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Truncated harmonic oscillator and parasupersymmetric quantum mechanics

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Abstract. We discuss in detail the parasupersymmetric quantum mechanics of arbitrary order where the parasupersymmetry is between the normal bosons and those corresponding to the truncated harmonic oscillator. We show that even though the parasusy algebra is different from that of the usual parasusy quantum mechanics, still the consequences of the two are identical. We further show that the parasupersymmetric quantum mechanics of arbitrary order p can also be rewritten in terms of p supercharges (i.e. all of which obey $Q_i^2 = 0$). However, the Hamiltonian cannot be expressed in a simple form in terms of the p supercharges except in a special case. A model of conformal parasupersymmetry is also discussed and it is shown that in this case, the p supercharges, the p conformal supercharges along with Hamiltonian H, conformal generator K and dilatation generator D form a closed algebra.

Keywords. Harmonic oscillator; parasupersymmetry; conformal generator; dilatation generator.

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A great deal of attention is now being paid to study [1–4] quantum mechanics in a finite dimensional Hilbert space (FHS). In particular, we would like to mention the recent developments [2, 3, 4] in quantum phase theory which deals [2] with a quantized harmonic oscillator in a FHS and which finds interesting applications [4] in problems of quantum optics.

Recently we studied [5] some basic properties of these oscillators. In particular it was pointed out that the raising and lowering operators of the truncated oscillator behave like parafermi oscillator. Inspired by this similarity, a parasupersymmetric quantum mechanics (PSQM) of order 2 was also written down where the parasusy is between the usual bosons and the truncated bosons. However, the explicit form of the charge was not written down. Further, the consequences were also not elaborated upon. The purpose of this note is to generalize this construction to arbitrary order. In particular, we show that for these PSQM models of arbitrary order p, the algebra is given by

$$Q^{p+1} = 0; \quad [H, Q] = 0, \tag{1}$$

$$Q^p Q^+ + Q^{p-1} Q^+ Q + \cdots + Q^+ Q^p = p(p+1) Q^{p-1} H \tag{2}$$

and the Hermitian conjugated relations and discuss their consequences in some detail. In particular, we show that the consequences following from this algebra are identical to those following from the well-known PSQM model of the same order p [6, 7] even though the two algebras are different. In particular, whereas eq. (1) is identical in the two schemes, eq. (2) is different in the two schemes in the sense that in the well-known case, the coefficient on the r.h.s. is $2p$ instead of $p(p+1)$ in eq. (2). In view of the identical consequences, it is worth examining as to why the PSQM of order p can be written down in an alternative way. To that end we show that one can in fact express PSQM of order p in terms of p super (rather than parasuper) charges all of which satisfy $Q_i^2 = 0$ and further all of them commute with the Hamiltonian. However, unlike the usual supersymmetric (SUSY) quantum mechanics (QM), here H cannot be simply expressed in terms of the p supercharges except in a special case. In the special case we show that the Hamiltonian has a very simple expression in terms of the p supercharges

$$Q_1 Q_1^+ + \sum_{j=1}^{p} Q_j^+ Q_j = 2H. \tag{3}$$

We also discuss a parasuperconformal model of order p and show that the dilatation and conformal operators also can be similarly expressed in quadratic form in terms of the p SUSY and p parasuperconformal charges.

Let us start with the truncated raising and lowering operators a^+ and a. It is well-known that if one truncates at $(p+1)$th level ($p > 0$ is an integer) then a and a^+ can be represented by $(p+1) \times (p+1)$ matrices and they satisfy the commutation relation [8]

$$[a, a^+] = I - (p+1)K, \tag{4}$$

where I is $(p+1) \times (p+1)$ unit matrix while $K = \mathrm{diag}(0, 0, \ldots, 0, 1)$ with $Ka = 0$ and further $K^2 = K \neq 0$. As shown by Kleeman [9], the irreducible representations of eq. (4) are the same as those for the scheme

$$[a, a^+ a] = a; \; a^{p+1} = 0; \quad a^j \neq 0 \quad \text{if} \quad j < (p+1). \tag{5}$$

A convenient set of representation of the matrices a and a^+ is given by

$$(a)_{\alpha,\beta} = \sqrt{\alpha}\,\delta_{\alpha+1,\beta} \tag{6}$$

$$(a^+)_{\alpha,\beta} = \sqrt{\beta}\,\delta_{\alpha,\beta+1} \tag{7}$$

where $\alpha, \beta = 1, 2, \ldots, (p+1)$. As shown in [5], the nontrivial multilinear relation between a and a^+ is given by

$$a^p a^+ + a^{p-1} a^+ a + \cdots + a a^+ a^{p-1} + a^+ a^p = \frac{p(p+1)}{2} a^{p-1}. \tag{8}$$

These relations are strikingly similar to those of parafermi oscillator of order p [7] except that in the latter case, the coefficient on the right hand side is $p(p+1)(p+2)/6$ unlike $p(p+1)/2$ in eq. (8). As expected, for the case of the Fermi oscillator ($p = 1$), both the coefficients are same while they are different otherwise.

