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DISPLACEMENT AND STRESS PRODUCED BY A PRESSURIZED CYLINDRICAL MAGMA CHAMBER SURROUNDED BY A VISCOELASTIC SHELL AND AN ELASTIC MEDIUM

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The elastic solution for the displacement and stress due to a pressurized cylindrical cavity surrounded by a viscoelastic shell and an unbounded elastic medium is first obtained. The viscoelastic solution is then obtained by using the correspondence principle of linear viscoelasticity, assuming the shell to be elastic dilatational and Maxwell deviatoric. The variation of the displacement and stress fields with distance and with time is studied. It is found that in the case of a cylindrical magma chamber there are two characteristic times, in contrast to only one characteristic time for a spherical magma chamber.

1. INTRODUCTION

Large and often times rapid ground deformations take place in areas of volcanic activity. In recent years, measurements of surface deformations have provided useful means of studying the volcanic processes. Observations and measurements have shown that surface deformations are intimately related to eruptive activity and reflect the movement of the magma either within or under the volcanic edifice.

The problem of modelling of volcanic activity has attracted the attention of several investigators. Mogi[1] applied a centre of dilatation in an unbounded elastic medium to interpret the ground deformation produced in volcanic areas. Bonafede *et al.*[2] derived analytical expressions for the static displacement field produced by a centre of dilatation and by a pressure source in a viscoelastic half-space. The rheology of a standard linear solid was adopted for the shear modulus, while the incompressibility was kept elastic. The model was then applied to volcanic area of Campi, Flegrei, south of Italy. An approximate solution for the displacement and stress fields due to a pressurized spherical cavity in an elastic half-space has been given by McTigue[3]. Bianchi *et al.*[4] used a finite element method to calculate the surface deformation, assuming axial symmetry about the vertical axis through the caldera centre.

Dragoni and Magnanensi[5] discussed the problem of a pressurized spherical magma chamber surrounded by a viscoelastic shell. The shell was assumed to be elastic dilatational and Maxwell deviatoric, the outer medium being elastic. Since the magma chamber is not always spherical, we have discussed the problem of a pressurized cylindrical magma chamber surrounded by a viscoelastic shell and an elastic unbounded medium. Analytical expressions for the displacement and stress fields in the shell as well as outside the shell have been obtained. The variation of the field with distance and with time is studied numerically. It is found that, while in the case of a spherical magma chamber surrounded by a Maxwell shell, there is only one characteristic time, the cylindrical magma chamber model leads to two characteristic times.

2. THEORY

Consider a cylindrical cavity of radius R_1 representing the magma chamber surrounded by a co-axial cylindrical shell with outer radius R_2, representing the viscoelastic shell (Fig. 1). Cylindrical co-ordinates (r, θ, z) are used with z-axis along the axis of the cylindrical cavity. For $r > R_2$, the medium is elastic. A pressure $p(t)$ is applied within the magma chamber. To solve this viscoelastic problem, we first solve the corresponding elastic problem.

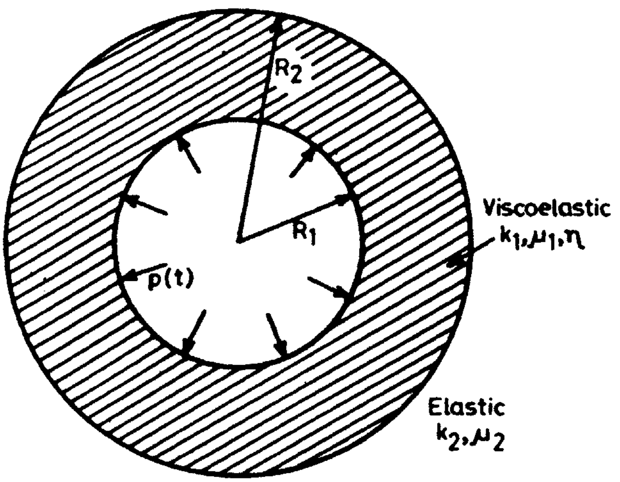

FIG. 1. A pressurized cylindrical cavity of radius R_1 surrounded by a viscoelastic shell ($R_1 < r < R_2$) and an elastic unbounded medium ($r > R_2$).

