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# A note on the dispersion of Love waves in layered monoclinic elastic media

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**Abstract.** The dispersion equation for Love waves in a monoclinic elastic layer of uniform thickness overlying a monoclinic elastic half-space is derived by applying the traction-free boundary condition at the surface and continuity conditions at the interface. The dispersion curves showing the effect of anisotropy on the calculated phase velocity are presented. The special cases of orthotropic and transversely isotropic media are also considered. It is shown that the well-known dispersion equation for Love waves in an isotropic layer overlying an isotropic half-space follows as a particular case.

**Keywords.** Dispersion; half-space; Love waves; monoclinic media.

## 1. Introduction

The study of surface wave dispersion in an isotropic half-space containing anisotropic layers is important in seismology for determining the presence or absence of anisotropic layers within the Earth. Such studies play a significant role in in-seam seismic exploration as well. The propagation of surface waves in an anisotropic half-space has been considered by many investigators. Wave propagation in a half-space with cubic symmetry has been discussed by Buchwald and Davies [3], and with orthorhombic symmetry by Stoneley [10]. Elastic wave propagation in transversely isotropic media has been reviewed by Payton [8]. Van der Hijden [12] discussed in great detail the propagation of transient elastic waves in stratified anisotropic media. Recent investigations on the propagation of elastic waves through anisotropic media include, among others, papers by Mench and Rasolofosaon [7], Savers [9] and Thomsen [11].

In an isotropic medium, SH type motion is decoupled from the P-SV type motion [6]. Surface waves of the SH type are known as Love waves and surface waves of the P-SV type are known as Rayleigh waves. The dispersion relation for Love waves in an isotropic elastic layer of uniform thickness $H$ overlying an isotropic elastic half-space can be written in the form ([2], Sec. 3.6.2)

$$\tan\left[kH\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2}\right]=\frac{\mu_2}{\mu_1}\cdot\frac{(1-c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2-1)^{1/2}}, \tag{1}$$

where $k$ denotes the wave number, $c$ the phase velocity, $\mu_1$ and $\mu_2$ the rigidities of the layer and the half-space, respectively, and $\beta_1$ and $\beta_2$ the shear-wave velocities of the layer and the half-space, respectively ($\beta_1 < c < \beta_2$). Equation (1) is also the dispersion relation for Love waves in an isotropic elastic layer of uniform thickness $2H$ sandwiched between two isotropic elastic half-spaces of identical properties [5]. The purpose of the present study is to derive the corresponding dispersion relation when the media are anisotropic of the monoclinic type possessing one plane of elastic symmetry.

## 2. Love waves in a monoclinic layer overlying a monoclinic half-space

We consider the propagation of Love waves in a monoclinic elastic layer of uniform thickness $H$ overlying a monoclinic elastic half-space. The layer $(0 \le x_3 \le H)$ is designated as medium (1) with displacement $u_1^{(1)}$ $(x_2, x_3, t)$, density $\rho_1$ and elastic constants $c_{ij}$ [6] and the half-space $(x_3 \ge H)$ is designated as medium (2) with displacement $u_1^{(2)}(x_2, x_3, t)$, density $\rho_2$ and elastic constants $d_{ij}$. A monoclinic medium has one plane of elastic symmetry [4]. We assume that the plane of symmetry is parallel to the $x_2x_3$-plane. For Love waves propagating in the positive $x_2$-direction with phase velocity $c$, we assume

$$u_1^{(1)} = f(x_3) \exp[ik(x_2 - ct)]. \tag{2}$$

The horizontal displacement $u_1^{(1)}$ satisfies the equation

$$c_{66} \frac{\partial^2 u_1}{\partial x_2^2} + 2c_{56} \frac{\partial^2 u_1}{\partial x_2 \partial x_3} + c_{55} \frac{\partial^2 u_1}{\partial x_3^2} = \rho_1 \frac{\partial^2 u_1}{\partial t^2}. \tag{3}$$

Equations (2) and (3) yield

$$c_{55} f''(x_3) + 2c_{56} ikf'(x_3) - k^2(c_{66} - \rho_1 c^2) f(x_3) = 0. \tag{4}$$

The general solution of eq. (4) is

$$f(x_3) = A_1 \exp(ikb_1 x_3) + B_1 \exp(-ikb_2 x_3),$$

where $A_1$, $B_1$ are arbitrary constants and

$$\begin{aligned} b_1 &= (\sqrt{A} - c_{56})/c_{55}, \quad b_2 = (\sqrt{A} + c_{56})/c_{55}, \\ A &= c_{55}(\rho_1 c^2 - c_{66}) + c_{56}^2. \end{aligned} \tag{5}$$

