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Toroidal oscillations of a transradially isotropic elastic sphere

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Abstract. In order to consider the effect of anisotropy on the periods of the oscillations of the Earth, the problem of toroidal oscillations of a transradially isotropic elastic sphere is considered. At each point, the medium is assumed to be transversely isotropic about the radius through the point. The roots of the frequency equation are obtained for different values of the anisotropy parameter α. It is found that, for large order oscillations, the percentage change in the frequency of the toroidal oscillations on account of the anisotropy is nearly equal to $|\alpha - 1| \times 100$.

Keywords. Anisotropy parameter; elastic sphere; frequency equation; order of oscillations; toroidal oscillations; transradial isotropy.

1. Introduction

Backus (1967) investigated the problem of wave propagation in transradially isotropic elastic media. He showed that, for a transradially isotropic sphere, the toroidal and spheroidal modes of vibrations are independent of each other. Sato *et al* (1971) considered waves and rays in a transradially isotropic homogeneous sphere. Singh (1974) applied the method of layer matrices to study the problem of the toroidal oscillations of a transradially isotropic stratified earth model. Wave propagation in transradially isotropic media has also been discussed by Helbig (1966), Vlaar (1969) and Datta and Shah (1970).

In order to consider the effect of anisotropy on the periods of the oscillations of the Earth, we consider the problem of the toroidal oscillations of a transradially isotropic elastic sphere. At each point, the medium is assumed to be transversely isotropic about the radius through the point. Thus, the stress-strain relations at any point are invariant under all rotations about the radius through the point.

2. Theory

Using spherical polar co-ordinates (r, θ, ϕ), with origin at the centre of the sphere, we can write the stress-strain relations for transradial isotropy as (Backus 1967)

$$\tau_{\theta\theta} = c_{11} e_{\theta\theta} + c_{12} e_{\phi\phi} + c_{13} e_{rr},$$

$$\tau_{\phi\phi} = c_{12} e_{\theta\theta} + c_{11} e_{\phi\phi} + c_{13} e_{rr},$$

$$\tau_{rr} = c_{13}(e_{\theta\theta} + e_{\phi\phi}) + c_{33} e_{rr},$$

$$\tau_{r\theta} = 2c_{44} e_{r\theta},$$

$$\tau_{\theta\phi} = 2c_{55} e_{\theta\phi},$$

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where

$$c_{55} = \tfrac{1}{2}(c_{11} - c_{12}). \tag{1}$$

τ_{ij} and e_{ij} $(i, j = r, \theta, \phi)$ are the stress and strain components, respectively, in spherical polar co-ordinates (r, θ, ϕ). c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are the five elastic constants of the medium. For an isotropic medium, $c_{12} = c_{13} = \lambda$, $c_{11} = c_{33} = \lambda + 2\mu$, $c_{44} = c_{55} = \mu$, where λ, μ are the Lame constants.

Denoting the (r, θ, ϕ) components of the displacement by (u_r, u_θ, u_ϕ), the strain-displacement relations are

$$e_{rr} = \partial u_r / \partial r,$$

$$e_{\theta\theta} = (u_r/r) + (1/r)(\partial u_\theta / \partial \theta),$$

$$e_{\phi\phi} = \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi},$$

$$2e_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right),$$

$$2e_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},$$

$$2e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}. \tag{2}$$

As shown by Backus (1967), the oscillations of a transradially isotropic sphere are of two kinds, toroidal oscillations and spheroidal oscillations. For toroidal oscillations, the dilatation (div **u**) and the radial component of the displacement vanish identically. On the other hand, for spheroidal oscillations, the radial component of the curl of the displacement vanishes identically. Therefore, for the toroidal oscillations, we can assume

$$\mathbf{u} = \nabla \times (\mathbf{r}\psi), \tag{3}$$

where **u** denote the displacement vector and ψ is the potential function. Therefore

$$u_r = 0, \quad u_\theta = (1/\sin\theta)(\partial\psi/\partial\phi), \quad u_\phi = -\partial\psi/\partial\theta. \tag{4}$$

Equations (1), (2) and (4) yield the stresses in terms of ψ. Substituting the expressions for the stresses in the equations of motion in spherical polar coordinates (Love 1927, p. 91), we find that these equations are identically satisfied if ψ is a solution of the equation (Singh 1974)