2.1. *Elastic Shell Model*

Suppose that a constant pressure p_0 is applied within the magma chamber ($r < R_1$). Assume that the shell is elastic with rigidity μ_1 and compressibility k_1 and the surrounding medium is also elastic with parameters μ_2 and k_2. Assuming axial symmetry, we have

$$\frac{\partial}{\partial \theta} = 0, \ \frac{\partial}{\partial z} = 0, \ u_\theta = u_z = 0, \ u_r = u_r(r),$$

where $\vec{u} = (u_r, u_\theta, u_z)$ is the displacement vector. Navier's equation of motion in the absence of body forces may be solved to obtain[6]

$$u_r(r) = \begin{cases} ar + b/r, & R_1 < r < R_2 \quad \text{... (1a)} \\ c/r, & r > R_2 \quad \text{... (1b)} \end{cases}$$

where a, b and c are arbitrary constants to be determined from the boundary conditions. From (1a) and (1b), the stress components are

$$\sigma_{rr}(r) = \begin{cases} \frac{2}{3}(3k_1 + \mu_1)a - \frac{2\mu_1 b}{r^2}, & R_1 < r < R_2 \quad \text{... (2a)} \\ -\frac{2\mu_2 c}{r^2}, & r > R_2 \quad \text{... (2b)} \end{cases}$$

$$\sigma_{\theta\theta}(r) = \begin{cases} \frac{2}{3}(3k_1 + \mu_1)a + \frac{2u_1 b}{r^2}, & R_1 < r < R_2 \quad \text{... (3a)} \\ \frac{2\mu_2 c}{r^2}, & r > R_2 \quad \text{... (3b)} \end{cases}$$

$$\sigma_{zz}(r) = \begin{cases} \frac{2}{3}(3k_1 - 2\mu_1)a, & R_1 < r < R_2 \quad \text{... (4a)} \\ 0, & r > R_2. \quad \text{... (4b)} \end{cases}$$

Also

$$\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zr} = 0. \quad \text{... (5)}$$

The boundary conditions are

(i) $\sigma_{rr}(r) = -p_0$ at $r = R_1$,

(ii) continuity of $u_r(r)$ at $r = R_2$,

(iii) continuity of $\sigma_{rr}(r)$ at $r = R_2$.

Using these boundary conditions, we get a system of three equations in three unknowns which can be solved to obtain

$$a = -3p_0 R_1^2 (\mu_1 - \mu_2)/2D, \quad \text{... (6)}$$

$$b = -p_0 R_1^2 R_2^2 (3k_1 + 3\mu_2 + \mu_1)/2D, \quad \text{... (7)}$$

$$c = -p_0 R_1^2 R_2^2 (4\mu_1 + 3k_1)/2D, \quad \text{... (8)}$$

where

$$D = (\mu_1 - \mu_2)(3k_1 + \mu_1) R_1^2 - \mu_1 (3k_1 + 3\mu_2 + \mu_1) R_2^2. \qquad \text{... (9)}$$

The elastic solution can be obtained by using these values of a, b and c in (1)-(5).

2.2. *Viscoelastic Shell Model*

If the region $R_1 < r < R_2$ is viscoelastic, then the viscoelastic solution can be obtained from the elastic solution by using correspondence principle (Fung[7]). According to this principle, if the shell is Maxwell deviatoric and elastic dilatational, we should replace μ_1 by $\bar{\mu}_1$, where

$$\bar{\mu}_1 = \frac{\mu_1 \eta s}{\mu_1 + \eta s} \qquad \text{... (10)}$$

and η is the viscosity of the shell material and s is the Laplace transform variable. Replacing μ_1 by $\bar{\mu}_1$ in (6)-(9) and replacing p_0 by $\bar{p}(s)$, where $\bar{p}(s)$ is the Laplace transform of $p(t)$, the source function representing the pressure within the magma chamber, we obtain the expressions for $\bar{a}(s), \bar{b}(s)$ and $\bar{c}(s)$. These expressions for $\bar{a}(s), \bar{b}(s)$ and $\bar{c}(s)$ are given in Appendix 1. The Laplace transform of the viscoelastic solution can be obtained by using these expressions for $\bar{a}(s), \bar{b}(s)$ and $\bar{c}(s)$ in (1)-(5).