We thus have

$$u_1^{(1)} = (A_1 e^{ikb_1 x_3} + B_1 e^{-ikb_2 x_3}) e^{ik(x_2 - ct)}. \tag{6}$$

For surface waves

$$u_1^{(2)} \to 0 \text{ as } x_3 \to \infty.$$

Therefore, we assume

$$u_1^{(2)} = A_2 e^{ikb_3 x_3} \cdot e^{ik(x_2 - ct)}, \tag{7}$$

with

$$\begin{aligned} b_3 &= (i\sqrt{B} - d_{56})/d_{55}, \\ B &= d_{55}(d_{66} - \rho_2 c^2) - d_{56}^2, \end{aligned} \tag{8}$$

$\operatorname{Im} b_3 > 0$ (i.e. $B > 0$). The boundary conditions are

$$\tau_{13}^{(1)} = 0 \text{ at } x_3 = 0, \tag{9}$$

$$u_1^{(1)} = u_1^{(2)} \text{ at } x_3 = H, \tag{10}$$

$$\tau_{13}^{(1)} = \tau_{13}^{(2)} \text{ at } x_3 = H. \tag{11}$$

But

$$\tau_{13}^{(1)} = c_{55}\frac{\partial u_1^{(1)}}{\partial x_3} + c_{56}\frac{\partial u_1^{(1)}}{\partial x_2}, \tau_{13}^{(2)} = d_{55}\frac{\partial u_1^{(2)}}{\partial x_3} + d_{56}\frac{\partial u_1^{(2)}}{\partial x_2}. \tag{12}$$

Equations (5) to (12) yield

$$\begin{aligned} &A_1 - B_1 = 0, \\ &A_1 \exp(ikb_1H) + B_1 \exp(-ikb_2H) = A_2 \exp(ikb_3H), \\ &A_1 \exp(ikb_1H) - B_1 \exp(-ikb_2H) = i(B/A)^{1/2} A_2 \exp(ikb_3H). \end{aligned} \tag{13}$$

Eliminating $A_1, B_1$ and $A_2$, we obtain

$$\tan\theta = (B/A)^{1/2}, \tag{14}$$

where

$$\theta = (b_1 + b_2)kH/2 = (\sqrt{A}/c_{55})kH. \tag{15}$$

Equation (14) is the dispersion equation (frequency equation) for Love waves. From eq. (8) we note that $B > 0$. As in the case of isotropic media [2], it can be shown that (14) has no relevant solution if $A < 0$. Therefore, for the existence of Love waves, we must have $A > 0$, $B > 0$. Using (5) and (8), the dispersion equation (14) may be written in the form

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2} - 1 + \varepsilon_1\right)^{1/2} kH\right] = \left(\frac{d_{55}d_{66}}{c_{55}c_{66}}\right)^{1/2} \frac{(1 - \varepsilon_2 - c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2 - 1 + \varepsilon_1)^{1/2}}, \tag{16}$$

where

$$\begin{aligned} &\beta_1^2 = c_{66}/\rho_1, \quad \beta_2^2 = d_{66}/\rho_2, \\ &\varepsilon_1 = c_{56}^2/(c_{55}c_{66}), \varepsilon_2 = d_{56}^2/(d_{55}d_{66}), \\ &\gamma_1^2 = c_{66}/c_{55}. \end{aligned} \tag{17}$$

The conditions $A > 0, B > 0$ imply

$$(1 - \varepsilon_1)^{1/2}\beta_1 < c < (1 - \varepsilon_2)^{1/2}\beta_2.$$

Equation (16) is the dispersion relation for Love waves in a monoclinic layer of thickness $H$ overlying a monoclinic half-space. It is also the dispersion relation for a monoclinic layer of thickness $2H$ sandwiched between two monoclinic half-spaces of identical elastic properties.

From (16), we note that the dispersion relation for Love waves in a free monoclinic plate of thickness $H$ reduces to

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2} - 1 + \varepsilon_1\right)^{1/2} kH\right] = 0,$$

which implies $c > (1-\varepsilon_1)^{1/2}\beta_1$ and

$$\gamma_1\left(\frac{c^2}{\beta_1^2}-1+\varepsilon_1\right)^{1/2} kH = n\pi, \quad n=0,1,2,\ldots. \tag{18}$$

This relation also holds for a monoclinic layer in contact with a fluid layer on one or both sides.