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\alpha^2}{r^2}\left[\frac{\partial^2\psi}{\partial\theta^2} + \cot\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + 2(1-\alpha^{-2})\psi\right]$$

$$= \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}, \tag{5}$$

where

$$\alpha^2 = c_{55}/c_{44}, \quad c^2 = c_{44}/\rho. \tag{6}$$

Obviously, α is the ratio of the velocity perpendicular to the radius to the velocity along the radius at any point of the medium. For an isotropic medium, $\alpha = 1$. Assuming a harmonic time-dependence, a solution of (5) can be expressed in the form

$$\psi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} W_{nm}(r) P_n^m(\cos\theta) \exp[i(m\phi - \omega t)], \tag{6a}$$

where $W_{nm}(r)$ is a function of r only and $P_n^m(\cos\theta)$ is the associated Legendre function. From (5) and (6), we note that $W_{nm}(r)$ satisfies the equation

$$\frac{d^2 W_{nm}}{dr^2} + \frac{2}{r}\frac{dW_{nm}}{dr} + \left[\beta^2 - \frac{\alpha^2}{r^2}\{n(n+1) - 2(1-\alpha^{-2})\}\right] W_{nm} = 0, \tag{7}$$

where

$$\beta^2 = \omega^2/c^2. \tag{8}$$

For the solution of (7), we put,

$$l(l+1) = \alpha^2[n(n+1) - 2(1-\alpha^{-2})], \tag{9}$$

i.e.

$$v^2 = (l+\tfrac{1}{2})^2 = \alpha^2(n+\tfrac{1}{2})^2 + \tfrac{9}{4}(1-\alpha^2). \tag{10}$$

From the above relation, we note that v^2 is positive for all $n \geqslant 1$, whatever the value of α. Therefore, v is real. Equation (7) now becomes

$$\frac{d^2 W_{nm}}{dr^2} + \frac{2}{r}\frac{dW_{nm}}{dr} + \left(\beta^2 - \frac{l(l+1)}{r^2}\right) W_{nm} = 0. \tag{11}$$

Inserting

$$W_{nm} = r^{-1/2} V_{nm}(r), \tag{12}$$

we see that $V_{nm}(r)$ satisfies the equation

$$\frac{d^2 V_{nm}}{dr^2} + \frac{1}{r}\frac{dV_{nm}}{dr} + \left(\beta^2 - \frac{v^2}{r^2}\right) V_{nm} = 0, \tag{13}$$

where $v^2 = (l+\frac{1}{2})^2$ is given by (10).

Equation (13) is the Bessel equation of order v. It has two independent solutions $J_v(\beta r)$ and $Y_v(\beta r)$. But $Y_v(\beta r)$ has a singularity at $r = 0$. Therefore, from (6), we have

$$\psi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} r^{-1/2} E_{nm} J_v(\beta r) P_n^m(\cos\theta) \exp[i(m\phi - \omega t)], \tag{14}$$

where E_{nm} is an arbitrary constant to be determined from boundary conditions.

For the toroidal oscillations of a homogeneous transradially isotropic sphere of radius a, the boundary conditions are

$$\tau_{r\theta} = 0, \quad \tau_{r\phi} = 0 \text{ at } r = a. \tag{15}$$

From (1), (2) and (4)

$$\tau_{r\theta} = \frac{c_{44}}{\sin\theta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\frac{\partial\psi}{\partial\phi},$$

$$\tau_{r\phi} = -c_{44}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\frac{\partial\psi}{\partial\theta}. \tag{16}$$

Therefore, the boundary conditions are identically satisfied if

$$(\partial\psi/\partial r) - (\psi/r) = 0 \quad \text{at } r = a. \tag{17}$$

Equations (14) and (17) yield the frequency equation

$$\eta J_\nu'(\eta) - \tfrac{3}{2}J_\nu(\eta) = 0, \tag{18}$$

where

$$\eta = \beta a = \omega a/c. \tag{19}$$

On using the recurrence relation

$$J_\nu'(\eta) = \frac{\nu}{\eta}J_\nu(\eta) - J_{\nu+1}(\eta), \tag{20}$$

equation (19) can also be expressed in the form

$$(\nu - \tfrac{3}{2})J_\nu(\eta) - \eta J_{\nu+1}(\eta) = 0, \tag{21}$$

or

$$(l-1)J_{l+\frac{1}{2}}(\eta) - \eta J_{l+\frac{3}{2}}(\eta) = 0. \tag{22}$$

For an isotropic sphere ($\alpha = 1$), the frequency equation (22) becomes

$$(n-1)J_{n+\frac{1}{2}}(\eta) - J_{n+\frac{3}{2}}(\eta) = 0. \tag{23}$$

Equation (23) coincides with the frequency equation for the toroidal oscillations of a homogeneous isotropic sphere (Ben-Menahem and Singh 1981; p. 340).

