Plane strain deformation of a multi-layered poroelastic half-space by surface loads

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The Biot linearized quasi-static theory of fluid-infiltrated porous materials is used to formulate the problem of the two-dimensional plane strain deformation of a multi-layered poroelastic half-space by surface loads. The Fourier–Laplace transforms of the stresses, displacements, pore pressure and fluid flux in each homogeneous layer of the multi-layered half-space are expressed in terms of six arbitrary constants. Generalized Thomson–Haskell matrix method is used to obtain the deformation field. Simplified explicit expressions for the elements of the 6×6 propagator matrix for the poroelastic medium are obtained. As an example of the possible applications of the analytical formulation developed, formal solution is given for normal strip loading, normal line loading and shear line loading.

1. Introduction

Poroelasticity is concerned with heterogeneous media consisting of an elastic solid skeleton infiltrated by a diffusing pore fluid. The theory of poroelasticity studies the time-dependent coupling between the deformation of rock and fluid flow within the rock. The study of quasi-static deformation of a fluid-infiltrated porous half-space by surface loads is important for its geophysical and engineering applications. Biot (1941, 1956) developed linearized constitutive and field equations for porous media which have been used very extensively (see, e.g., Rice and Cleary 1976; Bell and Nur 1978; Roeloffs 1988; Kalpna and Chander 1997; Pan 1999). Further references can be found in Wang (2000) and Rudnicki (2001).

Singh (1970) used a generalization of the Thomson–Haskell matrix method to study the deformation of a multi-layered elastic half-space by buried sources. The corresponding two-dimensional problem has been discussed by Singh and Garg (1985). The formulation of Singh (1970) and Singh and Garg (1985) has been used very extensively. Pan (1999) discussed the deformation of a multi-layered poroelastic half-space by buried sources. However, the elements of the propagator matrix given in this paper are complicated functions of the poroelastic parameters used. Wang and Fang (2003) studied the consolidation problem for a multi-layered poroelastic half-space. Only three poroelastic parameters are involved in their formulation as against five poroelastic parameters which define a general, homogeneous, isotropic, poroelastic medium.

The aim of the present analytical study is to formulate the two-dimensional plane strain problem of the quasi-static deformation of a multi-layered poroelastic half-space by surface loads. The mathematical analysis consists of two parts: (i) The derivation of the poroelastic solution of the coupled system, and (ii) the derivation of the explicit expressions for the elements of the 6×6 Haskell propagator matrix. For finding the

Keywords. Multi-layered half-space; plane strain; poroelastic; propagator matrix; surface loads.

J. Earth Syst. Sci. **115**, No. 6, December 2006, pp. 685–694

poroelastic solution, the stresses and the pore pressure are taken as the basic state variables. The displacements are obtained by integrating the coupled constitutive relations. The Fourier–Laplace transforms of the stresses, displacements, pore pressure and fluid flux in a homogeneous isotropic medium are expressed in terms of six arbitrary constants. The expressions for the elements of the propagator matrix obtained are simple functions of the five poroelastic parameters.

Geomechanic problems, such as loading by a reservoir lake or seabed structure that is very extensive in one direction on the Earth's surface, can be solved as two-dimensional plane strain problems. Bell and Nur (1978) used two-dimensional half-space models with surface loading to study the change in strength produced by reservoir-induced pore pressure and stresses for thrust, normal and strike-slip faulting.

2. Basic equations

We consider a two-dimensional approximation in which the displacement components (u_1, u_2, u_3) are independent of the Cartesian coordinate x_2 so that $\partial/\partial x_2 \equiv 0$. Under this assumption, the plane strain problem ($u_2 = 0$) and the antiplane strain problem ($u_1 = u_3 = 0$) get decoupled and can therefore be treated independently. Since the antiplane deformation is not affected by pore pressure, we shall confine our discussion to the plane strain problem only.

A homogeneous, isotropic, poroelastic medium can be characterized by five poroelastic parameters: shear modulus (G), drained Poisson's ratio (ν), undrained Poisson's ratio (ν_u), Skempton's coefficient (B) and hydraulic diffusivity (c). Darcy conductivity (χ) and Biot–Willis coefficient (α) can be expressed in terms of these five parameters.

For plane strain deformation of a poroelastic medium in the x_1x_3-plane, the displacement components in the solid skeleton are of the form

$$u_1 = u_1(x_1, x_3, t),$$

$$u_2 = 0,$$

$$u_3 = u_3(x_1, x_3, t). \tag{1}$$

Let σ_{ij} denote the total stresses in the fluid-infiltrated porous elastic material, ε_{ij} the corresponding strains and p the pore pressure (compression negative). For plane strain, these quantities are related through the following coupled system of equations (Rice and Cleary 1976; Roeloffs 1988) in which the total stresses σ_{ij} and the pore pressure p are taken as the basic state variables.

