Proc. Indian Acad. Sci. (Earth Planet. Sci.), Vol. 91, Number 3, November 1982, pp. 241–245.

Rayleigh wave group velocity in a spherically symmetric gravitating earth model

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MS received 23 November 1981

Abstract. Expressions for kinetic energy, elastic potential energy and gravitational potential energy for the spheroidal oscillations of a spherically symmetric, self-gravitating, elastic earth model have been obtained. Some inconsistencies in the expressions given by earlier authors have been pointed out. The principle of equipartition of energy and the Rayleigh principle have been used to derive a formula for Rayleigh wave group velocity in terms of energy integrals. This formula can be used to compute the group velocity without the numerical differentiation implied in its definition.

Keywords. Energy integrals; group velocity; Rayleigh waves; spheroidal oscillations.

1. Introduction

After the fundamental work by Pekeris and Jarosch (1958), numerous investigators have studied the analytical and numerical aspects of the problem of the spheroidal oscillations of a spherically symmetric, self-gravitating, elastic model of the earth. Recent theoretical studies on the subject include the works of Saito (1967), Dahlen (1968), Singh and Ben-Menahem (1969), Phinney and Burridge (1973), Luh (1974), and Dahlen and Smith (1975). Ottelet (1966) proved a variational principle applicable to these oscillations.

In this paper, we have derived the expressions for the kinetic energy, elastic potential energy and gravitational potential energy for the spheroidal oscillations of a spherically symmetric, self-gravitating, elastic model of the earth. The principle of equipartition of energy asserts that in any normal mode mean kinetic energy is equal to the mean potential energy. This principle together with the Rayleigh principle are applied to get a formula for the Rayleigh wave group velocity in terms of energy integrals. This formula is very useful for computing the group velocity.

2. Energy integrals

In the case of spheroidal oscillations, the radial component of the curl of the displacement vanishes identically. We consider a spherical coordinate system

(r, θ, ϕ) with its origin at the centre of the earth. Then, for the spheroidal mode ω_l, the displacement may be taken in the form

$$\mathbf{u} = [U_l(r)\,\mathbf{P}^{\epsilon}_{ml} + V_l(r)\,\{l(l+1)\}^{1/2}\,\mathbf{B}^{\epsilon}_{ml}] \cos(\omega_l\, t) = \mathbf{q} \cos(\omega_l\, t), \quad (1)$$

where $\epsilon = c$ (for cosine) or s (for sine),

$$\mathbf{P}^{\epsilon}_{ml} = \mathbf{e}_r\, Y^{\epsilon}_{ml},$$

$$\{l(l+1)\}^{1/2}\;\mathbf{B}^{\epsilon}_{ml} = \left(\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) Y^{\epsilon}_{ml}, \quad (2)$$

$$Y^{c,\,s}_{ml} = \mathbf{P}^m_l(\cos\theta)\,(\cos m\phi,\ \sin m\phi).$$

The vector spherical harmonics satisfy the following orthogonality relations:

$$\mathbf{P}^{\epsilon}_{ml} \cdot \mathbf{B}^{\epsilon'}_{m'l'} = 0,$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{P}^{\epsilon}_{ml} \cdot \mathbf{P}^{\epsilon'}_{m'l'} \sin\theta\; d\theta\; d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} \mathbf{B}^{\epsilon}_{ml} \cdot \mathbf{B}^{\epsilon'}_{m'l'} \sin\theta\; d\theta\; d\phi$$

$$= \frac{1}{\epsilon_m}\,\delta_{\epsilon\epsilon'}\,\delta_{mm'}\,\delta_{ll'}\;\Omega_{ml}, \quad (3)$$

where δ_{mn} denotes the Kronecker delta,

$$\Omega_{ml} = \frac{4\pi}{2l+1} \cdot \frac{(l+m)\,!}{(l-m)\,!}, \quad (4)$$

and ϵ_m is the Neumann factor.

$$(\epsilon_m = 1, \text{ if } m = 0 \text{ and } \epsilon_m = 2, \text{ if } m > 0).$$