When $n = 1$, we find from (10) that $\nu = 3/2$, $l = 1$ for all α. This shows that the anisotropy has no effect on the frequency of the toroidal oscillations of order $n = 1$ of a solid sphere.

For $n > 1$, we note from (10) that

$$(l + \tfrac{1}{2})^2 - (n + \tfrac{1}{2})^2 = (\alpha^2 - 1)[(n + \tfrac{1}{2})^2 - (\tfrac{3}{2})^2], \tag{24}$$

which is greater than zero if $\alpha > 1$. This shows that, if $\alpha > 1$, the zeros for the anisotropic sphere coincide with the zeros for an isotropic sphere with larger parameter n.

3. Numerical results

We have computed the values of the parameter l from equation (10) for $n = 1$ to 100 and $\alpha = 0{\cdot}6$, $0{\cdot}8$, $1{\cdot}0$, $1{\cdot}2$, $1{\cdot}4$, $1{\cdot}6$, $1{\cdot}8$ and $2{\cdot}0$. These values are given in table 1 for $n = 1$ to 50. From this table we find that $l = \alpha n$ is a good approximation for large

Table 1. Values of the parameter l for different values of n and α.

n	α = 0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
1	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
2	1·421	1·693	2·000	2.330	2·676	3·034	3·400	3·772
3	1·919	2·441	3·000	3·580	4·174	4·777	5·386	6·000
4	2·455	3·211	4·000	4·808	5·626	6·452	7·283	8·117
5	3·011	3·991	5·000	6·025	7·058	8·098	9·142	10·189
6	3·580	4·777	6·000	7·236	8·481	9·730	10·983	12·238
7	4·157	5·567	7·000	8·445	9·897	11·353	12·812	14·273
8	4·739	6·359	8·000	9·651	11·309	12·970	14·634	16·300
9	5·325	7·153	9·000	10·856	12·719	14·584	16·452	18·322
10	5·913	7·948	10·000	12·061	14·126	16·195	18·266	20·339
11	6·504	8·744	11·000	13·264	15·533	17·804	20·078	22·353
12	7·095	9·540	12·000	14·467	16·938	19·412	21·888	24·365
13	7·688	10·337	13·000	15·669	18·343	21·019	23·696	26·375
14	8·282	11·135	14·000	16·872	19·747	22·624	25·503	28·383
15	8·877	11·933	15·000	18·073	21·150	24·229	27·310	30·391
16	9·472	12·731	16·000	19·275	22·553	25·833	29·115	32·398
17	10·068	13·529	17·000	20·476	23·956	27·437	30·920	34·403
18	10·665	14·327	18·000	21·678	25·358	29·041	32·724	36·409
19	11·261	15·126	19·000	22·879	26·760	30·644	34·528	38·413
20	11·858	15·925	20·000	24·080	28·162	32·246	36·332	40·418
21	12·456	16·724	21·000	25·281	29·564	33·849	38·135	42·421
22	13·053	17·522	22·000	26·482	30·966	35·451	39·938	44·425
23	13·651	18·322	23·000	27·682	32·367	37·053	41·740	46·428
24	14·249	19·121	24·000	28·883	33·768	38·655	43·543	48·431
25	14·847	19·920	25·000	30·084	35·170	40·257	45·345	50·434
26	15·445	20·719	26·000	31·284	36·571	41·859	47·147	52·436
27	16·044	21·518	27·000	32·485	37·972	43·460	48·949	54·439
28	16·642	22·318	28·000	33·686	39·373	45·061	50·751	56·441
29	17·241	23·117	29·000	34·886	40·774	46·663	52·553	58·443
30	17·839	23·917	30·000	36·086	42·175	48·264	54·354	60·445
31	18·438	24·716	31·000	37·287	43·576	49·865	56·156	62·446
32	19·037	25·516	32·000	38·487	44·976	51·466	57·957	64·448
33	19·636	26·315	33·000	39·688	46·377	53·067	59·758	66·450
34	20·235	27·115	34·000	40·888	47·778	54·668	61·559	68·451
35	20·834	27·914	35·000	42·088	49·178	56·269	63·361	70·452
36	21·433	28·714	36·000	43·289	50·579	57·870	65·162	72·454
37	22·032	29·513	37·000	44·489	51·979	59·471	66·963	74·455
38	22·631	30·313	38·000	45·689	53·380	61·072	68·764	76·456
39	23·230	31·113	39·000	46·890	54·780	62·672	70·565	78·457
40	23·830	31·913	40·000	48·090	56·181	64·273	72·365	80·458
41	24·429	32·712	41·000	49·290	57·581	65·874	74·166	82·459
42	25·028	33·512	42·000	50·490	58·982	67·474	75·967	84·460
43	25·628	34·312	43·000	51·691	60·382	69·075	77·768	86·461
44	26·227	35·111	44·000	52·891	61·783	70·675	79·569	88·462
45	26·826	35·911	45·000	54·091	63·183	72·276	81·369	90·463
46	27·426	36·711	46·000	55·291	64·583	73·876	83·170	92·464
47	28·025	37·511	47·000	56·491	65·984	75·477	84·971	94·464
48	28·625	38·310	48·000	57·691	67·384	77·077	86·771	96·465
49	29·224	39·110	49·000	58·892	68·784	78·678	88·572	98·466
50	29·824	39·910	50·000	60·092	70·185	80·278	90·372	100·467