2.1 *Equilibrium equations*

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{13}}{\partial x_3} = 0, \qquad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{33}}{\partial x_3} = 0. \tag{2}$$

2.2 *Compatibility equation*

$$\nabla^2(\sigma_{11} + \sigma_{33} + 2\eta p) = 0, \tag{3}$$

where

$$\eta = \frac{1-2\nu}{2(1-\nu)}\alpha \tag{4}$$

is the poroelastic stress coefficient.

2.3 *Pore fluid mass conservation equation*

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\left[\sigma_{11} + \sigma_{33} + \frac{3}{(1+\nu_u)B}p\right] = 0. \tag{5}$$

2.4 *Constitutive equations*

$$2G\varepsilon_{11} = (1-\nu)\sigma_{11} - \nu\sigma_{33} + \alpha_0 p, \tag{6}$$

$$2G\varepsilon_{33} = (1-\nu)\sigma_{33} - \nu\sigma_{11} + \alpha_0 p, \tag{7}$$

$$2G\varepsilon_{13} = \sigma_{13}, \tag{8}$$

where

$$\alpha_0 = (1-2\nu)\alpha. \tag{9}$$

Further,

$$\varepsilon_{21} = \varepsilon_{22} = \varepsilon_{23} = 0, \tag{10}$$

$$\sigma_{21} = \sigma_{23} = 0,$$

$$\sigma_{22} = \nu(\sigma_{11} + \sigma_{33}) - \alpha_0 p. \tag{11}$$

The coupled system of equations in (2) to (5) can be solved in terms of Biot's stress function (Wang 2000). We put

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_3^2}, \qquad \sigma_{33} = \frac{\partial^2 F}{\partial x_1^2},$$

$$\sigma_{13} = -\frac{\partial^2 F}{\partial x_1 \partial x_3}. \tag{12}$$

The equilibrium equations in (2) are then identically satisfied. Equations (3), (5) and (12) yield

$$\nabla^2(\nabla^2 F + 2\eta p) = 0, \tag{13}$$

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\left[\nabla^2 F + \frac{3}{(1+\nu_u)B}p\right] = 0. \tag{14}$$

Eliminating F and p in turn, equations (13) and (14) lead us to the following decoupled equations

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^2 p = 0, \tag{15}$$

$$\left(c\nabla^2 - \frac{\partial}{\partial t}\right)\nabla^4 F = 0. \tag{16}$$

The general solution of equation (15) may be expressed in the form

$$p = p_1 + p_2, \tag{17}$$

where

$$c\nabla^2 p_1 = \frac{\partial p_1}{\partial t}, \tag{18}$$

$$\nabla^2 p_2 = 0. \tag{19}$$

Similarly, the general solution of equation (16) may be expressed in the form

$$F = F_1 + F_2, \tag{20}$$

where

$$c\nabla^2 F_1 = \frac{\partial F_1}{\partial t}, \tag{21}$$

$$\nabla^4 F_2 = 0. \tag{22}$$

Taking the Laplace transform of equations (18), (19), (21) and (22), we have

$$\nabla^2 \bar{p}_1 - \frac{s}{c}\bar{p}_1 = 0, \tag{23}$$

$$\nabla^2 \bar{p}_2 = 0, \tag{24}$$

$$\nabla^2 \bar{F}_1 - \frac{s}{c}\bar{F}_1 = 0, \tag{25}$$

$$\nabla^4 \bar{F}_2 = 0, \tag{26}$$

where, for example,

$$\bar{p}_1(x_1, x_2, s) = \int_0^\infty p_1(x_1,\, x_3,\, t)e^{-st}dt \tag{27}$$

is the Laplace transform of $p_1(x_1,\, x_3,\, t)$. In the following, we shall omit the overbar from the Laplace transform of a function. Thus, for example, $\bar{p}_1(x_1,\, x_3,\, s)$ can be written as $p_1(x_1,\, x_3,\, s)$.