3. Solution for Step-like Source Function

Consider a step-like increase in pressure in the magma chamber

$$p(t) = p_0 H(t), \qquad \text{... (11)}$$

where $H(t)$ is the Heaviside unit step function. The Laplace transform of (11) is

$$\bar{p}(s) = p_0/s. \qquad \text{... (12)}$$

On replacing $\bar{p}(s)$ by p_0/s in the Laplace transform of the viscoelastic solution, we obtain the expressions for $\bar{u}_r(s), \bar{\sigma}_{rr}(s)$, $\bar{\sigma}_{\theta\theta}(s)$ and $\bar{\sigma}_{zz}(s)$ for a step source. These can be inverted to get the displacement and stress as functions of time. We obtain

$$u_r(r, t) = \frac{1}{D_0} [A_1(r) + A_2(r) e^{-t/\tau_1} + A_3(r) e^{-t/\tau_2}], \qquad \text{... (13)}$$

$$\sigma_{rr}(r, t) = \frac{1}{D_0} [B_1(r) + B_2(r) e^{-t/\tau_1} + B_3(r) e^{-t/\tau_2}], \qquad \text{... (14)}$$

$$\sigma_{\theta\theta}(r, t) = \frac{1}{D_0} [C_1(r) + C_2(r) e^{-t/\tau_1} + C_3(r) e^{-t/\tau_2}], \qquad \text{... (15)}$$

$$\sigma_{zz}(r,t) = \frac{1}{D_0}\,[\,D_1(r) + D_2(r)\,e^{-t/\tau_1} + D_3(r)\,e^{-t/\tau_2}\,], \qquad \text{... (16)}$$

where D_0 is defined in Appendix 1 and A_1, B_1, etc. in Appendix 2. τ_1 and τ_2 are the two characteristic times :

$$\tau_1 = -\frac{1}{s_1}\,,\ \tau_2 = -\frac{1}{s_2}\,,$$

where s_1 and s_2 are the roots of the equation

$$D_0\,s^2 + E_0\,s + F_0 = 0$$

and D_0, E_0, F_0 are defined in Appendix 1.

4. Numerical Results

For numerical work, we assume that

$$\mu_1 = \mu_2 = \mu\,,\ k_1 = \frac{5}{3}\,\mu_1\,,\ R_2 = 2R_1, \qquad \text{... (17)}$$

so that the media involved are Poissonian. We then find

$$\tau_1 = 6.49\,(\eta/\mu)\,,\quad \tau_2 = 1.11\,(\eta/\mu).$$

The distance is measured in units of R_1 and time in units of $\tau_0 = \eta/\mu$. Let u_0 denote the radial displacement which would be produced at $r = R_1$ by a magma chamber with pressure p_0, in an elastic medium with rigidity μ, so that

$$u_0 = p_0\,R_1/2\mu. \qquad \text{... (18)}$$

We take u_0 as the unit of displacement. We define dimensionless distance (R), dimensionless time (T), and dimensionless radial displacement (U) as follows

$$R = r/R_1\,,\ T = t/\tau_0\,,\ U = u_r/u_0. \qquad \text{... (19)}$$

Figures 2-4 show the variation of the radial displacement and the stresses σ_{rr} and $\sigma_{\theta\theta}$ with radial distance for four values of T : 0, 1, 2, 10. $T = 0$ corresponds to the elastic shell model. We note that for the viscoelastic shell model, the stress $\sigma_{\theta\theta}$ is discontinuous at $r = R_2 = 2R_1$. The discontinuity increases with time.