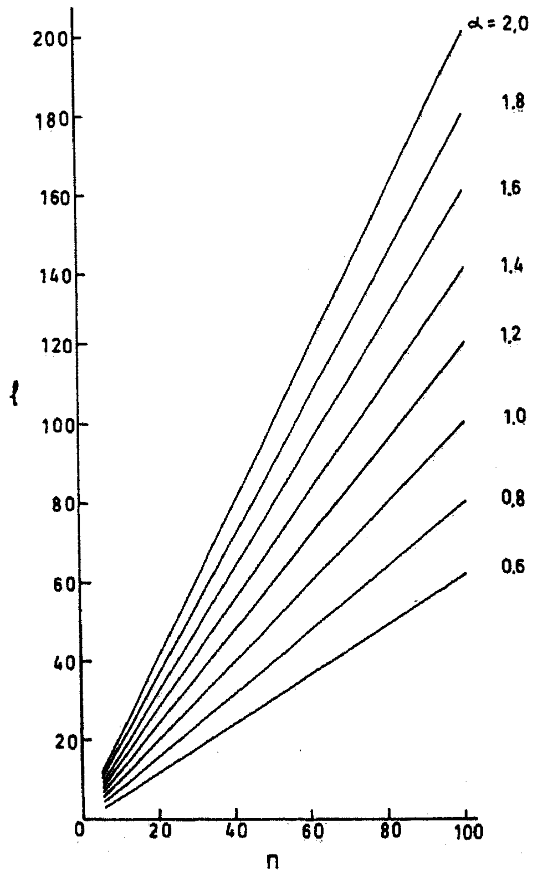

Figure 1. Variation of l with n for $\alpha = 0{\cdot}6$, 0·8, 1·0, 1·2, 1·4, 1·6, 1·8 and 2·0.

values of n for all the values of α considered. Figure 1 exhibits the variation of l with n for different values of α.

The roots of the frequency equation (22) have been given by Eringen and Suhubi (1975, Chapter 8) for integral values of l from 2 to 100. Figure 2 shows the variation of these roots with l (assumed integral). The roots of the frequency equation (22), denoted by F, for non-integral values of l have been obtained by interpolation, using Newton-Gregory forward and backward formulas. For calculating F for given n and α, we first calculate l from equation (10) and then calculate F by interpolation. These roots are given in table 2. Figure 3 shows the variation of F with n for $\alpha = 0{\cdot}6$, 0·8, 1·0, 1·2 and 1·4.

As explained above, for a given n, the root of the frequency equation (22) for the anisotropic case is greater than (less than) the corresponding root for the isotropic case if $\alpha > 1$ ($\alpha < 1$). For the isotropic case the root is denoted by f and for the anisotropic case by F. We have calculated the change in the frequency $|f - F|$ and the percentage change

$$|f - F|/f \times 100$$

on account of the anisotropy for different values of n and for $\alpha = 0{\cdot}6$, 0·8, 1·2, 1·4, 1·6, 1·8 and 2·0. It is found that, for large values of n, the percentage change is nearly equal to

$$|\alpha - 1| \times 100.$$

Table 2. The roots (F) of the frequency equation (22) for different values of n and α.