For plane parallel boundaries of the form $x_3 =$ const., suitable solutions of equations (23) to (26) are

$$p_1 = \int_0^\infty (A_1 e^{-mz} + C_1 e^{mz}) \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk, \tag{28}$$

$$p_2 = \int_0^\infty (A_2 e^{-kz} + C_2 e^{kz}) \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk, \tag{29}$$

$$F_1 = \int_0^\infty (B_1 e^{-mz} + D_1 e^{mz}) \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk, \tag{30}$$

$$F_2 = \int_0^\infty [(B_2 + B_3kz)e^{-kz} + (D_2 + D_3kz)e^{kz}] \times \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk, \tag{31}$$

where A_1, C_1, etc. may be functions of k, $x = x_1$, $z = x_3$ and

$$m = (k^2 + s/c)^{1/2}. \tag{32}$$

Using (17), (20), (24) to (31), equation (13) yields

$$A_1 = -\left(\frac{s}{2c\eta}\right)B_1, \qquad C_1 = -\left(\frac{s}{2c\eta}\right)D_1. \tag{33}$$

Similarly, equation (14) implies

$$A_2 = \frac{2}{3}(1+\nu_u)Bk^2B_3,$$

$$C_2 = -\frac{2}{3}(1+\nu_u)Bk^2D_3. \tag{34}$$

From equations (17), (28) and (29), we have

$$p = \int_0^\infty (A_1e^{-mz} + C_1e^{mz} + A_2e^{-kz} + C_2e^{kz}) \times \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk. \tag{35}$$

The fluid flux in the z-direction, q, is given by

$$q = -\frac{\chi\partial p}{\partial z} = \chi\int_0^\infty [m(A_1e^{-mz} - C_1e^{mz}) + k(A_2e^{-kz} - C_2e^{kz})] \begin{pmatrix}\sin kx\\ \cos kx\end{pmatrix} dk. \tag{36}$$

Using (33) and (34), equations (35) and (36) become

$$p = -\int_0^\infty \left[\frac{s}{2c\eta}(B_1 e^{-mz} + D_1 e^{mz}) + \xi k^2(-B_3 e^{-kz} + D_3 e^{kz})\right] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk, \quad (37)$$

$$q = \chi \int_0^\infty \left[\frac{ms}{2c\eta}(-B_1 e^{-mz} + D_1 e^{mz}) + \xi k^3(B_3 e^{-kz} + D_3 e^{kz})\right] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk, \quad (38)$$

where

$$\xi = \frac{2}{3}(1+\nu_u)B. \quad (39)$$

From equations (12), (20), (30) and (31), we find

$$\sigma_{11} = \int_0^\infty [m^2(B_1 e^{-mz} + D_1 e^{mz}) + k^2(B_2 e^{-kz} + D_2 e^{kz}) + k^2\{(kz-2)B_3 e^{-kz} + (kz+2)D_3 e^{kz}\}] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk, \quad (40)$$

$$\sigma_{33} = -\int_0^\infty [B_1 e^{-mz} + D_1 e^{mz} + (B_2 + B_3 kz)e^{-kz} + (D_2 + D_3 kz)e^{kz}] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} k^2 dk, \quad (41)$$

$$\sigma_{13} = \int_0^\infty [m(B_1 e^{-mz} - D_1 e^{mz}) + k(B_2 e^{-kz} - D_2 e^{kz}) + k\{B_3(kz-1)e^{-kz} - D_3(kz+1)e^{kz}\}] \begin{pmatrix} \cos kx \\ -\sin kx \end{pmatrix} kdk. \quad (42)$$

Corresponding to the stresses given by equations (40) to (42), the displacements are found by integrating the coupled constitutive relations given in equations (6) to (8). We find (Singh and Garg 1985)

$$2Gu_1 = -\int_0^\infty [B_1 e^{-mz} + D_1 e^{mz} + B_2 e^{-kz} + D_2 e^{kz} + B_3(2\nu_u - 2 + kz)e^{-kz} + D_3(-2\nu_u + 2 + kz)e^{kz}] \begin{pmatrix} \cos kx \\ -\sin kx \end{pmatrix} kdk,$$

$$2Gu_3 = \int_0^\infty [m(B_1 e^{-mz} - D_1 e^{mz}) + k(B_2 e^{-kz} - D_2 e^{kz}) + B_3(1 - 2\nu_u + kz)ke^{-kz} + D_3(1 - 2\nu_u - kz)ke^{kz}] \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk. \quad (43)$$