The kinetic energy at time t is given by

$$K(t) = \frac{1}{2}\,\omega_l^2\,\sin^2(\omega_l\, t) \int_V \rho\,|\,\mathbf{q}\,|^2\, dV,$$

where V denotes the volume and ρ the density. Therefore, the kinetic energy averaged over a cycle is

$$K = \frac{1}{T}\int_0^T K(t)\; dt$$

$$= \frac{1}{4}\,\omega_l^2 \int_V \rho\,|\,\mathbf{q}\,|^2\; dV, \quad (5)$$

where T is the period of the oscillation. From (1) to (5), we find

$$K = \frac{1}{4\epsilon_m}\,\Omega_{ml}\,\omega_l^2\,[I_1 + l(l+1)\,I_2], \quad (6)$$

where

$$I_1 = \int_0^a \rho\; U_l^2\; r^2\; dr,$$

$$I_2 = \int_0^a \rho\; V_l^2\; r^2\; dr, \quad (7)$$

a being the radius of the earth.

The potential energy of deformation may be expressed in the form

$$W_1(t) = \frac{1}{2}\cos^2(\omega_l t)\int_V [\lambda \mid \operatorname{div} \mathbf{q} \mid^2 + 2\mu\, \mathbf{E} : \mathbf{E}]\, dV,$$

where $2\mathbf{E} = \nabla\mathbf{q} + \mathbf{q}\nabla$,

λ, μ are the Lamé parameters and the colon (:) denotes the double dot product (Ben-Menahem and Singh 1981) Averaging over a cycle we find

$$W_1 = \frac{1}{T}\int_0^T W_1(t)\, dt$$

$$= \frac{1}{4}\int_V [\lambda \mid \operatorname{div} \mathbf{q} \mid^2 + 2\mu\, \mathbf{E} : \mathbf{E}]\, dV. \tag{8}$$

Substituting for $\mathbf{q}$ from (1) and using (3), we get

$$W_1 = \frac{1}{4\epsilon_m}\Omega_{ml}[I_3 + I_6 + l(l+1)(I_7 - 2I_4)$$

$$+ \{l(l+1)\}^2 (I_5 + 2I_8)], \tag{9}$$

where

$$I_3 = \int_0^a \lambda (2U_l + r\dot{U}_l)^2\, dr,$$

$$I_4 = \int_0^a \lambda (2U_l + r\dot{U}_l) V_l\, dr,$$

$$I_5 = \int_0^a \lambda V_l^2\, dr,$$

$$I_6 = 2\int_0^a \mu (2U_l^2 + r^2\dot{U}_l^2)\, dr,$$

$$I_7 = \int_0^a \mu \{U_l^2 - V_l^2 - 6U_l V_l + 2r(U_l - V_l)V_l + r^2 V_l^2\}\, dr,$$

$$I_8 = \int_0^a \mu V_l^2\, dr,$$

and $\dot{U}_l = dU_l/dr$, etc.

The gravitational potential energy is given by

$$W_2(t) = -\tfrac{1}{2}\int_V (\mathbf{F}.\mathbf{u})\, dV, \tag{10}$$

where

$$\mathbf{F} = \rho\,[\operatorname{grad}(\psi - gu_r) + g\mathbf{e}_r \operatorname{div} \mathbf{u}] \tag{11}$$

is the body force contributed by the gravity, g denotes the acceleration due to gravity and ψ is the perturbation in the gravitational potential. Putting

$$\psi = P_l(r)\, Y_{ml}^e \cos(\omega_l t), \tag{12}$$

and averaging over a cycle, we find

$$W_2 = \frac{1}{T}\int_0^T W_2(t)\, dt$$

$$= \frac{1}{4\epsilon_m}\Omega_{ml}[4I_9 - I_{10} + l(l+1)(2I_{11} - I_{12})], \tag{13}$$

where

$$I_9 = \int_0^a \rho\,(\pi\rho r G - g)\,U_l^2\, r\,\mathrm{d}r,$$

$$I_{10} = \int_0^a \rho U_l P_l\, r^2\,\mathrm{d}r,$$

$$I_{11} = \int_0^a \rho\, g U_l V_l\, r\,\mathrm{d}r,$$

$$I_{12} = \int_0^a \rho V_l P_l r\,\mathrm{d}r. \tag{14}$$

The above expression for the gravitational potential energy agrees with the expressions given by Kovach and Anderson (1967) and Ward (1980) but differs by a factor of half from the one given by Ben-Menahem and Singh (1981). However, the definitions of the displacements, the spherical harmonics and gravitational potential energy of Kovach and Anderson (1967). are not consistent with their subsequent results.