Motivated by the nontrivial relation between a and a^+ as given by eq. (8) it is worth enquiring if one can construct a kind of PSQM of order p in which there will be symmetry between bosons and truncated bosons of order p. It turns out that the answer to the question is yes. In particular, on choosing the parasusy charges Q and Q^+ as $(p+1)\times(p+1)$ matrices as given by

$$(Q)_{\alpha\beta} = b^+a = \sqrt{\alpha}(P+iW_\alpha)\delta_{\alpha+1,\beta}, \tag{9}$$

$$(Q^\dagger)_{\alpha\beta} = ba^+ = \sqrt{\beta}(P-iW_\beta)\delta_{\alpha,\beta+1}, \tag{10}$$

where b, b^+ denote the bosonic annihilation and creation operators and $\alpha,\beta = 1,2,\ldots,(p+1)$, so that Q and $Q^\dagger$ automatically satisfy $Q^{p+1} = 0 = (Q^\dagger)^{p+1}$. Further, it is easily shown that the Hamiltonian $\hbar = m = 1$)

$$(H)_{\alpha\beta} = H_\alpha\delta_{\alpha\beta}, \tag{11}$$

where $(r = 1,2,\ldots,p)$

$$H_r = \frac{P^2}{2} + \frac{1}{2}(W_r^2 - W_r') + \frac{1}{2}C_r,$$

$$H_{p+1} = \frac{P^2}{2} + \frac{1}{2}(W_p^2 + W_p') + \frac{1}{2}C_p \tag{12}$$

commutes with the parasusy charges Q and $Q^\dagger$ (i.e. $[H,Q] = 0 = [H,Q^+]$) provided $(s = 2,3,\ldots,p)$

$$W_{s-1}^2 + W_{s-1}' + C_{s-1} = W_s^2 - W_s' + C_s. \tag{13}$$

Here $C_1, C_2, \ldots, C_p$ are arbitrary constants with the dimension of energy. It turns out that the nontrivial relation given by eq. (2) between $Q, Q^\dagger$ and H is satisfied provided

$$C_1 + 2C_2 + \cdots + pC_p = 0. \tag{14}$$

It is interesting to notice that the parasusy charge as well as the algebra as given by eqs (1), (2), (9), (10) and (14) is very similar to that of standard PSQM of order p [7] except that in the standard case the coefficient on the r.h.s. of eq. (2) is $2p$ instead of $p(p+1)/2$ and instead of eq. (14) in the standard case one has

$$C_1 + C_2 + \cdots + C_p = 0. \tag{15}$$

Besides, unlike in eq. (9), in the standard case, Q is defined without the factor of $\sqrt{\alpha}$. However, the Hamiltonian and the relation between the superpotentials as given by eqs (11) to (13) are identical in the two cases. As a result the consequences following from the two different PSQM schemes of order p are identical. In particular, as shown in [7], in both the cases (i) the spectrum is not necessarily positive semidefinite unlike in SUSY QM, (ii) the spectrum is $(p+1)$-fold degenerate atleast above the first p levels while the ground state could be $1,2,\ldots,p$ fold degenerate depending on the form of the superpotentials and (iii) one can associate p ordinary SUSY QM Hamiltonians.

Why do the two seemingly different PSQM schemes given the same consequences? The point is that in the case of parasusy of order p, one has p independent parasusy charges and in the two schemes one has merely used two of the p independent forms of Q. It is then clear that one can in fact define p seemingly different PSQM schemes of order p but all of them will have identical consequences. For example, the parafermi operators are usually defined by the following $(p+1) \times (p+1)$ matrices [7]

$$(a)_{\alpha\beta} = \sqrt{\alpha(p-\alpha+1)}\delta_{\alpha+1,\beta}, \tag{16}$$

$$(a^{+})_{\alpha,\beta} = \sqrt{\beta(p-\beta+1)}\delta_{\alpha,\beta+1}. \tag{17}$$