Equations (37), (38) and (41) to (43) may be written in the form

$$p = \int_0^\infty P \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk, \quad (44a)$$

$$q = \int_0^\infty T \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk, \quad (44b)$$

$$\sigma_{33} = \int_0^\infty N \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} kdk, \quad (45a)$$

$$\sigma_{13} = \int_0^\infty S \begin{pmatrix} \cos kx \\ -\sin kx \end{pmatrix} kdk, \quad (45b)$$

$$u_1 = \int_0^\infty U \begin{pmatrix} \cos kx \\ -\sin kx \end{pmatrix} dk, \quad (46a)$$

$$u_3 = \int_0^\infty W \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix} dk. \quad (46b)$$

The functions $P(k, z, s)$, etc. are given by the matrix relation

$$\mathbf{A}(z) = \mathbf{Z}(z)\mathbf{E}(z)\mathbf{K}, \quad (47)$$

where **A, K** are the column vectors

$$\mathbf{A}(z) = [U, W, S, N, P, T]^T,$$

$$\mathbf{K} = [B_1, D_1, B_2, D_2, B_3, D_3]^T, \quad (48)$$

and **E** is the diagonal matrix

$$\mathbf{E}(z) = \mathrm{diag}(e^{-mz}, e^{mz}, e^{-kz}, e^{kz}, e^{-kz}, e^{kz}). \quad (49)$$

The elements of the 6×6 matrix $\mathbf{Z}(z)$ are

$$(11) = (12) = (13) = (14) = -\frac{k}{2G},$$

$$(15) = -\left(\frac{k}{2G}\right)(2\nu_u - 2 + kz),$$

$$(16) = -\left(\frac{k}{2G}\right)(-2\nu_u + 2 + kz),$$

$$(21) = -(22) = \frac{m}{2G},$$

$$(23) = -(24) = \frac{k}{2G},$$

$$(25) = \left(\frac{k}{2G}\right)(1 - 2\nu_u + kz),$$

$$(26) = \left(\frac{k}{2G}\right)(1 - 2\nu_u - kz),$$

$$(31) = -(32) = m,$$

$$(33) = -(34) = k,$$

$$(35) = k(kz - 1),\ (36) = -k(kz + 1),$$

$$(41) = (42) = (43) = (44) = -k,$$

$$(45) = (46) = -k^2 z,$$

$$(51) = (52) = -s/2c\eta,$$

$$(53) = (54) = 0,$$

$$(55) = -(56) = \xi k^2,$$

$$(61) = -(62) = -\frac{ms\chi}{2c\eta},$$

$$(63) = (64) = 0,$$

$$(65) = (66) = \xi\chi k^3. \qquad (50)$$

The 4×4 matrix obtained on deleting the first and the second columns and the fifth and the sixth rows of the matrix $\mathbf{Z}$ for a poroelastic medium given by equation (50) coincides with the corresponding matrix for an elastic medium given by Singh and Garg (1985) with the drained Poisson's ratio ν replaced by the undrained Poisson's ratio ν_u.

3. Multi-layered half-space

We consider a semi-infinite poroelastic medium consisting of $j-1$ parallel, homogeneous, isotropic layers lying over a homogeneous, isotropic half-space. The layers are numbered serially, the top-most layer being layer 1 and the half-space, layer j. The origin of the Cartesian system (x, y, z) is placed at the surface with the z-axis drawn vertically downwards into the medium. The nth layer is of thickness d_n, the corresponding poroelastic parameters are denoted by $G_n, \nu_n, \nu_{un}, B_n, c_n$ and is bounded by the interfaces $z = z_{n-1}, z_n$. Evidently, $d_n = z_n - z_{n-1}, z_0 = 0$ and $z_{j-1} = H$, where H is the depth of the last interface. From equation (47), we have

$$\mathbf{A}_n(z) = [U_n,\ W_n,\ S_n,\ N_n,\ P_n,\ T_n]^T$$
$$= \mathbf{Z}_n(z)\mathbf{E}_n(z)\mathbf{K}_n, \quad (z_{n-1} \le z \le z_n), \quad (51)$$

$$\mathbf{K}_n = [B_{1n},\ D_{1n},\ B_{2n},\ D_{2n},\ B_{3n},\ D_{3n}]^T.$$

Equation (51) yields

$$\mathbf{A}_n(z_{n-1}) = \mathbf{a}_n \mathbf{A}_n(z_n), \qquad (52)$$

where $\mathbf{a}_n$ is the propagator matrix

$$\mathbf{a}_n(z_{n-1}, z_n) = \mathbf{Z}_n(z_{n-1})\mathbf{E}_n(z_{n-1})\mathbf{E}_n^{-1}(z_n)\mathbf{Z}_n^{-1}(z_n).$$