Figures 5-7 show the variation of the radial displacement and the stresses σ_{rr} and $\sigma_{\theta\theta}$ with time for three values of R : 1.5, 2.5, 5. $R = 1.5$ corresponds to a point half-way through the shell, $R = 2.5$ and 5 correspond to points in the outer elastic medium.

The characteristic times τ_1 and τ_2 depend on the ratio R_2/R_1. Figure 8 shows the variation of τ_1 and Fig. 9 shows the variation of τ_2 with R_2/R_1.

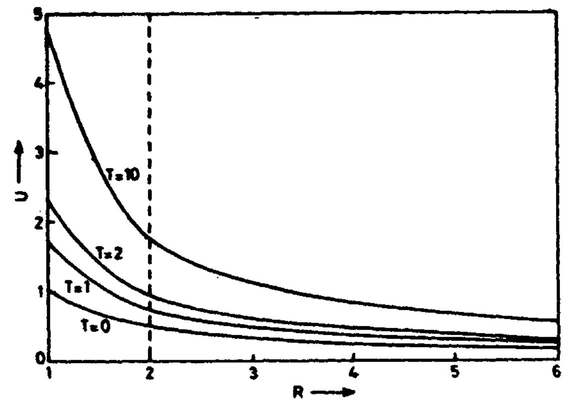

FIG. 2. Variation of the radial displacement with radial distance. $T = 0$ corresponds to the elastic shell model. The dotted line indicates the outer shell boundary.

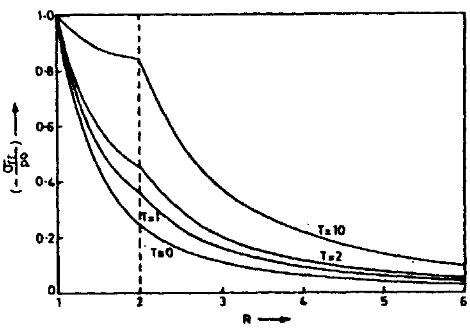

FIG. 3. Variation of the radial stress σ_{rr} with radial distance.

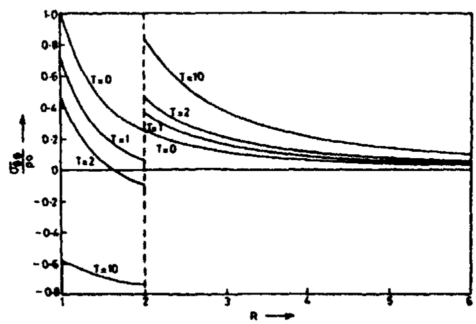

FIG. 4. Variation of the transverse stress $\sigma_{\theta\theta}$ with radial distance. For viscoelastic shell model, $\sigma_{\theta\theta}$ is discontinuous at the shell boundary.

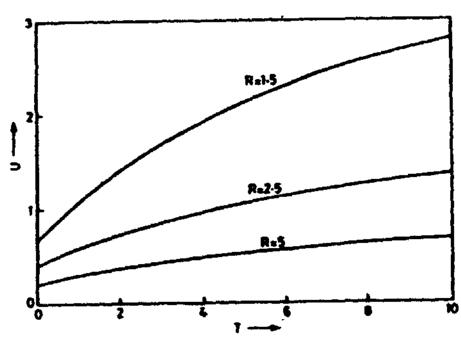

FIG. 5. Variation of the radial displacement with time. $R = 1.5$ corresponds to a point half-way through the shell and $R = 2.5$ and 5 correspond to points in the outer elastic medium.

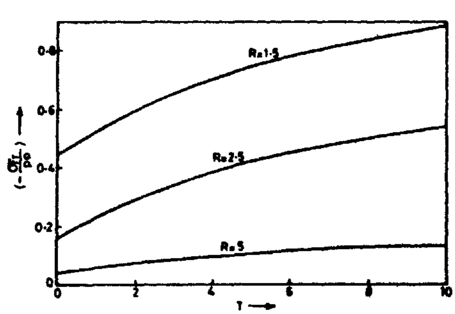

FIG. 6. Variation of the radial stress σ_{rr} with time.

