Consolidation of a poroelastic half-space with anisotropic permeability and compressible constituents by axisymmetric surface loading

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The fully coupled Biot quasi-static theory of linear poroelasticity is used to study the consolidation of a poroelastic half-space caused by axisymmetric surface loads. The fluid and solid constituents of the poroelastic medium are compressible and its permeability in the vertical direction is different from its permeability in the horizontal direction. An analytical solution of the governing equations is obtained by taking the displacements and the pore pressure as the basic state variables and using a combination of the Laplace and Hankel transforms. The problem of an axisymmetric normal load is discussed in detail. An explicit analytical solution is obtained for normal disc loading. Detailed numerical computations reveal that the anisotropy in permeability as well as the compressibilities of the fluid and solid constituents of the poroelastic medium have significant effects on the consolidation of the half-space. The anisotropy in permeability may accelerate the consolidation process and may lead to a dilution in the theoretical prediction of the Mandel–Cryer effect. The compressibility of the solid constituents may also accelerate the consolidation process.

1. Introduction

Biot's theory of linear poroelasticity (Biot 1941, 1956) has been used very extensively to study the consolidation of a homogeneous or layered half-space by surface loads or buried sources (see, e.g., Rudnicki 1986; Pan 1999; Wang and Kuempel 2003; Singh and Rani 2006). However, most of these studies did not take into account the anisotropy in hydraulic permeability. Permeability determines the ability of the porous medium to conduct fluid flow in its pores and, therefore, can be different in different directions. In most cases, the soil deposits are the result of a sedimentation process which produces horizontal stratification. Consequently, permeabilities in the horizontal and vertical directions may differ. It has been found experimentally that the horizontal permeability of soil may be an order of magnitude, or more, greater than the vertical permeability. This degree of hydraulic anisotropy will have significant effect on the consolidation process. As mentioned by Ganbe and Kurashige (2001), the permeability is anisotropic in many oil/gas reservoirs. In some structures, the permeability anisotropy in the horizontal direction to the vertical direction is as much as 100:1. Anisotropy from 2:1 to 3:1 is common, especially in naturally fissured media. It is, therefore, useful to study the effect of the anisotropy in permeability on the consolidation of a poroelastic half-space under an externally applied load due to its possible geophysical or engineering applications.

Booker and Randolph (1984) discussed the consolidation of a soil medium with transversely isotropic deformation and flow properties due to surface loading. Both the pore water and the soil

grains were assumed to be incompressible. Booker and Carter (1986) obtained a closed-form solution for the long-term deformation and pore pressure in a poroelastic half-space caused by withdrawal of pore fluid at a constant rate from a point sink. The soil was assumed to be transversely isotropic with respect to its flow properties. Booker and Carter (1987a) presented a solution for the transient effects of pumping fluid from a point sink embedded in a poroelastic half-space possessing anisotropic permeability and incompressible fluid and solid constituents. In a subsequent study, Booker and Carter (1987b) obtained the corresponding solution for a poroelastic half-space with compressible fluid but incompressible solid constituents.

Chen (2004) studied the consolidation of a multi-layered poroelastic half-space with anisotropic permeability and compressible fluid constituents. The solid constituents were assumed to be incompressible. For such a poroelastic material, the Biot-Willis coefficient $\alpha = 1$. In a subsequent study, Chen (2005) discussed the steady-state response of multilayered poroelastic half-space to a point sink. Both the permeability and the poroelasticity of the medium were assumed to be transversely isotropic, but its fluid and solid constituents were assumed to be incompressible. In a recent study, Singh *et al* (2007) discussed the quasi-static plane strain deformation of a poroelastic half-space with anisotropic permeability and compressible constituents by two-dimensional surface loads. An analytical solution was obtained by using a pure compliance formulation in which the stresses and the pore pressure are taken as the basic state variables. Biot's stress function was used to decouple the governing equations. The problem of normal strip loading was discussed in detail.

The purpose of the present paper is to study the consolidation of a poroelastic half-space by an axisymmetric surface loading. The permeability in the vertical direction may be different from the permeability in the horizontal direction. The fluid and solid constituents of the poroelastic medium are compressible. A stiffness formulation is used, taking the displacements and the pore pressure as the basic state variables. An analytical solution of the governing equations is obtained by taking the Laplace transform with respect to time and the Hankel transform with respect to the space variables. The problem of normal disc loading is discussed in detail. An explicit analytical solution in the Laplace–Hankel transforms domain is obtained. Detailed numerical computations are performed to study the effects of the anisotropy in permeability and the compressibilities of the fluid and solid constituents.

2. Basic equations

Let (r, θ, z) denote the cylindrical polar co-ordinates and (u_r, u_θ, u_z) the corresponding displacement components. For axial symmetry, $\partial/\partial\theta \equiv 0$, and the strain components are:

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z},$$

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad \epsilon_{r\theta} = \epsilon_{\theta z} = 0. \quad (1)$$

A homogenous elastically isotropic poroelastic material with compressible fluid and solid constituents can be characterized by four constitutive constants. Let these constants be: the shear modulus (G), the drained Poisson's ratio (ν), the undrained Poisson's ratio (ν_u) and the Biot–Willis coefficient (α). In terms of the bulk moduli, α and ν_u can be expressed in the form:

$$\alpha = 1 - \frac{K}{K_s} \quad (0 \le \alpha \le 1),$$

$$\nu_u = \frac{3K_u - 2G}{2(3K_u + G)} \quad \left(0 \le \nu \le \nu_u \le \frac{1}{2}\right),$$

where K is the drained bulk modulus, K_u is the undrained bulk modulus and K_s is the bulk modulus of the solid phase of the poroelastic material. For axial symmetry the components of the total stress tensor $(\overrightarrow{\overrightarrow{\sigma}})$, the total strain tensor $(\overrightarrow{\overrightarrow{\epsilon}})$, and the pore pressure (p) are related by the following system of equations (Detournay and Cheng 1993; Wang 2000):