To evaluate $\mathbf{a}_n$, we temporarily shift the origin to the interface $z = z_n$. This yields

$$\mathbf{a}_n(d_n) = \mathbf{Z}_n(-d_n)\mathbf{E}_n(-d_n)\mathbf{Z}_n^{-1}(0). \qquad (53)$$

The continuity of u_1, u_3, σ_{13}, σ_{33}, p and q at the interface $z = z_{n-1}$ implies

$$\mathbf{A}_{n-1}(z_{n-1}) = \mathbf{A}_n(z_{n-1}).$$

Equation (52) can now be written in the form

$$\mathbf{A}_{n-1}(z_{n-1}) = \mathbf{a}_n \mathbf{A}_n(z_n). \qquad (54)$$

The propagator matrix $\mathbf{a}_n$ is a function of the five poroelastic parameters of the nth layer and its thickness $d_n = z_n - z_{n-1}$. It does not depend upon z_{n-1} or z_n *per se*. The elements of the matrix $\mathbf{Z}^{-1}(0)$ are given in Appendix A and the elements of the propagator matrix $\mathbf{a}_n(d_n)$ are given in Appendix B.

A repeated use of the relation (54) yields

$$\mathbf{A}_1(0) = \mathbf{V}\mathbf{A}_j(H), \qquad (55)$$

where

$$\mathbf{V} = \mathbf{a}_1\mathbf{a}_2 \cdots \mathbf{a}_{j-1}. \qquad (56)$$

In the half-space (layer j), we apply the finiteness condition that the displacements, stresses and pore pressure remain finite as $z \to \infty$. This requires

$$D_{1j} = D_{2j} = D_{3j} = 0, \tag{57}$$

so that

$$\mathbf{K}_j = [B_{1j},\, 0,\, B_{2j},\, 0,\, B_{3j},\, 0]^T. \tag{58}$$

From equations (51), (55) and (58), we have

$$[U_0,\, W_0,\, S_0,\, N_0,\, P_0,\, T_0]^T = \mathbf{J}[B_{1j},\, 0,\, B_{2j},\, 0,\, B_{3j},\, 0]^T, \tag{59}$$

where

$$\mathbf{J} = \mathbf{V}\mathbf{Z}_j(H)\mathbf{E}_j(H). \tag{60}$$

4. Surface loads

For prescribed surface loads, S_0, N_0 are known. Moreover, $p = 0$ ($P_0 = 0$) at the surface $z = 0$. Equation (59) gives

$$\begin{aligned} S_0 &= J_{31}B_{1j} + J_{33}B_{2j} + J_{35}B_{3j},\\ N_0 &= J_{41}B_{1j} + J_{43}B_{2j} + J_{45}B_{3j},\\ 0 &= J_{51}B_{1j} + J_{53}B_{2j} + J_{55}B_{3j}. \end{aligned} \tag{61}$$

These equations can be solved for B_{1j}, B_{2j}, B_{3j}. We find

$$B_{1j} = \frac{\Delta_1}{\Delta}, \qquad B_{2j} = \frac{\Delta_2}{\Delta}, \qquad B_{3j} = \frac{\Delta_3}{\Delta}, \tag{62}$$

where Δ, Δ_1, Δ_2, Δ_3 are the determinants

$$\begin{aligned} \Delta &= |C_1C_2C_3|, \qquad \Delta_1 = |C_4C_2C_3|,\\ \Delta_2 &= |C_1C_4C_3|, \qquad \Delta_3 = |C_1C_2C_4|, \end{aligned} \tag{63}$$

and C_1, C_2, C_3, C_4 are the columns of the 3×4 matrix

$$\begin{bmatrix} J_{31} & J_{33} & J_{35} & S_0\\ J_{41} & J_{43} & J_{45} & N_0\\ J_{51} & J_{53} & J_{55} & 0 \end{bmatrix}. \tag{64}$$

Using (62), equation (59) yields

$$\begin{aligned} U_0 &= \frac{1}{\Delta}(J_{11}\Delta_1 + J_{13}\Delta_2 + J_{15}\Delta_3),\\ W_0 &= \frac{1}{\Delta}(J_{21}\Delta_1 + J_{23}\Delta_2 + J_{25}\Delta_3). \end{aligned} \tag{65}$$

Inserting these expressions for U_0, W_0 in equation (46), we can find the surface displacements $u_1(0)$, $u_3(0)$.