3. Group velocity

According to the principle of equipartition of energy, the mean kinetic and potential energies in any normal mode are equal. The mean kinetic energy is K while the mean potential energy is $W = W_1 + W_2$. Therefore, (6), (9) and (13) yield

$$\begin{aligned}\omega_l^2\,[I_1 + l\,(l+1)\;\; I_2] = I_3 + I_6 + 4I_9 - I_{10} \\ + l\,(l+1)\,(I_7 - 2I_4 + 2I_{11} - I_{12}) \\ + [l\,(l+1)]^2\,(I_5 + 2I_8).\end{aligned} \tag{15}$$

By Jeans formula, we have the following approximation

$$l\,(l+1) = a^2 k_l^2 = a^2\,\omega_l^2/c_R^2, \tag{16}$$

valid for large values of l. Here, k_l denotes the wave number and c_R the Rayleigh wave phase velocity corresponding to the eigen frequency ω_l. With the above approximation, (15) becomes

$$\begin{aligned}\omega_l^2\,(I_1 + a^2 k_l^2 I_2) = I_3 + I_6 + 4I_9 - I_{10} \\ + a^2 k_l^2\,(I_7 - 2I_4 + 2I_{11} - I_{12}) + a^4 k_l^4\,(I_5 + 2I_8).\end{aligned} \tag{17}$$

According to Rayleigh's principle, when ω_l changes to $\omega_l + \delta\omega_l$ and k_l to $k_l + \delta k_l$, the corresponding changes in the eigen functions (*i.e.*, in I_1 to I_{12}) are of the second order and, therefore, may be neglected. Applying this principle to (17), we get the following formula for the group velocity (U_R) of Rayleigh waves

$$\begin{aligned}c_R U_R &= c_R\,\frac{\mathrm{d}\omega_l}{\mathrm{d}k_l} \\ &= \frac{a^2\,[I_7 - 2I_4 + 2I_{11} - I_{12} + 2l\,(l+1)\,(I_5 + 2I_8) - \omega_l^2 I_2]}{I_1 + l\,(l+1)\,I_2}.\end{aligned} \tag{18}$$

In the absence of gravity, (18) reduces to the corresponding result for a non-gravitating earth given by Ben-Menahem and Singh (1981).

The main advantage of the expression (18) for group velocity lies in the fact that one can use it to compute the group velocity without the differentiation implied in its definition $U = \mathrm{d}\omega/\mathrm{d}k$.

4. Conclusions

Equation (10) gives the group velocity of Rayleigh waves on a spherical earth model in terms of the energy integrals. Since these integrals can be computed easily this gives a convenient method of calculating the group velocity.

References

Ben-Menahem A and Singh S J 1981 *Seismic waves and sources* (New York : Springe r-Verlag)
Dahlen F A 1968 *Geophys. J. R. Astr. Soc.* **16** 329
Dahlen F A and Smith M L 1975 *Phil. Trans. Roy. Soc.* **A279** 583
Kovach R and Anderson D L 1967 *J. Geophys. Res.* **72** 2155
Luh P C 1974 *Geophys. J. R. Astr. Soc.* **38** 187
Ottelet I 1966 *Astrophys. J.* **143** 253
Pekeris C L and Jarosch H 1958 in *Contributions in geophysics ; In honour of Beno Gutenberg* (New York : Pergamon) 171–192
Phinney R A and Burridge R 1973 *Geophys. J. R. Astr. Soc.* **34** 451
Saito M 1967 *J. Geophys. Res.* **72** 3689
Singh S J and Ben-Menahem A 1969 *Geophys. J. R. Astr. Soc.* **17** 333
Ward S N 1980 *J. Phys. Earth* **28** 441