2.1 *Equilibrium equations*

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}$$

$$+ \frac{1}{1-2\nu}\frac{\partial\epsilon}{\partial r} - \frac{\alpha}{G}\frac{\partial p}{\partial r} = 0, \quad (2)$$

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial\epsilon}{\partial z} - \frac{\alpha}{G}\frac{\partial p}{\partial z} = 0, \quad (3)$$

where

$$\epsilon = \text{div } \mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, \quad (4)$$

denotes the dilatation. From equations (2) to (4), we obtain

$$\eta \nabla^2 p = G \nabla^2 \epsilon, \quad (5)$$

where

$$\eta = \frac{1 - 2\nu}{2(1 - \nu)}\alpha, \quad (6)$$

is the poroelastic stress coefficient.

2.2 *Constitutive equations*

$$\sigma_{ij} = 2G\left(\epsilon_{ij} + \frac{\nu}{1 - 2\nu}\epsilon\delta_{ij}\right) - \alpha p\delta_{ij}, \quad (7)$$

where the subscripts i and j can be r, θ or z.

2.3 *Darcy's law*

According to Darcy's law of fluid flow in a poroelastic medium with anisotropic permeability

$$q_r = -\chi_r \partial p/\partial r, \quad q_\theta = 0, \quad q_z = -\chi_z \partial p/\partial z, \quad (8)$$

where $\mathbf{q}$ is the fluid flux and (χ_r, χ_z) is the Darcy conductivity in the (r, z) direction.

2.4 *Continuity equation*

$$\text{div}\mathbf{q} = -\frac{\partial}{\partial t}\left(\alpha\epsilon + \frac{1}{M}p\right), \quad (9)$$

where

$$M = \frac{2G(\nu_u - \nu)}{\alpha^2(1 - 2\nu)(1 - 2\nu_u)}, \quad (10)$$

is the Biot modulus.

2.5 *Fluid diffusion equation*

Combining the continuity equation and Darcy's law, we obtain the fluid diffusion equation in the form

$$\chi_r\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right) + \chi_z\frac{\partial^2 p}{\partial z^2} = \frac{\partial}{\partial t}\left(\alpha\epsilon + \frac{1}{M}p\right). \quad (11)$$

This is a coupled diffusion equation in pore pressure for a poroelastic material possessing anisotropic permeability and compressible fluid and solid constituents. If we put $\alpha = 1$ in equations (7) and (11), we get the corresponding equations of Chen (2004) for axial symmetry. Since $\alpha = 1$ for incompressible solid constituents, the results of Chen (2004) are valid for the particular case of a poroelastic material with incompressible solid constituents.

3. Solution of the governing equations

3.1 *Pore pressure*

Eliminating ϵ from equations (5) and (11), we obtain:

$$\left[c_r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + c_z\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}\right]\nabla^2 p = 0, \quad (12)$$

where

$$(c_r, c_z) = \frac{2G(1 - \nu)(\nu_u - \nu)}{\alpha^2(1 - 2\nu)^2(1 - \nu_u)}(\chi_r, \chi_z), \quad (13)$$

is the hydraulic diffusivity. Taking the Laplace transform of equation (12), we have

$$\left[c_r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + c_z\frac{\partial^2}{\partial z^2} - s\right]\nabla^2 \tilde{p} = 0, \quad (14)$$

where

$$\tilde{p}(r, z, s) = \int_0^\infty p(r, z, t)e^{-st}dt, \quad (15)$$

is the Laplace transform of $p(r, z, t)$.

We define the nth order Hankel transform $\bar{f}_n(k, z, s)$ of a function $\tilde{f}(r, z, s)$ by the relation

$$\bar{f}_n(k, z, s) = \int_0^\infty \tilde{f}(r, z, s)J_n(kr)r\,dr, \quad (16)$$

so that

$$\tilde{f}(r, z, s) = \int_0^\infty \bar{f}_n(k, z, s)J_n(kr)k\,dk. \quad (17)$$

Taking the zeroth-order Hankel transform of equation (14), we obtain

$$\left(\frac{d^2}{dz^2} - m^2\right)\left(\frac{d^2}{dz^2} - k^2\right)\bar{p}_0(k, z, s) = 0, \quad (18)$$

where

$$m^2 = \left(\frac{c_r}{c_z}\right)k^2 + \frac{s}{c_z}. \quad (19)$$

Suitable solutions of equation (18) which are bounded for $z \to \infty$ are of the form

$$\bar{p}_0 = A_1 e^{-mz} + A_2 e^{-kz}, \quad (20)$$

where the arbitrary constants A_1, A_2 may be functions of s and k.

3.2 *Dilatation*

Equations (5) and (20) yield

$$G\left(\frac{d^2}{dz^2} - k^2\right)\bar{\epsilon}_0 = \eta(m^2 - k^2)A_1 e^{-mz}.$$

Solving, we have

$$\bar{\epsilon}_0(k, z, s) = \left(\frac{\eta}{G}\right)A_1 e^{-mz} + A_3 e^{-kz}. \quad (21)$$

Inserting the expressions for $\bar{p}_0$ and $\bar{\epsilon}_0$ from equations (20) and (21) into the Laplace–Hankel transformed equation (11), we have

$$\frac{A_3}{A_2} = \frac{\alpha(1 - 2\nu)(1 - 2\nu_u)}{2G(\nu_u - \nu)}$$

$$\times \left[(c_z - c_r)\frac{(1 - 2\nu)(1 - \nu_u)}{(1 - \nu)(1 - 2\nu_u)}\frac{k^2}{s} - 1\right]. \quad (22)$$