4.1 *Normal strip loading*

Consider a strip $-L \le x \le L$ of infinite length in the y-direction on the surface of the multi-layered half-space. Let a normal load σ_0 per unit length acting in the positive z-direction be uniformly distributed over this strip. The boundary conditions at the surface $z = 0$ are

$$\begin{aligned} \sigma_{13} &= 0,\\ \sigma_{33} &= 0 \quad \text{for } |x| > L\\ &= -\frac{\sigma_0}{2L} \quad \text{for } |x| < L. \end{aligned} \tag{66}$$

We may write

$$\begin{aligned} \sigma_{13} &= 0,\\ \sigma_{33} &= -\frac{\sigma_0}{\pi}\int_0^\infty \frac{\sin kL}{kL}\cos kx\; dk, \end{aligned} \tag{67}$$

for $z = 0$. Comparing equations (45) and (67), we have

$$S_0 = 0, \qquad N_0 = -\frac{\sigma_0}{\pi k}\left(\frac{\sin kL}{kL}\right), \tag{68}$$

and the lower solution in equations (44) to (46) is to be chosen. From equations (46) and (65), we find

$$u_1(0) = -\int_0^\infty \frac{1}{\Delta}(J_{11}\Delta_1 + J_{13}\Delta_2 + J_{15}\Delta_3) \times \sin kx\; dk, \tag{69}$$

$$u_3(0) = \int_0^\infty \frac{1}{\Delta}(J_{21}\Delta_1 + J_{23}\Delta_2 + J_{25}\Delta_3)\cos kx\; dk, \tag{70}$$

where Δ, Δ_1, Δ_2, Δ_3 are given by equations (63) and (64) with S_0, N_0 of equation (68).

4.2 *Normal line loading*

Taking the limit as $L \to 0$ with σ_0 fixed, equation (67) becomes

$$\begin{aligned} \sigma_{13} &= 0,\\ \sigma_{33} &= -\frac{\sigma_0}{\pi}\int_0^\infty \cos kx\; dk = -\sigma_0\delta(x), \end{aligned} \tag{71}$$

at $z = 0$. This represents a line load σ_0 per unit length acting at the origin to the surface $z = 0$ in the positive z-direction. Equation (68) now becomes

$$S_0 = 0, \qquad N_0 = -\frac{\sigma_0}{\pi k}, \tag{72}$$

and the lower solution is to be chosen.

4.3 *Shear line loading*

Suppose a shear line loading τ_0 per unit length is applied at the origin to the surface $z = 0$ in the positive x-direction. The boundary conditions at the surface $z = 0$ yield

$$\sigma_{33} = 0,$$

$$\sigma_{13} = -\tau_0\delta(x) = -\frac{\tau_0}{\pi}\int_0^\infty \cos kx\; dk. \tag{73}$$

Comparing equations (45) and (73), we find

$$N_0 = 0, \qquad S_0 = -\frac{\tau_0}{\pi k}, \tag{74}$$

and the upper solution is to be chosen. Equations (46) and (65) yield

$$u_1(0) = \int_0^\infty \frac{1}{\Delta}(J_{11}\Delta_1 + J_{13}\Delta_2 + J_{15}\Delta_3)\cos kx\; dk, \tag{75}$$

$$u_3(0) = \int_0^\infty \frac{1}{\Delta}(J_{21}\Delta_1 + J_{23}\Delta_2 + J_{25}\Delta_3)\sin kx\; dk, \tag{76}$$

where Δ, Δ_1, Δ_2, Δ_3 are given by equations (63) and (64) with S_0, N_0 of equation (74).

4.4 *Uniform half-space*

In the case of a uniform half-space $\mathbf{J} = \mathbf{Z}(0)$. Equations (50), (69), (70) and (72) yield the following expressions for the surface displacements due to a normal line load acting on the surface of a homogeneous poroelastic half-space

$$u_1(0) = \frac{\sigma_0}{2\pi G}\int_0^\infty \frac{1}{\Omega} \times \left[k - m + (1 - 2\nu_u)\frac{s}{k\gamma_1}\right]\sin kx\; dk, \tag{77}$$

$$u_3(0) = -\frac{(1-\nu_u)\sigma_0 s}{\pi G\gamma_1}\int_0^\infty \frac{1}{k\Omega}\cos kx\; dk, \tag{78}$$

where

$$\Omega = k^2 - mk - \frac{s}{\gamma_1},$$

$$\gamma_1 = -2c\eta\xi = 2c\frac{(\nu - \nu_u)}{(1-\nu)}. \tag{79}$$

Similarly, for a shear line load

$$u_1(0) = -\frac{(1-\nu_u)\tau_0 s}{\pi G\gamma_1}\int_0^\infty \frac{1}{k\Omega}\cos kx\; dk, \tag{80}$$

$$u_3(0) = -\frac{\tau_0}{2\pi G}\int_0^\infty \frac{1}{\Omega}\left[k - m + (1 - 2\nu_u)\frac{s}{k\gamma_1}\right] \times \sin kx\; dk. \tag{81}$$

The solutions for a line load can be used as Green's functions for finding the response for general surface loading.