	α							
n	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
2			2·501	2·971	3·423	3·908	4·366	4·826
3		3·124	3·865	4·595	5·302	6·006	6·709	7·404
4	3·143	4·130	5·095	6·042	6·981	7·911	8·832	9·749
5	3·879	5·085	6·266	7·432	8·585	9·729	10·865	11·995
6	4·595	6·006	7·404	8·781	10·146	11·501	12·847	14.189
7	5·282	6·914	8·520	10·107	11·681	13·244	14·799	16·347
8	5·962	7·807	9·621	11·416	13·197	14·968	16·729	18·484
9	6·639	8·689	10·711	12·712	14·700	16·675	18·643	20·603
10	7·306	9·564	11·792	14·000	16·192	18·373	20·545	22·710
11	7·968	10·433	12·866	15·279	17·677	20·061	22·438	24·806
12	8·626	11·296	13·935	16·552	19·153	21·743	24·322	26·895
13	9·279	12·155	14·999	17·820	20·625	23·418	26·202	28·977
14	9·930	13·010	16·058	19·083	22·093	25·088	28·076	31·054
15	10·577	13·863	17·115	20·343	23·555	26·755	29·944	33·126
16	11·223	14·713	18·168	21·600	25·014	28·418	31·810	35·194
17	11·866	15·560	19·218	22·854	26·471	30·076	33·671	37·258
18	12·507	16·405	20·266	24·104	27·926	31·733	35·531	39·320
19	13·146	17·248	21·313	25·353	29·377	33·387	37·386	41·378
20	13·784	18·089	22·357	26·600	30·826	35·039	39·241	43·434
21	14·421	18·928	23·399	27·845	32·273	36·687	41·092	45·487
22	15·055	19·766	24·439	29·088	33·719	38·336	42·941	47·539
23	15·689	20·603	25·479	30·330	35·163	39·982	44·790	49·588
24	16·322	21·439	26·517	31·570	36·605	41·626	46·636	51·636
25	16·953	22·273	27·554	32·809	38·047	43·269	48·480	53·683
26	17·584	23·106	28·590	34·048	39·486	44·910	50·324	55·727
27	18·214	23·938	29·624	35·284	40·925	46·551	52·166	57·771
28	18·842	24·770	30·658	36·519	42·362	48·190	54·007	59·814
29	19·470	25·601	31·691	37·755	43·799	49·828	55·846	61·854
30	20·098	26·431	32·723	38·989	45·234	51·466	57·685	63·894
31	20·725	27·260	33·754	40·222	46·670	53·102	59·523	65·933
32	21·352	28·088	34·785	41·454	48·103	54·737	61·359	67·971
33	21·977	28·916	35·814	42·685	49·536	56·372	63·195	70·009
34	22·602	29·743	36·843	43·916	50·969	58·006	65·030	72·045
35	23·226	30·569	37·872	45·146	52·400	59·639	66·864	74·080
36	23·849	31·396	38·900	46·376	53·831	61·271	68·697	76·116
37	24·472	32·221	39·927	47·604	55·261	62·902	70·531	78·150
38	25·095	33·046	40·954	48·832	56·691	64·533	72·363	80·182
39	25·718	33·870	41·980	50·060	58·120	66·163	74·194	82·216
40	26·340	34·695	43·005	51·288	59·549	67·793	76·026	84·247
41	26·962	35·518	44·031	52·515	60·977	69·422	77·856	86·279
42	27·583	36·341	45·055	53·741	62·403	71·051	79·684	88·310
43	28·204	37·164	46·080	54·966	63·830	72·679	81·516	90·340
44	28·825	37·987	47·104	56·191	65·258	74·307	83·344	92·370
45	29·445	38·809	48·127	57·417	66·683	75·935	85·172	94·399
46	30·064	39·630	49·150	58·641	68·109	77·562	87·001	96·428
47	30·684	40·451	50·173	59·865	69·534	79·186	88·827	98·456
48	31·304	41·273	51·196	61·089	70·960	80·814	90·655	100·484
49	31·923	42·093	52·218	62·312	72·384	82·440	92·481	102·512
50	32·541	42·913	53·240	63·535	73·808	84·064	94·307	104·066