3.3 *Displacements*

Taking the Laplace transform followed by the first order Hankel transform of equation (2), using

equation (20) and solving the resulting ordinary differential equation of the second order we obtain:

$$\bar{u}_{r,1}(k, z, s) = \int_0^\infty \tilde{u}_r(r, z, s)J_1(kr)r\,dr$$

$$= -\frac{\eta k}{G(m^2 - k^2)}A_1 e^{-mz}$$

$$+ \frac{1}{2}\left(\frac{\alpha}{G}A_2 - \frac{A_3}{1 - 2\nu}\right)ze^{-kz} + A_4 e^{-kz}, \quad (23)$$

where A_4 is an arbitrary constant. Similarly, equation (3) yields

$$\bar{u}_{z,0} = \frac{-\eta m}{G(m^2 - k^2)}A_1 e^{-mz}$$

$$+ \frac{1}{2}\left(\frac{\alpha}{G}A_2 - \frac{A_3}{1 - 2\nu}\right)ze^{-kz} + A_5 e^{-kz}. \quad (24)$$

Equation (4) implies

$$\bar{\epsilon}_0 = k\bar{u}_{r,1} + \frac{d}{dz}\bar{u}_{z,0}. \quad (25)$$

From equations (21), (23) and (24), we find

$$k(A_4 - A_5) = -\frac{\alpha}{2G}A_2 + \frac{3 - 4\nu}{2(1 - 2\nu)}A_3. \quad (26)$$

3.4 *Stresses*

Equations (1), (7), (20), and (21) yield

$$\frac{\bar{\sigma}_{rz,1}}{2G} = \frac{\eta mk}{G(m^2 - k^2)}A_1 e^{-mz}$$

$$+ \frac{1}{4}\left(\frac{\alpha}{G}A_2 - \frac{A_3}{1 - 2\nu}\right)(1 - 2kz)e^{-kz}$$

$$- \frac{1}{2}k(A_4 + A_5)e^{-kz}, \quad (27)$$

$$\frac{\bar{\sigma}_{zz,0}}{2G} = \frac{\eta k^2}{G(m^2 - k^2)} A_1 e^{-mz}$$

$$-\frac{1}{2}\left(\frac{\alpha}{G}A_2 - \frac{A_3}{1 - 2\nu}\right)(1 + kz)e^{-kz}$$

$$+\left(\frac{1}{2}A_3 - kA_4\right)e^{-kz}. \tag{28}$$

Using equations (17), (22) and (26), the Laplace transform of the solution can be expressed in the form

$$\tilde{p}(r, z, s) = \int_0^\infty (A_1 e^{-mz} + A_2 e^{-kz}) J_0(kr)k\,dk, \tag{29}$$

$$\tilde{q}_z(r, z, s) = \chi_z \int_0^\infty (mA_1 e^{-mz} + kA_2 e^{-kz}) J_0(kr)k\,dk, \tag{30}$$

$$\tilde{\epsilon}(r, z, s) = \int_0^\infty \left(\frac{\eta}{G}A_1 e^{-mz} + A_3 e^{-kz}\right) J_0(kr)k\,dk, \tag{31}$$

$$\tilde{u}_r(r, z, s) = \int_0^\infty \left[\frac{-\eta k}{G(m^2 - k^2)}A_1 e^{-mz}\right.$$

$$\left. + A_6 z e^{-kz} + A_4 e^{-kz}\right]J_1(kr)k\,dk, \tag{32}$$

$$\tilde{u}_z(r, z, s) = \int_0^\infty \left[\frac{-\eta m}{G(m^2 - k^2)}A_1 e^{-mz}\right.$$

$$\left. + A_6 z e^{-kz} + A_5 e^{-kz}\right]J_0(kr)k\,dk, \tag{33}$$

$$\tilde{\sigma}_{rz}(r, z, s) = 2G \int_0^\infty \left[\frac{\eta mk}{G(m^2 - k^2)}A_1 e^{-mz}\right.$$

$$\left. -A_6 kz e^{-kz} + \frac{1}{2}(A_3 - 2kA_4)e^{-kz}\right]$$

$$\times J_1(kr)k\,dk, \tag{34}$$

$$\tilde{\sigma}_{zz}(r, z, s) = 2G \int_0^\infty \left[\frac{\eta k^2}{G(m^2 - k^2)}A_1 e^{-mz}\right.$$

$$- A_6(1 + kz)e^{-kz}$$

$$\left. +\frac{1}{2}(A_3 - 2kA_4)e^{-kz}\right]J_0(kr)k\,dk, \tag{35}$$

where

$$A_3 = \frac{\eta}{G}\left[1 - (1 - 2\nu)\frac{(1 - \nu_u)}{(\nu_u - \nu)}\frac{s_a}{s}\right]A_2, \tag{36}$$

$$A_5 = A_4 + \frac{\eta}{2Gk}\left[-1 + (3 - 4\nu)\frac{(1 - \nu_u)}{(\nu_u - \nu)}\frac{s_a}{s}\right]A_2, \tag{37}$$

$$A_6 = \frac{1}{2}\left(\frac{\alpha}{G}A_2 - \frac{A_3}{1 - 2\nu}\right) = \frac{\eta}{2G}\left[1 + \frac{(1 - \nu_u)s_a}{(\nu_u - \nu)s}\right]A_2, \tag{38}$$

$$s_a = s + (c_r - c_z)k^2 = c_z(m^2 - k^2). \tag{39}$$

Three arbitrary constants, A_1, A_2 and A_4, appear in the solution given by equations (29) to (35). These constants have to be determined from the boundary conditions.