5. Conclusions

We have formulated the plane strain problem of the quasi-static deformation of a poroelastic half-space by surface loads. Poroelastic solution of the coupled system has been obtained using the Biot stress function. Explicit expressions for the Laplace–Fourier transforms of the displacements, stresses, pore pressure and fluid flux in a homogeneous isotropic medium have been expressed in terms of six arbitrary constants. These expressions can be used for studying the deformation of a uniform half-space or two half-spaces in contact by surface loads or buried sources. Simplified explicit expressions for the elements of the propagator matrix have been obtained for the poroelastic case. This is a generalization of the propagator matrix obtained by Singh (1970) and Singh and Garg (1985) for the elastic case. The propagator matrix can be used for studying the quasi-static plane strain deformation of a multi-layered poroelastic half-space by surface loads or buried sources. We have found the solution in the Fourier–Laplace domain. Two integrations are required to be performed to get the solution in the space–time domain. These integrations have to be performed numerically.

List of notations

- B Skempton's coefficient
- c hydraulic diffusivity
- d layer thickness
- F Biot's stress function
- G shear modulus

m	$= (k^2 + s/c)^{1/2}$
p	pore pressure (compression negative)
q	fluid flux in the z-direction
s	Laplace transform variable
t	time
u_i	displacement components
α	Biot–Willis coefficient
	$= \dfrac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}$
α_0	$= \dfrac{3(\nu_u - \nu)}{B(1+\nu_u)} = (1-2\nu)\alpha$
β_1	$= \dfrac{1}{2(1-\nu_u)}$
β_2	$= \dfrac{c(\nu - \nu_u)}{2(1-\nu)(1-\nu_u)} = \dfrac{1}{2}\gamma_1\beta_1$
β_3	$= \dfrac{1-2\nu_u}{4(1-\nu_u)} = \dfrac{1}{2}(1-2\nu_u)\beta_1 = \dfrac{1}{2}(1-\beta_1)$
β_4	$= \dfrac{BG(1+\nu_u)}{3(1-\nu_u)} = \dfrac{c\eta}{\chi} = G\xi\beta_1$
γ_1	$= \dfrac{2c(\nu - \nu_u)}{(1-\nu)} = -2c\eta\xi$
$\delta(x)$	Dirac delta function
ϵ_{ij}	strain components
η	$= \dfrac{3(\nu_u - \nu)}{2B(1-\nu)(1+\nu_u)} = \dfrac{1-2\nu}{2(1-\nu)}\alpha$
ν	drained Poisson's ratio
ν_u	undrained Poisson's ratio
ξ	$= \dfrac{2}{3}B(1+\nu_u)$
σ_{ij}	stress components
χ	Darcy conductivity
	$= \dfrac{9c(1-\nu_u)(\nu_u - \nu)}{2GB^2(1-\nu)(1+\nu_u)^2}$

Acknowledgement

One of the authors (SJS) is thankful to the Indian National Science Academy, New Delhi for financial support under its Senior Scientist Scheme.

Appendix A

The elements of the matrix $\mathbf{Z}^{-1}(0)$ are:

$$(11) = (21) = -(31) = -(41) = -2G\beta_2 k/s,$$

$$(12) = -(22) = -2G\beta_2 k^2/(ms),$$

$$(13) = -(23) = \beta_2 k^2/(ms),$$

$$(14) = (24) = \beta_2 k/s,$$

$$(15) = (25) = -(35) = -(45) = -c\eta/s,$$

$$(16) = -(26) = -\beta_4/(ms),$$

$$(32) = -(42) = G\beta_1/k + 2G\beta_2 k/s,$$

$$(33) = -(43) = \beta_3/k - \beta_2 k/s,$$

$$(34) = (44) = -1/(2k) - \beta_2 k/s,$$

$$(36) = -(46) = \beta_4/(ks),$$

$$(51) = (52) = -(61) = (62) = \beta_1 G/k,$$

$$(53) = (54) = (63) = -(64) = -\beta_1/(2k),$$

$$(55) = (56) = (65) = (66) = 0.$$

The constants β_i $(i = 1, 2, 3, 4)$ are defined in the List of notations.