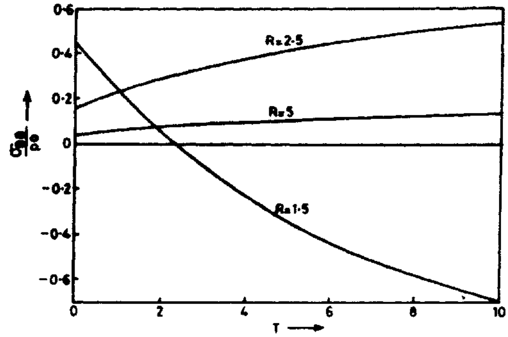

FIG. 7. Variation of the transverse stress $\sigma_{\theta\theta}$ with time.

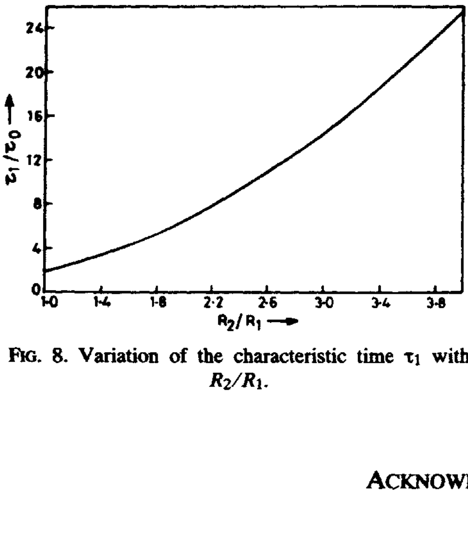

FIG. 8. Variation of the characteristic time τ_1 with R_2/R_1.

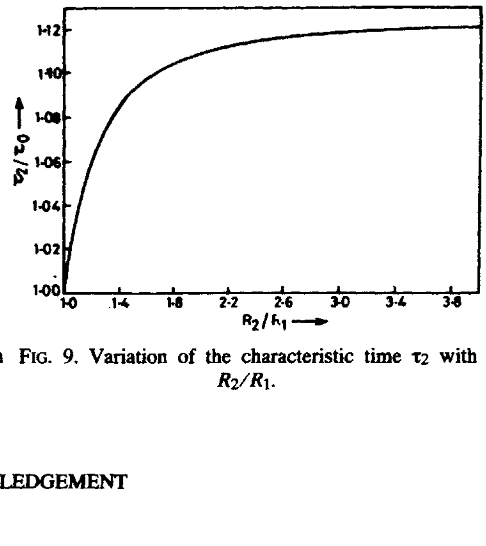

FIG. 9. Variation of the characteristic time τ_2 with R_2/R_1.

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APPENDIX 1

$$\bar{a}(s) = \frac{-3\left\{(\mu_1-\mu_2)\eta-\mu_1\mu_2\right\}(\mu_1+\eta s)R_1^2\bar{p}(s)}{2(D_0s^2+E_0s+F_0)},$$

$$\bar{b}(s) = \frac{-\left\{(3k_1+3\mu_2+\mu_1)\eta+3(k_1+\mu_2)\mu_1\right\}(\mu_1+\eta s)R_1^2R_2^2\bar{p}(s)}{2(D_0s^2+E_0s+F_0)},$$

$$\bar{c}(s) = \frac{-\left\{(4\mu_1+3k_1)\eta+3k_1\mu_1\right\}(\mu_1+\eta s)R_1^2R_2^2\bar{p}(s)}{2(D_0s^2+E_0s+F_0)},$$

where

$$D_0=\eta^2\left[(\mu_1-\mu_2)(3k_1+\mu_1)R_1^2-\mu_1(\mu_1+3k_1+3\mu_2)R_2^2\right],$$
$$E_0=\eta\left[\left\{3k_1\mu_1(\mu_1-\mu_2)-(3k_1+\mu_1)\mu_1\mu_2\right\}R_1^2-3\mu_1^2(k_1+\mu_2)R_2^2\right],$$
$$F_0=-3\mu_1^2\mu_2k_1R_1^2.$$