4. Isotropic permeability

For isotropic permeability, $\chi_r = \chi_z = \chi$, $c_r = c_z = c$, $s_a = s$ and the solution is given by equations (29) to (35) with

$$A_3 = -\frac{\eta(1 - \nu)(1 - 2\nu_u)}{G(\nu_u - \nu)}A_2,$$

$$A_5 = A_4 + \frac{\eta(1 - \nu)(3 - 4\nu_u)}{2Gk(\nu_u - \nu)}A_2,$$

$$A_6 = \frac{\eta(1 - \nu)}{2G(\nu_u - \nu)}A_2,$$

$$m^2 = k^2 + \frac{s}{c}. \tag{40}$$

We have verified that the poroelastic solution for isotropic permeability obtained by us as a particular case of the general solution for anisotropic permeability coincides with the solution given by Wang (2000)[*].

5. Surface loading

Consider a uniform poroelastic half-space $z \geq 0$ with z-axis drawn vertically downwards, into the medium. For prescribed surface loads, boundary conditions are of the form

$$\sigma_{rz} = (\sigma_{rz})_0, \quad \sigma_{zz} = (\sigma_{zz})_0, \quad p = 0, \qquad (41)$$

at $z = 0$, assuming permeable surface. Let

$$(\sigma_{rz})_0 = \int_0^\infty S_0(k,s) J_1(kr) k \, dk, \qquad (42)$$

$$(\sigma_{zz})_0 = \int_0^\infty N_0(k,s) J_0(kr) k \, dk. \qquad (43)$$

Equations (29) and (34)–(38) yield

$$A_1 = -A_2 = (\nu - \nu_u)(m + k)(N_0 - S_0)\frac{s}{\eta\Omega},$$

$$A_3 = [s(\nu_u - \nu) - s_a(1 - \nu_u)(1 - 2\nu)]$$
$$\times (N_0 - S_0)\frac{m + k}{G\Omega},$$

$$A_4 = -\frac{S_0}{2Gk} + [s(\nu_u - \nu)(k^2 - m^2 + 2mk)$$
$$+ s_a(1 - \nu_u)(1 - 2\nu)(m^2 - k^2)]$$
$$\times \frac{(S_0 - N_0)}{2Gk(m - k)\Omega},$$

$$A_5 = -\frac{S_0}{2Gk} + [s(\nu_u - \nu)mk$$
$$- s_a(1 - \nu)(1 - \nu_u)(m^2 - k^2)]\frac{(S_0 - N_0)}{Gk(m - k)\Omega},$$

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[*]The expression for $\tilde{\sigma}_{zr}$ given in equation (9.48) of Wang (2000) is incorrect. The correct expression is obtained by replacing C_0 and D_0 by $\frac{1}{2}C_0$ and $\frac{1}{2}D_0$, respectively, in equation (9.48) of Wang (2000).

$$A_6 = [s(\nu_u - \nu) + s_a(1 - \nu_u)](N_0 - S_0)\frac{m + k}{2G\Omega}, \qquad (44)$$

where

$$\Omega = s(\nu_u - \nu)(k - m) - s_a(1 - \nu_u)(k + m). \quad (45)$$

5.1 *Normal disc loading*

Suppose a total normal force Q_0 is uniformly applied over a circular surface area ($z = 0$, $r \leq a$) of radius a with its centre at the origin. If the surface is permeable and the load is applied in the positive z-direction, the boundary conditions yield

$$p = 0, \quad \sigma_{rz} = 0,$$

$$\sigma_{zz} = \begin{cases} -\frac{Q_0}{\pi a^2}, & r \leq a \\ 0, & r > a \end{cases} \qquad (46)$$

at $z = 0$. This implies

$$S_0 = 0, \quad N_0 = -\frac{Q_0}{\pi}\frac{J_1(ak)}{ak}. \qquad (47)$$

Equations (29) to (39) yield the solution in the transform domain:

$$\tilde{p}(r,z,s) = \frac{(\nu_u - \nu)sQ_0}{\pi a\eta}\int_0^\infty (e^{-mz} - e^{-kz})J_0(kr)$$
$$\times J_1(ka)\frac{(m + k)}{\Omega}dk, \qquad (48)$$

$$\tilde{q}_z(r,z,s) = \frac{(\nu_u - \nu)\chi_z sQ_0}{\pi a\eta}\int_0^\infty (me^{-mz} - ke^{-kz})$$
$$\times J_0(kr)J_1(ka)\frac{(m + k)}{\Omega}dk, \qquad (49)$$

$$\tilde{\epsilon}(r,z,s) = \frac{Q_0}{\pi aG}\int_0^\infty [(\nu_u - \nu)se^{-mz} - \{s(\nu_u - \nu)$$
$$- s_a(1 - 2\nu)(1 - \nu_u)\}e^{-kz}]$$
$$\times J_0(kr)J_1(ka)\frac{(m + k)}{\Omega}dk, \qquad (50)$$

$$\tilde{u}_r(r,z,s) = \frac{Q_0}{\pi aG} \int\limits_0^\infty \left[\frac{(\nu_u - \nu)s}{(k-m)}(ke^{-mz} - me^{-kz}) \right.$$

$$- \frac{1}{2}\{s(\nu_u - \nu) + s_a(1 - \nu_u)\}(m+k)ze^{-kz}$$

$$- \frac{1}{2}\{s(\nu_u - \nu) - s_a(1 - 2\nu)$$

$$\times (1 - \nu_u)\}\frac{(m+k)}{k}e^{-kz} \Bigg]$$

$$\times J_1(kr)J_1(ka)\frac{1}{\Omega}dk, \tag{51}$$

$$\tilde{u}_z(r,z,s) = \frac{Q_0}{\pi aG} \int\limits_0^\infty \left[\frac{(\nu_u - \nu)ms}{k-m}(e^{-mz} - e^{-kz}) \right.$$