Appendix B

The elements of the propagator matrix $\mathbf{a}_n(d_n)$ are given below (suppressing the suffix n of the layer thickness d_n and the poroelastic parameters G_n, ν_n, ν_{un}, B_n and c_n):

$$(11) = (33) = CKD + \beta_1 kd\, SKD + \frac{2\beta_2 k^2}{s}(CMD - CKD),$$

$$(12) = -(43) = 2\beta_3 SKD + \beta_1 kd\, CKD + \frac{2\beta_2 k^2}{s}\left(\frac{k}{m}SMD - SKD\right),$$

$$(13) = \frac{1}{2G}\left[(\beta_1 - 2)SKD - \beta_1 kd\, CKD + \frac{2\beta_2 k^2}{s}\left(-\frac{k}{m}SMD + SKD\right)\right],$$

$$(14) = -(23) = -\frac{1}{2G} \times \left[\beta_1 kd\ SKD + \frac{2\beta_2 k^2}{s}(CMD - CKD)\right],$$

$$(15) = \frac{1}{2G}(45) = \frac{c\eta k}{Gs}(CMD - CKD),$$

$$(16) = \frac{1}{2G}(46) = \frac{\beta_4}{Gs}\left(\frac{k}{m}SMD - SKD\right),$$

$$(21) = -(34) = 2\beta_3 SKD - \beta_1 kd\ CKD + \frac{2\beta_2 k}{s} \times(-m\ SMD + kSKD),$$

$$(22) = (44) = CKD - \beta_1 kd\ SKD + \frac{2\beta_2 k^2}{s} \times(-CMD + CKD),$$

$$(24) = \frac{1}{2G}\left[(\beta_1 - 2)SKD + \beta_1 kd\ CKD + \frac{2\beta_2 k}{s}(m\ SMD - k\ SKD)\right],$$

$$(25) = \frac{1}{2G}(35) = \frac{c\eta}{Gs}(-m\ SMD + k\ SKD),$$

$$(26) = \frac{1}{2G}(36) = \frac{\beta_4}{Gs}(-CMD + CKD),$$

$$(31) = -2G\beta_1\left[SKD + kd\ CKD + \frac{\gamma_1 k}{s}(m\ SMD - k\ SKD)\right],$$

$$(32) = -(41) = -2G\beta_1\left[kd\ SKD + \frac{\gamma_1 k^2}{s}(CMD - CKD)\right],$$

$$(42) = 2G\beta_1\left[kd\ CKD - SKD + \frac{\gamma_1 k^2}{s}\left(\frac{k}{m}SMD - SKD\right)\right],$$

$$(51) = -2G(54) = 2G\xi\beta_1 k(-CMD + CKD),$$

$$(52) = -2G(53) = 2G\xi\beta_1 k\left(-\frac{k}{m}SMD + SKD\right),$$

$$(55) = (66) = CMD,$$

$$(56) = \frac{1}{m\chi}SMD,$$

$$(61) = -2G(64) = 2c\eta k^2\left(SKD - \frac{m}{k}SMD\right),$$

$$(62) = -2G(63) = 2c\eta k^2(CKD - CMD),$$

$$(65) = m\chi SMD,$$

where

$$CKD = \cosh(kd), \qquad SKD = \sinh(kd),$$
$$CMD = \cosh(md), \qquad SMD = \sinh(md).$$

The 4×4 matrix obtained by deleting the fifth and the sixth columns and the fifth and the sixth rows of $\mathbf{a}_n$ and putting $\nu_u = \nu$ in the surviving 4×4 matrix coincides with the propagator matrix for an elastic body given by Singh and Garg (1985). We have obtained the elements of the propagator matrix directly by using the definition given in equation (53). Pan (1999) obtained these elements by using the Laplace transform technique. On comparison, we find that the elements (14), (16), (23), (25), (35), (46), (52), (53), (61), (64) given by Pan (1999, Appendix III) are incorrect. To get the correct elements, the algebraic expressions appearing in lines 8 and 11, p. 1649 (Appendix III) of Pan (1999) should be replaced by the expressions

$$\frac{1}{2G}[\rho_1\ \mathrm{ch}(\lambda z) - \rho_1\ \mathrm{ch}(\lambda\rho z) - \rho_2\lambda z\ \mathrm{sh}(\lambda z)]$$

and

$$\frac{\rho_4\alpha_2}{\kappa}[\mathrm{ch}(\lambda\rho z) - \mathrm{ch}(\lambda z)],$$

respectively.

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MS received 12 August 2005; revised 14 March 2006; accepted 16 March 2006