APPENDIX 2

The expressions for A_1, B_1, etc. appearing in (13)-(16) are given below. In each case, the upper expression holds for $R_1 \le r \le R_2$ and the lower for $r > R_2$

$$A_1(r)=\begin{cases}\frac{3}{2}p_0\mu_1^2\tau_1\tau_2R_1^2\left[\mu_2r-(k_1+\mu_2)R_2^2/r\right]\\ -\frac{3}{2}p_0\mu_1^2k_1\tau_1\tau_2R_1^2R_2^2/r\end{cases}$$

$$A_2(r)=\begin{cases}\frac{3p_0(\eta-\mu_1\tau_1)\tau_2R_1^2}{2(\tau_1-\tau_2)}\left[\left\{(\mu_1-\mu_2)\eta+\mu_1\mu_2\tau_1\right\}r\right.\\ \qquad\left.-\left[\left\{(k_1+\mu_2)\mu_1\tau_1-\eta\left(k_1+\mu_2+\frac{\mu_1}{3}\right)\right\}\frac{R_2^2}{r}\right]\right.\\ \frac{-p_0(\eta-\mu_1\tau_1)\tau_2R_1^2R_2^2}{2(\tau_1-\tau_2)r}\left[3k_1^*\mu_1\tau_1-(4\mu_1+3k_1)\eta\right]\end{cases}$$

$$A_3(r)=\begin{cases}\frac{3p_0(\eta-\mu_1\tau_2)\tau_1R_1^2}{2(\tau_2-\tau_1)}\left[\left\{(\mu_1-\mu_2)\eta+\mu_1\mu_2\tau_2\right\}r\right.\\ \qquad\left.-\left[\left\{(k_1+\mu_2)\mu_1\tau_2-\eta\left(k_1+\mu_2+\frac{\mu_1}{3}\right)\right\}\frac{R_2^2}{r}\right]\right.\\ \frac{-p_0(\eta-\mu_1\tau_2)\tau_1R_1^2R_2^2}{2(\tau_2-\tau_1)r}\left[3k_1\mu_1\tau_2-(4\mu_1+3k_1)\eta\right]\end{cases}$$

$$B_1(r) = \begin{cases} 3p_0\,\mu_1^2\,k_1\,\mu_2\,\tau_1\,\tau_2\,R_1^2 \\[2ex] 3p_0\,\mu_1^2\,k_1\,\mu_2\,\tau_1\,\tau_2\,R_1^2\,R_2^2/r^2 \end{cases}$$

$$B_2(r) = \begin{cases} \dfrac{p_0\,\tau_2\,R_1^2}{\tau_1-\tau_2}\Big[\Big\{(3k_1+\mu_1)\,\eta - 3k_1\,\mu_1\,\tau_1\Big\}\Big\{(\mu_1-\mu_2)\,\eta + \mu_1\,\mu_2\,\tau_1\Big\} \\ \qquad + \Big[\Big\{\tau_1\,(k_1+\mu_2)\,\mu_1 - \Big(k_1+\mu_2+\dfrac{\mu_1}{3}\Big)\eta\Big\}\dfrac{3\mu_1\,\eta\,R_2^2}{r^2}\Big] \\[2ex] \dfrac{p_0\,\mu_2\,\tau_2\,(\mu_1\,\tau_1-\eta)\,R_1^2\,R_2^2}{(\tau_1-\tau_2)r^2}\,[(4\mu_1+3k_1)\,\eta - 3k_1\,\mu_1\,\tau_1] \end{cases}$$