$$- \frac{1}{2}\{s(\nu_u - \nu) + s_a(1 - \nu_u)\}(m+k)ze^{-kz}$$

$$- (1-\nu)(1-\nu_u)s_a\left(\frac{m+k}{k}\right)e^{-kz} \Bigg]$$

$$\times J_0(kr)J_1(ka)\frac{1}{\Omega}dk, \tag{52}$$

$$\tilde{\sigma}_{rz}(r,z,s) = \frac{Q_0}{\pi a} \int\limits_0^\infty \left[\frac{2(\nu_u - \nu)ms}{m-k}(e^{-mz} - e^{-kz}) \right.$$

$$+ \{s(\nu_u - \nu) + s_a(1 - \nu_u)\}$$

$$\times (m+k)ze^{-kz} \Bigg] J_1(kr)J_1(ka)\frac{k}{\Omega}dk, \tag{53}$$

$$\tilde{\sigma}_{zz}(r,z,s) = \frac{Q_0}{\pi a} \int\limits_0^\infty \left[\frac{2(\nu_u-\nu)ks}{m-k}(ke^{-mz}-me^{-kz}) \right.$$

$$+ \{s(\nu_u - \nu) + s_a(1 - \nu_u)\}(m+k)$$

$$\times (1+kz)e^{-kz} \Bigg] J_0(kr)J_1(ka)\frac{1}{\Omega}dk. \tag{54}$$

The results for a concentrated normal load Q_0 can be obtained from equations (48) to (54) by

using the limit

$$\lim_{a \to 0} \frac{J_1(ak)}{ak} = \frac{1}{2}. \tag{55}$$

6. Numerical results and discussion

Equations (48) to (54) constitute a solution of the problem of consolidation of a poroelastic half-space with anisotropic permeability and compressible fluid and solid constituents. The consolidation is caused by a normal disc load Q_0 distributed uniformly over the circular region $r \le a$ of the permeable surface $z = 0$ of the half-space. The solution obtained is in the Laplace–Hankel transforms domain. Two integrations are required to be performed to get the solution in the physical domain. Schapery (1962) proposed a very simple and efficient approximate formula for finding the Laplace inversion numerically. According to this formula

$$\phi(t) \approx [s\,\tilde{\phi}(s)]_{s=1/(2t)},$$

where $\tilde{\phi}(s)$ is the Laplace transform of $\phi(t)$. Accuracy of this formula has been demonstrated, amongst others, by Rajapakse and Senjuntichai (1993), Senjuntichai and Rajapakse (1995) and Chau (1996). In view of its computational efficiency and simplicity, we have used Schapery's approximate formula for the Laplace inversion. The Hankel transform inversion has been performed numerically by using the extended Simpson's rule.

We have computed the surface subsidence u_z (also known as settlement) at the centre of the normal disc load and pore pressure at various points on the z-axis (central line). We define the following dimensionless quantities

$$P = \left(\frac{\pi a^2}{Q_0}\right)p, \quad W = \left(\frac{\pi aG}{Q_0}\right)u_z,$$

$$Z = \frac{z}{a}, \quad T = \left(\frac{G\chi_z}{a^2}\right)t, \quad \gamma^2 = \frac{c_r}{c_z} = \frac{\chi_r}{\chi_z}. \tag{56}$$

Therefore, W denotes the dimensionless subsidence, P the dimensionless pore pressure, T the dimensionless time, Z the dimensionless depth below the surface of the half-space and γ^2 the ratio of the horizontal permeability χ_r to the vertical permeability χ_z.
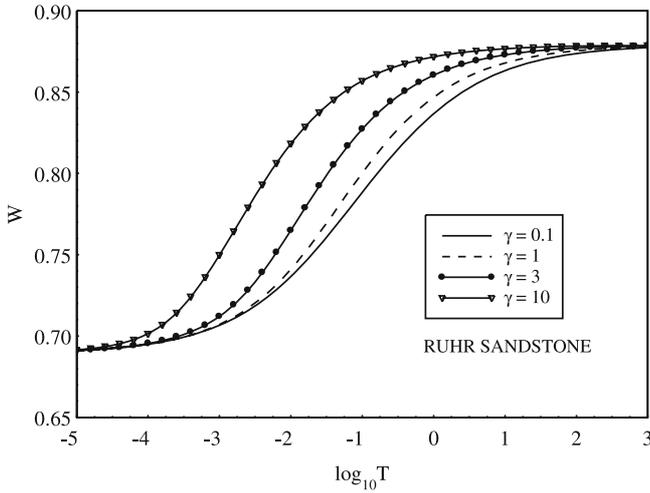
Figure 1. Effect of the permeability anisotropy on the time-settlement. W is the dimensionless displacement at the origin $(r = z = 0)$ in the vertical (down) direction, T is the dimensionless time and $\gamma = (\chi_r/\chi_z)^{1/2}$ is the permeability anisotropy parameter. $\gamma = 1$ for a medium with isotropic permeability. For $\gamma > 1$, the anisotropy in permeability accelerates the consolidation process.

Figure 2. Effect of the compressibility of the solid constituents on the time-settlement for $\nu = 0.25$, $\nu_u = 0.27$, $\gamma = 1$. For a poroelastic model with incompressible solid constituents, $\alpha = 1$. The compressibility of the solid constituents accelerates the consolidation process.

Figure 3. Effect of the compressibility of the fluid constituents on the time-settlement for $\nu = 0.12$, $\alpha = 0.65$, $\gamma = 1$. For a poroelastic model with incompressible fluid constituents, $\nu_u = 0.5$. The influence of the compressibility of the fluid constituents is to increase the initial settlement. The final settlement is not affected.