## 3. Particular cases

### 3.1 *Orthotropic media*

For orthotropic media, $c_{56}=d_{56}=0$. Therefore, $\varepsilon_1=\varepsilon_2=0$ and the dispersion equation (16) reduces to

$$\tan\left[\gamma_1\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2} kH\right] = \left(\frac{d_{55}d_{66}}{c_{55}c_{66}}\right)^{1/2}\frac{(1-c^2/\beta_2^2)^{1/2}}{(c^2/\beta_1^2-1)^{1/2}}, \quad (\beta_1<c<\beta_2). \tag{19}$$

The dispersion relation (18) for Love waves in a free plate of thickness $H$ becomes

$$\gamma_1\left(\frac{c^2}{\beta_1^2}-1\right)^{1/2} kH = n\pi, \quad n=0,1,2,\ldots. \tag{20}$$

The corresponding equation given by Stoneley [10] is in error.

### 3.2 *Transversely isotropic media*

The dispersion equations (19) and (20) are also valid when the two media are transversely isotropic. These coincide with the dispersion equations given by Anderson [1] for transversely isotropic media.

### 3.3 *Isotropic media*

For isotropic media,

$$c_{55}=c_{66}=\mu_1, \quad d_{55}=d_{66}=\mu_2, \quad c_{56}=d_{56}=0, \quad \varepsilon_1=\varepsilon_2=0, \quad \gamma_1=\gamma_2=1.$$

Using these relations, the dispersion eq. (16) reduces to the form (1) valid for isotropic media.

## 4. Numerical results and discussion

Equation (16) is the dispersion equation for Love waves propagating in the plane of symmetry of a monoclinic elastic layer of thickness $H$ overlying a monoclinic half-space. This is also the dispersion equation for Love waves propagating in the plane of symmetry of a monoclinic elastic layer of thickness $2H$ sandwiched between two monoclinic elastic half-spaces of identical properties. For computing the dispersion curves, we assume

$$c_{55}=c_{66}, \quad d_{55}=d_{66}, \quad \varepsilon_1=\varepsilon_2=\varepsilon, \quad d_{55}/c_{55}=a, \quad \beta_2/\beta_1=b. \tag{21}$$

The dispersion equation (16) can now be written as

$$\tan[(C^2-1+\varepsilon)^{1/2}K] = a\frac{(1-\varepsilon-C^2/b^2)^{1/2}}{(C^2-1+\varepsilon)^{1/2}}, \tag{22}$$

