace transform inversion. The Hankel transform inversion has been performed numerically by using the extended Simpson's rule.

We have computed the surface settlement u_z at the centre of the disc load (r = z = 0) and pore pressure at various points on the z-axis (central line). We define the following dimensionless quantities

$$P = \left(\frac{pa^2}{Q_0}\right)p\,, \qquad W = \left(\frac{paG}{Q_0}\right)u_z\,, \qquad Z = \frac{z}{a}, \qquad T = \left(\frac{G c_z}{a^2}\right)t\,, \qquad g^2 = \frac{c_r}{c_z}\,. \qquad (20)$$

Figure 1 shows the time-settlement W for five values of the permeability anisotropy parameter γ for Ruhr sandstone (ν = 0.12, ν_u = 0.31, α = 0.65). When γ = 1, the vertical permeability is equal to the horizontal permeability. We notice that the permeability anisotropy has no effect on the initial settlement or the final settlement. However, if the horizontal permeability c_r is greater than the vertical permeability c_z, the permeability anisotropy accelerates the consolidation process. Figure 2 depicts the effect of the value of the Biot-Willis coefficient α on the time-settlement W for γ = 1, ν = 0.25, ν_u = 0.27. For a poroelastic material with incompressible solid constituents, α = 1. As α decreases, the compressibility of the solid constituents increases. Figure 2 shows that the compressibility of the solid constituents accelerates the consolidation process. The influence of the value of the undrained Poisson's ratio ν_u on the time-settlement W is displayed in Figure 3 for γ =1, ν = 0.12, α = 0.65. It is known that $\nu \le \nu_u \le 0.5$. The upper limit ν_u = 0.5 corresponds to a poroelastic material with incompressible fluid constituents. As expected, the final settlement is independent of the value of ν_u. However, the compressibility of the fluid constituents of the poroelastic medium has a strong influence on the consolidation process. The initial settlement for a compressible fluid constituents model is greater than the initial settlement for the corresponding incompressible fluid constituents model.

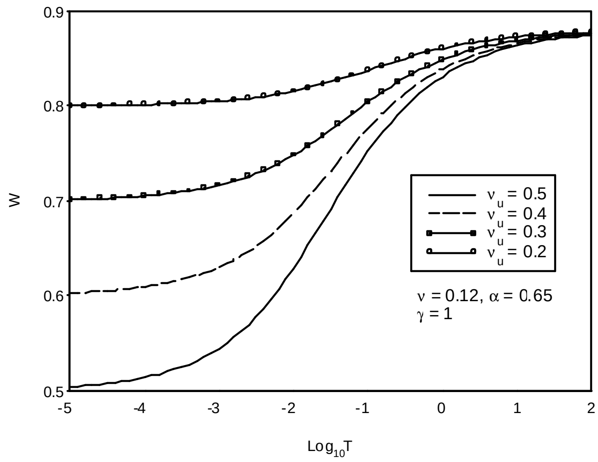

Figure 3. Effect of the compressibility of the fluid constituents on the time-settlement.

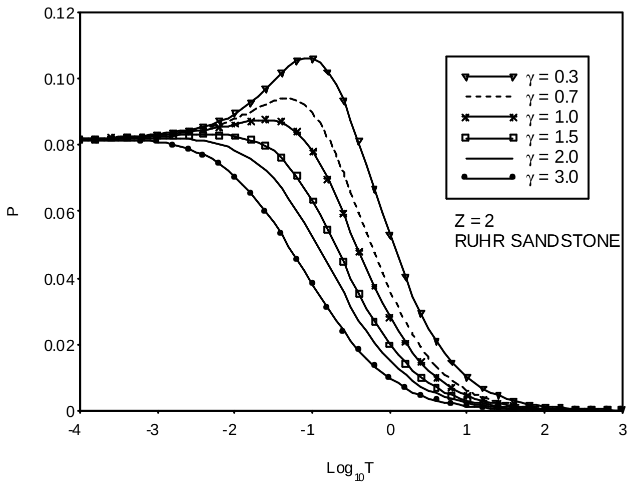

Figure 4. Effect of the permeability anisotropic on the diffusion of the pore pressure.

Figure 4 shows the effect of the permeability anisotropy on the diffusion of the pore pressure at the point r = 0, z = 2a. The pore pressure vanishes in the drained state ($T \rightarrow \infty$). Moreover, anisotropy has no effect in the undrained state ($T \rightarrow 0$). From Figure 4 we notice that instead of decreasing monotonically with time, the pore pressure rises above the initial undrained value before it decays to zero as $T \rightarrow \infty$. This is in accordance with the Mandal-Cryer Effect (Cryer, 1963). This effect is more pronounced at greater depths and for smaller values of the permeability anisotropy parameter $\gamma = (c_r / c_z)^{1/2}$. Since, in general, $c_r > c_z$, the theoretical prediction of the Mandel-Cryer Effect may get diluted in materials with anisotropic permeability.

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