Figure 1 shows the time-history of the dimensionless subsidence W at the centre of the disc load $(r = z = 0)$ for four values of the permeability anisotropy parameter γ for Ruhr sandstone for which $\nu = 0.12$, $\nu_u = 0.31$ and $\alpha = 0.65$ (Detournay and Cheng 1993; table 4).When $\gamma = 1$, the vertical permeability is equal to the horizontal permeability. We notice that the permeability anisotropy has no effect on the initial settlement or the final settlement. However, if the horizontal permeability χ_r is greater than the vertical permeability χ_z (i.e., if $\gamma > 1$), the permeability anisotropy accelerates the consolidation process.

Figure 2 depicts the effect of the value of the Biot–Willis coefficient α on the time-history of the dimensionless subsidence W for $\gamma = 1$, $\nu = 0.25$ and $\nu_u = 0.27$. For a poroelastic material with incompressible solid constituents, $\alpha = 1$. As α decreases, the compressibility of the solid constituents increases. Figure 2 shows that the compressibility of the solid constituents accelerates the consolidation process.

The influence of the value of the undrained Poisson's ratio ν_u on the time-history of the dimensionless subsidence W is displayed in figure 3 for $\gamma = 1$, $\nu = 0.12$ and $\alpha = 0.65$. It is known that $\nu \leq \nu_u \leq 0.5$. The upper limit $\nu_u = 0.5$ corresponds to a poroelastic material with incompressible fluid constituents. As expected, the final settlement is independent of the value of ν_u. However, the compressibility of the fluid constituents of the poroelastic medium has a strong influence on the consolidation process. The initial settlement for a compressible fluid constituents model is greater

than the initial settlement for the corresponding incompressible fluid constituents model.

Figure 4 shows the effect of the permeability anisotropy on the diffusion of the pore pressure with time. The pore pressure vanishes in the drained state $(T \to \infty)$. Moreover, anisotropy has no effect in the undrained state $(T \to 0)$. From figure 4, we notice that instead of decreasing monotonically with time, the pore pressure rises above the initial undrained value before it decays to zero as $T \to \infty$. This is in accordance with the Mandal–Cryer effect (Cryer 1963; Detournay and Cheng 1993).
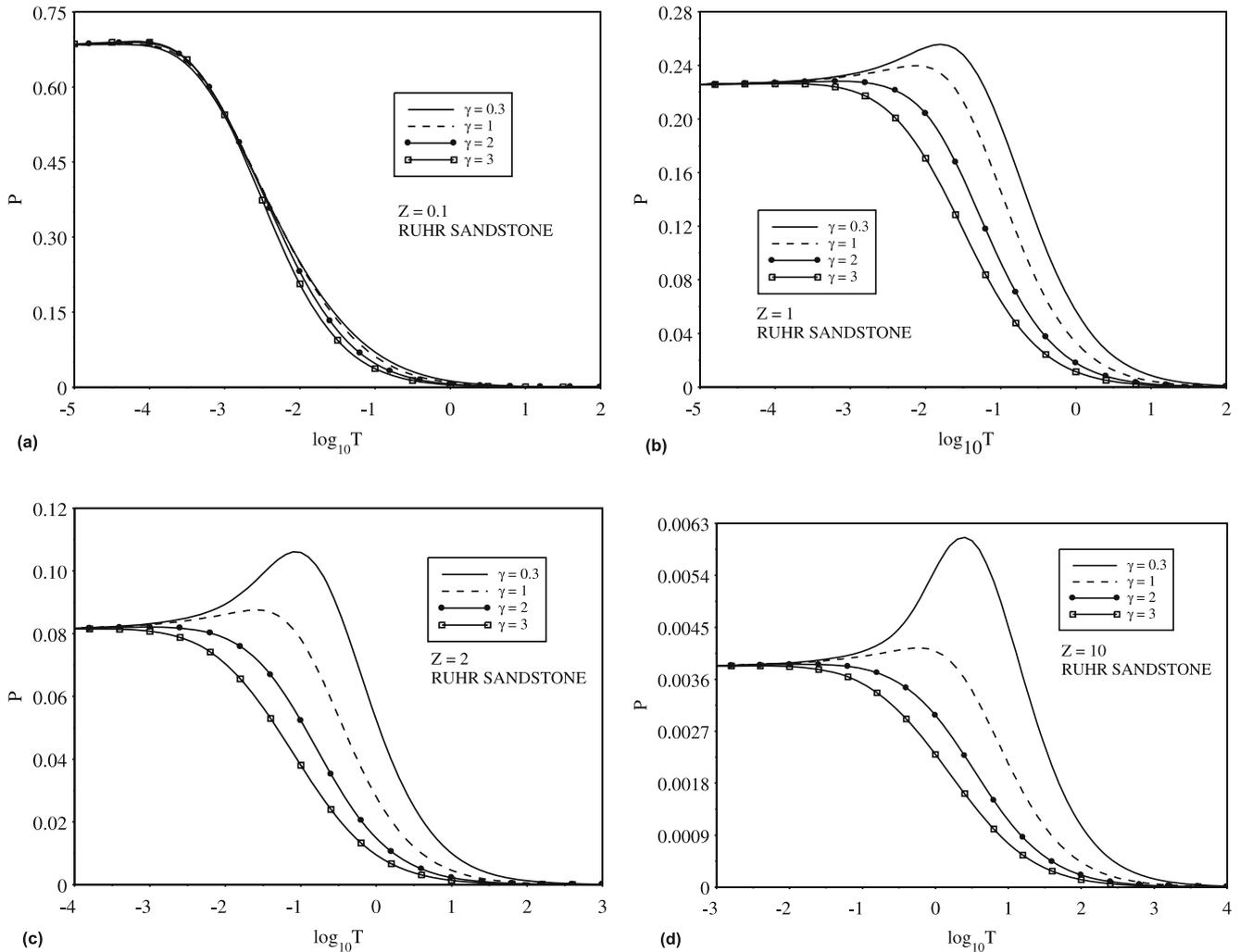
Figure 4. Influence of the value of the permeability anisotropy parameter $\gamma = (\chi_r/\chi_z)^{1/2}$ on the diffusion of the pore pressure P with time T for $r = 0$ and **(a)** $Z = 0.1$, **(b)** $Z = 1$, **(c)** $Z = 2$, **(d)** $Z = 10$. The pore pressure rises above the initial undrained value before it decays to zero as $T \to \infty$ (Mandel–Cryer effect). This effect is more pronounced for smaller values of γ and at greater depths.