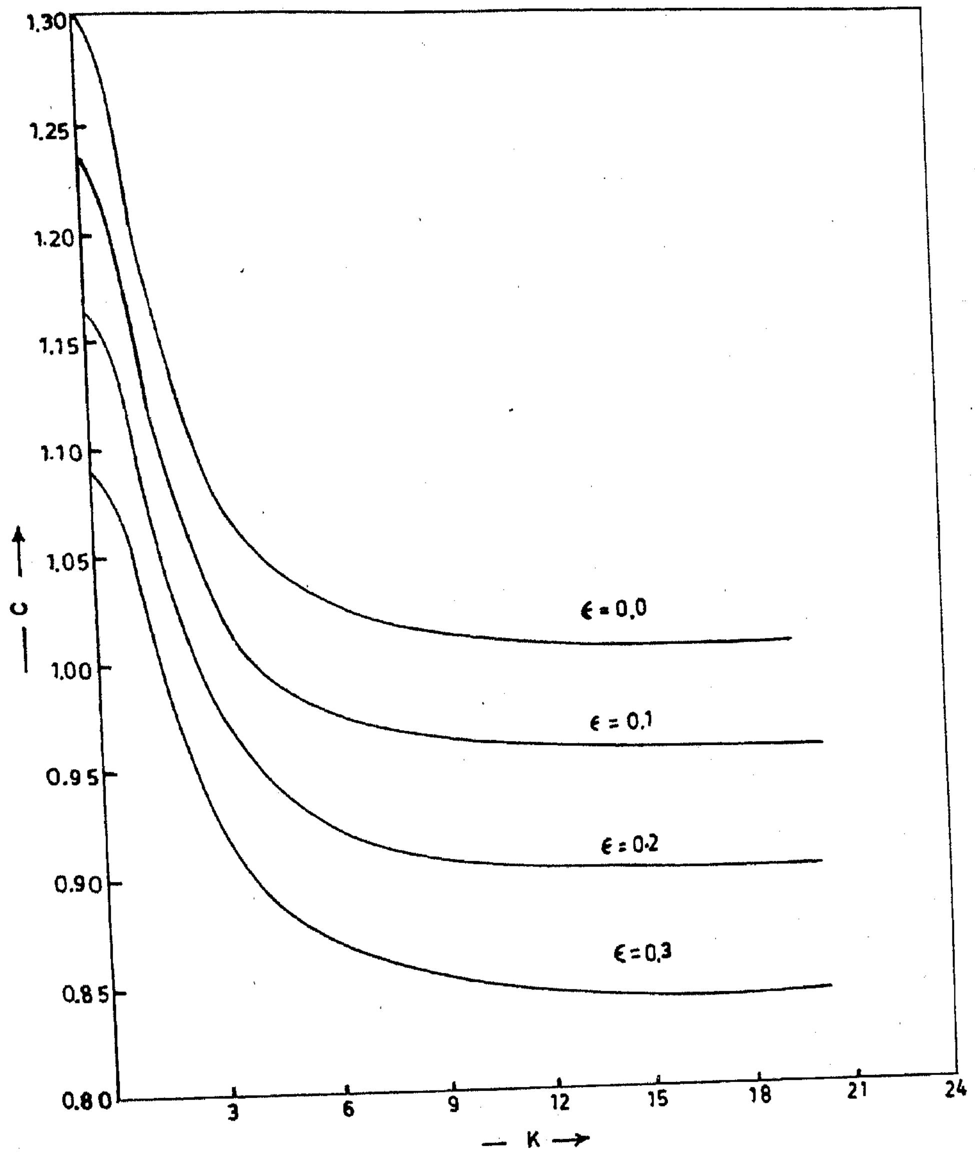


**Figure 1.** Variation of the dimensionless phase velocity $C = c/\beta_1$ with dimensionless wave number $K = kH$ for the fundamental Love mode for various values of the anisotropy parameter $\varepsilon$. The curve corresponding to $\varepsilon = 0$ is for isotropic media. The phase velocity lies in the range $(1-\varepsilon)^{1/2} < C < 1.3(1-\varepsilon)^{1/2}$. $K = 0$ for $C = (1-\varepsilon)^{1/2}$; $K \to \infty$ as $C \to 1.3(1-\varepsilon)^{1/2}$.

where $C = c/\beta_1$ is the dimensionless phase velocity, $K = kH$ is the dimensionless wave number and

$$(1-\varepsilon)^{1/2} < C < b(1-\varepsilon)^{1/2}. \tag{23}$$

Equation (22) has been used to obtain the dispersion curves showing the variation of the phase velocity with wave number for various values of the anisotropy parameter $\varepsilon$ assuming $a = 2$, $b = 1.3$ (figure 1). $\varepsilon = 0$ corresponds to the case when both the layer and the half-space are isotropic. From condition (23) we note that the effect of anisotropy is to reduce the range of the phase velocity from

$$\beta_1 < c < \beta_2,$$

**Table 1.** Cut-off period (in seconds) of the first three overtones ($n = 1, 2, 3$) for various values of the anisotropy parameter $\varepsilon_1$ of the layer.

| $n$ | Isotropic $\gamma_1 = 1, \varepsilon_1 = 0$ | Monoclinic, $\gamma_1 = 1.1$ | | |
|---|---|---|---|---|
| | | $\varepsilon_1 = 0.1$ | 0.2 | 0.3 |
| 1 | 12.57 | 14.85 | 15.80 | 16.70 |
| 2 | 6.29 | 7.42 | 7.90 | 8.35 |
| 3 | 4.19 | 4.95 | 5.27 | 5.57 |

valid for isotropic media, to

$$(1-\varepsilon)^{1/2}\beta_1 < c < (1-\varepsilon)^{1/2}\beta_2.$$

For a given $k$, the phase velocity decreases as the value of the anisotropy parameter $\varepsilon$ increases.

From (16), we find that the cut-off period for the $n$th overtone is given by

$$T_n = \frac{2\gamma_1 H}{n\beta_1}\left[1 - \left(\frac{1-\varepsilon_1}{1-\varepsilon_2}\right)\left(\frac{\beta_1}{\beta_2}\right)^2\right]^{1/2}. \tag{24}$$

For studying the effect of the anisotropy on the cut-off period, we assume that the half-space is isotropic ($\varepsilon_2 = 0$) and $H = 35\,\text{km}$, $\beta_1 = 3.5\,\text{km/s}$, $\beta_2 = 4.5\,\text{km/s}$.

Table 1 gives the values of the cut-off period for the first three overtones for various values of the anisotropy parameter $\varepsilon_1$ of the layer. We note that the cut-off period increases as the value of the anisotropy parameter increases.

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