$$B_3(r) = \begin{cases} \dfrac{p_0\,\tau_1\,R_1^2}{\tau_2-\tau_1}\Big[\Big\{(3k_1+\mu_1)\,\eta - 3k_1\,\mu_1\,\tau_2\Big\}\Big\{(\mu_1-\mu_2)\,\eta + \mu_1\,\mu_2\,\tau_2\Big\} \\ \qquad + \Big[\Big\{\mu_1\,(k_1+\mu_2)\,\tau_2 - \Big(k_1+\mu_2+\dfrac{\mu_1}{3}\Big)\eta\Big\}\dfrac{3\mu_1\,\eta\,R_2^2}{r^2}\Big] \\[2ex] \dfrac{p_0\,\mu_2\,\tau_2\,(\mu_1\,\tau_1-\eta)\,R_1^2\,R_2^2}{(\tau_1-\tau_2)r^2}\,[(4\mu_1+3k_1)\,\eta - 3k_1\,\mu_1\,\tau_1] \end{cases}$$

$$C_1(r) = \begin{cases} 3p_0\,\mu_1^2\,k_1\,\mu_2\,\tau_1\,\tau_2\,R_1^2 \\[2ex] -3p_0\,\mu_1^2\,k_1\,\mu_2\,\tau_1\,\tau_2\,R_1^2\,R_2^2/r^2 \end{cases}$$

$$C_2(r) = \begin{cases} \dfrac{p_0\,\tau_2\,R_1^2}{\tau_1-\tau_2}\Big[\Big\{(3k_1+\mu_1)\,\eta - 3k_1\,\mu_1\,\tau_1\Big\}\Big\{(\mu_1-\mu_2)\,\eta + \mu_1\,\mu_2\,\tau_1\Big\} \\ \qquad - \Big[\Big\{\tau_1\,(k_1+\mu_2)\,\mu_1 - \Big(k_1+\mu_2+\dfrac{\mu_1}{3}\Big)\eta\Big\}\dfrac{3\mu_1\,\eta\,R_2^2}{r^2}\Big] \\[2ex] \dfrac{-p_0\,\mu_2\,\tau_2\,(\mu_1\,\tau_1-\eta)\,R_1^2\,R_2^2}{(\tau_1-\tau_2)r^2}\,[(4\mu_1+3k_1)\,\eta - 3k_1\,\mu_1\,\tau_1] \end{cases}$$

$$C_3(r) = \begin{cases} \dfrac{p_0\,\tau_1\,R_1^2}{\tau_2-\tau_1}\Big[\Big\{(3k_1+\mu_1)\,\eta - 3k_1\,\mu_1\,\tau_2\Big\}\Big\{(\mu_1-\mu_2)\,\eta + \mu_1\,\mu_2\,\tau_2\Big\} \\ \qquad - \Big\{\tau_2\,(k_1+\mu_2)\,\mu_1 - \Big(k_1+\mu_2+\dfrac{\mu_1}{3}\Big)\eta\Big\}\dfrac{3\mu_1\,\eta\,R_2^2}{r^2}\Big] \\[2ex] \dfrac{-p_0\,\mu_2\,\tau_1\,(\mu_1\,\tau_2-\eta)\,R_1^2\,R_2^2}{(\tau_2-\tau_1)\,r^2}\,[(4\mu_1+3k_1)\,\eta - 3k_1\,\mu_1\,\tau_2] \end{cases}$$

$$D_1(r) = \begin{cases} 3p_0 \mu_1^2 k_1 \mu_2 \tau_1 \tau_2 R_1^2 \\ 0 \end{cases}$$

$$D_2(r) = \begin{cases} \dfrac{p_0 \tau_2 R_1^2}{\tau_1 - \tau_2} [(3k_1 - 2\mu_1)\eta - 3k_1 \mu_1 \tau_1] [(\mu_1 - \mu_2)\eta + \mu_1 \mu_2 \tau_1] \\ 0 \end{cases}$$

$$D_3(r) = \begin{cases} \dfrac{p_0 \tau_1 R_1^2}{\tau_2 - \tau_1} [(3k_1 - 2\mu_1)\eta - 3k_1 \mu_1 \tau_2] [(\mu_1 - \mu_2)\eta + \mu_1 \mu_2 \tau_2] \\ 0 \end{cases}$$