This effect is more pronounced at greater depths and for smaller values of the permeability anisotropy parameter $\gamma = (\chi_r/\chi_z)^{1/2}$. Since, in general, $\chi_r > \chi_z$, the theoretical prediction of the Mandel–Cryer effect may get diluted in materials with anisotropic permeability. A similar conclusion was drawn in the two-dimensional plane strain analysis (Singh *et al* 2007).

Figure 5 shows the influence of the compressibility of the solid constituents of the poroelastic medium on the diffusion of the pore pressure with time. The dimensionless pore pressure P at the point $r = 0$, $z = 2a$ on the central line is computed as a function of the dimensionless time T for four values of the Biot–Willis coefficient α for $\nu = 0.25$, $\nu_u = 0.27$ and $\gamma = 0.1, 1, 10$. For a poroelastic medium with incompressible solid constituents, $\alpha = 1$. We observe that the compressibility of the solid constituents of the poroelastic medium has a strong effect in the undrained limit $(T \to 0)$.

The effect of the compressibility of the fluid constituents of the poroelastic medium on the diffusion of the pore pressure with time is displayed in figure 6. The dimensionless pore pressure P at the point $r = 0$, $z = 2a$ on the central line is computed as a function of the dimensionless time T for four values of the undrained Poisson's ratio ν_u for $v = 0.12$, $\alpha = 0.65$ and $\gamma = 0.1, 1, 10$. For a poroelastic medium with incompressible fluid constituents, $\nu_u = 0.5$. We observe that the compressibility of the fluid constituents has a strong effect in the undrained state $(T \to 0)$. Figures 5 and 6 reveal that, in the short term, while the effect of the compressibility of the solid constituents is to increase the pore pressure, the effect of the compressibility of the fluid constituents is to decrease the pore pressure.
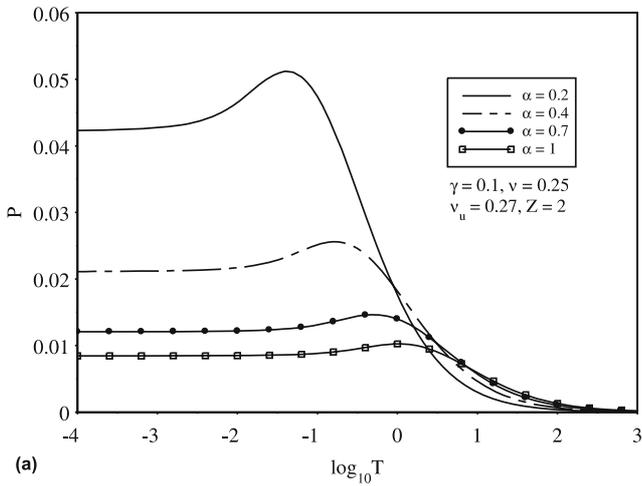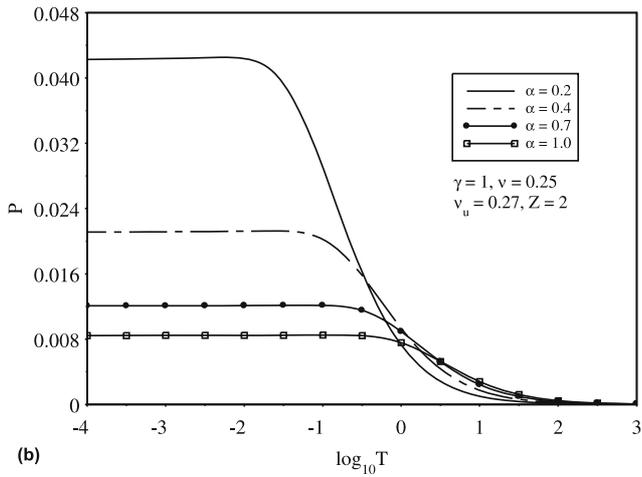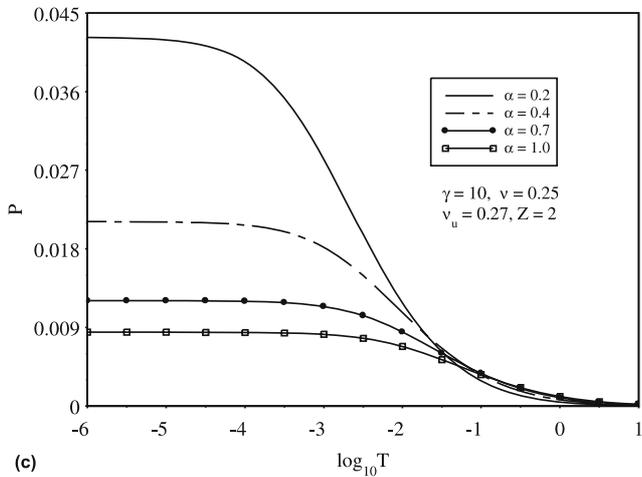
Figure 5. Effect of the value of the Biot–Willis coefficient α on the diffusion of the pore pressure P with time T for $Z = 2$, $\nu = 0.25$, $\nu_u = 0.27$ and for **(a)** $\gamma = 0.1$, **(b)** $\gamma = 1.0$, **(c)** $\gamma = 10$. For a poroelastic model with incompressible solid constituents, $\alpha = 1$. The influence of the compressibility of the solid constituents is to increase the short-term pore pressure.