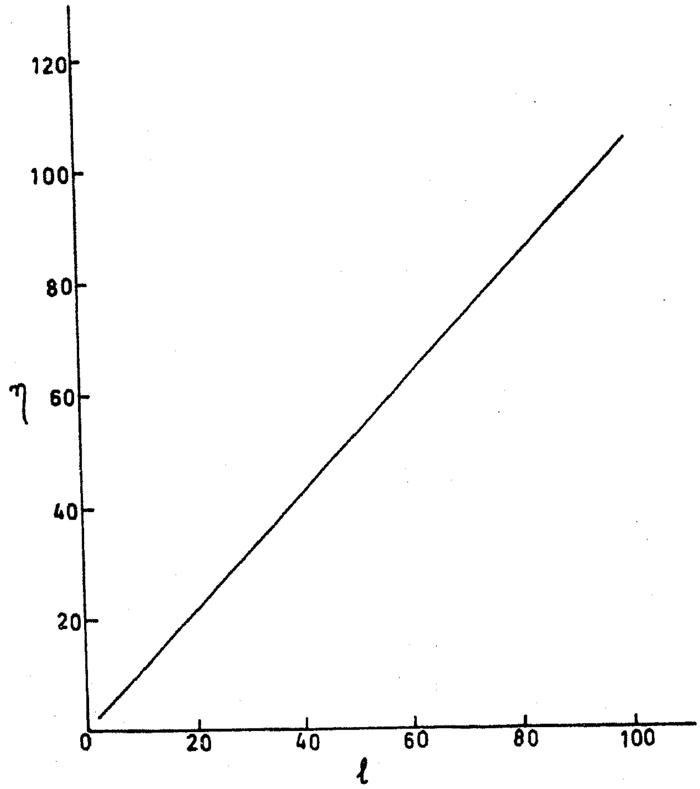

Figure 2. Variation of the roots of the frequency equation (22) with l (assumed integral).

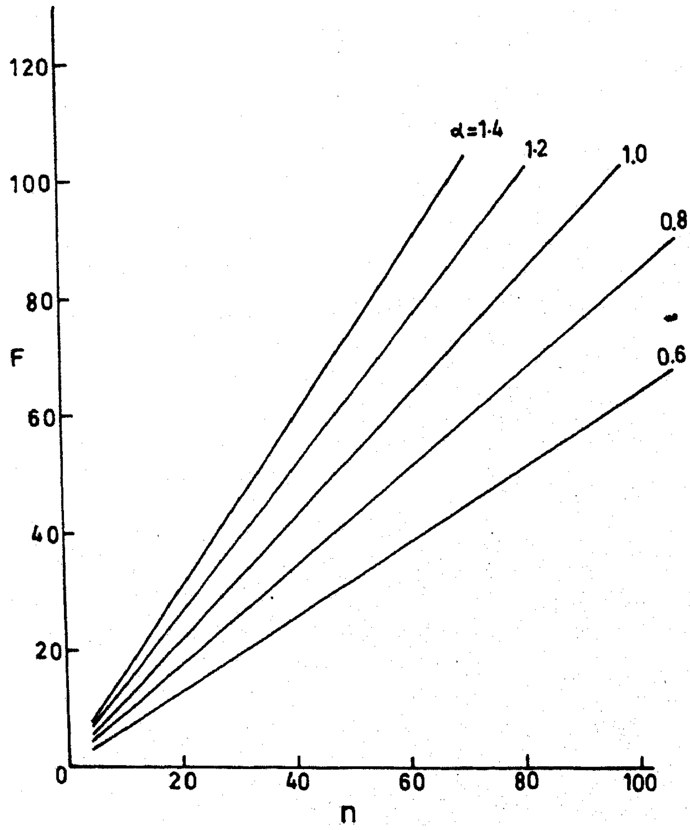

Figure 3. Variation of the roots of the frequency equation (22) with n for $\alpha = 0{\cdot}6$, $0{\cdot}8$, $1{\cdot}0$, $1{\cdot}2$ and $1{\cdot}4$.

4. Conclusions

(i) On account of the anisotropy, the order of oscillation n changes to an apparent order number l, given by (10). For large values of n, $l \approx \alpha\, n$, where α is the ratio of the velocity perpendicular to the radius to the velocity along the radius at any point of the medium.

(ii) The anisotropy has a strong effect on the frequencies of the toroidal oscillation of a homogeneous sphere. For large order oscillations, the percentage change in the frequency is approximately equal to $|\alpha - 1| \times 100$.

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