Figure 6. Effect of the value of the undrained Poisson's ratio ν_u on the diffusion of the pore pressure with time T for $Z = 2$, $\nu = 0.12$, $\alpha = 0.65$ and for **(a)** $\gamma = 0.1$, **(b)** $\gamma = 1.0$, **(c)** $\gamma = 10$. For a poroelastic model with incompressible fluid constituents, $\nu_u = 0.5$. The influence of the compressibility of the fluid constituents is to decrease the pore pressure.

Figure 7 displays the depth profile of the pore pressure for Ruhr sandstone ($\nu = 0.12$, $\nu_u = 0.31$, $\alpha = 0.65$) for three values of the dimensionless time T for $\gamma = 0.1$, 1, 10. Initially, a very large pore pressure develops near the surface resulting in a steep gradient of the pore pressure.
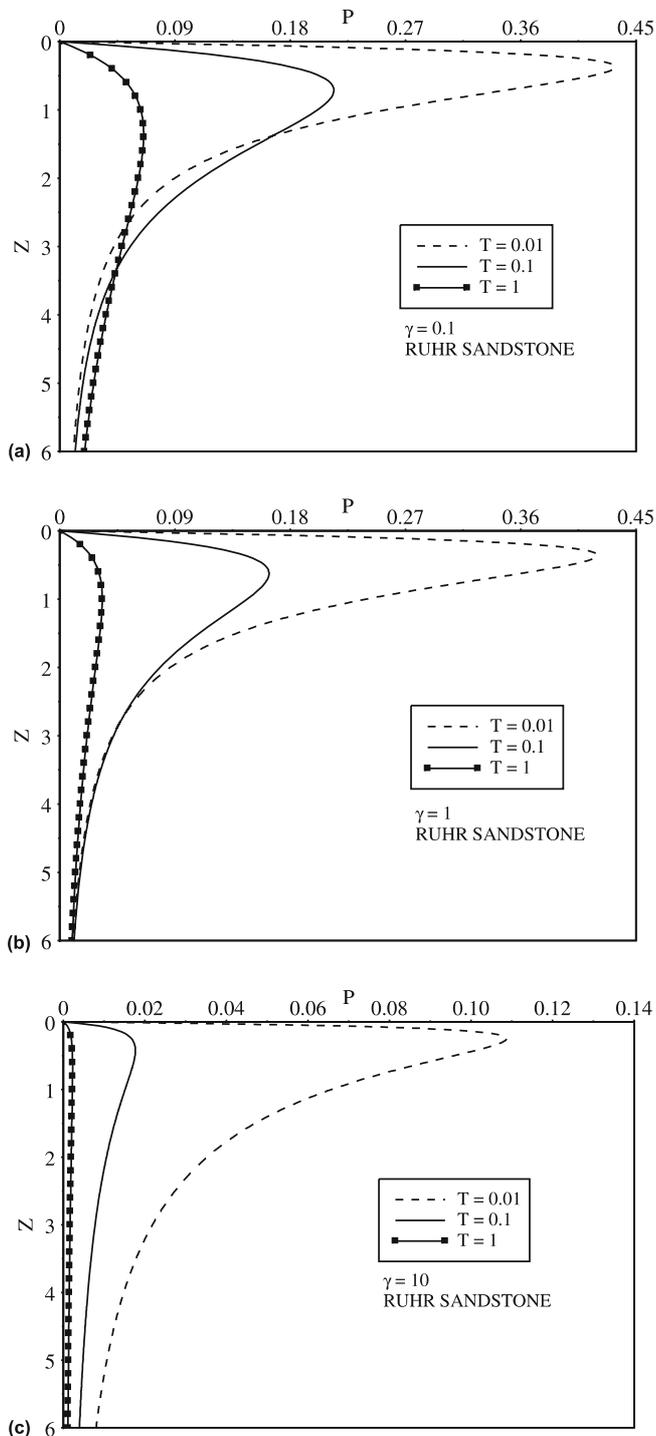
Figure 7. Depth profile of the dimensionless pore pressure P at three time-instants for $r = 0$ and (a) $\gamma = 0.1$, (b) $\gamma = 1.0$, (c) $\gamma = 10$.

7. Conclusions

An analytical solution has been obtained of the governing equations representing the diffusion-deformation of a poroelastic medium possessing anisotropic permeability and compressible fluid and solid constituents. This solution has been used

to study the quasi-static deformation of a half-space by surface loading. As an example, the problem of the consolidation of a uniform half-space caused by normal disc loading has been discussed in detail. The effect of anisotropy in permeability and compressibilities of the fluid and solid constituents have been investigated. The following observations are made in relation to the consolidation of a poroelastic half-space by normal loads:

- The anisotropy in permeability may accelerate the consolidation process. However, it has no effect on the initial and the final settlements.
- The anisotropy in permeability may lead to a dilution in the theoretical prediction of the Mandel–Cryer effect.
- The compressibility of the solid constituents of the poroelastic medium may accelerate the consolidation process. However, it has no influence on the initial and final settlements.
- In the short term, the compressibility of the solid constituents increases the pore pressure.
- The compressibility of the fluid constituents increases the initial settlement. It has no influence on the final settlement.
- The compressibility of the fluid constituents decreases the short-term pore pressure.

The theory developed can be used for modelling the time-settlement of a poroelastic soil medium possessing anisotropic permeability and compressible fluid and solid constituents. This relates to an important class of problems in geotechnical engineering. The theory may also find applications in studies involving oil/gas reservoirs. Because such reservoirs are usually sandwiched between impermeable formations, the oil or water injected from the bore well or from the induced fracture flows mainly in the horizontal direction (Watanabe and Kurashige 1997). Therefore, the permeability in the horizontal direction will be much more than the permeability in the vertical direction.

8. Acknowledgements

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