So one could as well have defined the parasusy charges by

$$(Q)_{\alpha\beta} = b^{+}a = \sqrt{\alpha(p-\alpha+1)}(P+iW_{\alpha})\delta_{\alpha+1,\beta}, \tag{18}$$

$$(Q^{\dagger})_{\alpha\beta} = ba^{+} = \sqrt{\beta(p-\beta+1)}(P-iW_{\beta})\delta_{\alpha,\beta+1} \tag{19}$$

instead of the usual choice without the square root factor [7]. It is easily shown that in this case too the parasusy charges Q and Q^{+} satisfy the algebra as given by eqs (1), (2) and (14) except that the factor on the r.h.s. of eq. (2) is now $p(p+1)(p+2)/3$ and the constants C_i satisfy

$$\begin{aligned} p(C_1+C_p) + 2(p-1)(C_2+C_{p-1}) + \cdots \\ + \frac{(p+1)}{2}C_{(p+1)/2} = 0, \quad p \text{ odd}, \end{aligned} \tag{20}$$

$$\begin{aligned} p(C_1+C_p) + 2(p-1)(C_2+C_{p-1}) + \cdots \\ + \frac{p(p+2)}{4}C_{p/2} + C_{(p+2)/2} = 0, \quad p \text{ even} \end{aligned} \tag{21}$$

instead of eq. (14). However, as before the Hamiltonian and the relation between the various superpotentials is unaltered and hence one would get the same consequences as in the standard PSQM case [7].

At this stage, it is worth asking if parasusy QM of order p can be put in an alternative form by making use of the fact that there are p independent parasupercharges [7]? If yes this would be analogous to the so called Green construction for parafermi and parabose operators [10]. We now show that the answer to the question is yes. Let us first note that the supercharge as given by eq. (9) can be written down as a linear combination of the following p supercharges [12]

$$Q = \sum_{j=1}^{p} \sqrt{j}Q_j, \tag{22}$$

where

$$(Q_j)_{\alpha\beta} = (P - iW_j)\delta_{\alpha+1,\beta=j+1}. \tag{23}$$

It is easily checked that these p charges Q_j are in fact supercharges in the sense that all of them satisfy $Q_j^2 = 0$. Further, all of them commute with the Hamiltonian as given by eq. (11) provided condition (13) is satisfied. Besides they satisfy

$$Q_iQ_j = 0 \quad \text{if} \quad j \neq i+1, \tag{24}$$

$$Q_iQ_j^{+} = 0 = Q_i^{+}Q_j \quad \text{if} \quad i \neq j. \tag{25}$$

However, there is one respect in which these charges are different from the usual SUSY charges: the nontrivial relation of the usual parasusy algebra (i.e. eq. (2) but with $2p$ on the r.h.s. instead of $p(p+1)$) now contains product of all p charges i.e.

$$Q_1[Q_1^+Q_1Q_2\cdots Q_{p-1}+Q_2Q_2^+Q_2\cdots Q_{p-1}+\cdots+Q_2Q_3\cdots Q_pQ_p^+]$$
$$=2pQ_1Q_2\cdots Q_{p-1}H, \tag{26}$$
$$[Q_1^+Q_1Q_2\cdots Q_{p-1}+Q_2Q_2^+Q_2\cdots Q_{p-1}+\cdots+Q_2Q_3\cdots Q_pQ_p^+]Q_p$$
$$=2pQ_2Q_3\cdots Q_pH, \tag{27}$$

provided eq. (15) is satisfied. If one considers other versions of PSQM of order p, then one would have similar relations but with different weight factors between the various terms and also different relations between C_i which can easily be worked out.

There is one special case however when the algebra takes a particularly simple form. In particular when all the constants C_i are zero then it is easily checked that the Hamiltonian can be written as a sum over quadratic pieces in Q as given by eq. (3) which is a generalization of the SUSY algebra in the case of p supercharges. In this case, clearly the spectrum is positive semidefinite and most of the results about SUSY breaking etc. would apply. Further all the excited states are always $(p+1)$-fold degenerate. It is amusing to note that in ortho supersymmetricsy QM too [11], the relation between H and charges is exactly as given by eq. (3).

Following the work of [7], we now consider a specific PSQM model of order p which in addition is conformally invariant and show that the conformal PSQM algebra is rather simple. Let us consider the choice

$$W_1=W_2=\cdots=W_p=-\frac{\lambda}{x}. \tag{28}$$

Note that in this case the condition (13) is trivially satisfied when all C_i are zero. The interesting point is that in this case, apart from the p parafermionic charges Q_i, we can also define the dilatation operator D, the conformal operator K and p para superconformal charges S_j so that they form a closed algebra. In particular, on defining

$$D=-\tfrac{1}{4}(xP+Px);\quad K=x^2/2, \tag{29}$$
$$(S_j)_{\alpha\beta}=-x\delta_{\alpha+1,\beta=j+1}, \tag{30}$$

it is easy to show that the algebra satisfied by D,H and K is standard, i.e.

$$[H,K]=2iD,\quad [D,K]=iK,\quad [D,H]=-iH. \tag{31}$$

Further

$$[K,S_j]=0,\quad [H,S_j]=iQ_j,\quad [K,Q_j]=iS_j, \tag{32}$$
$$[D,Q_j]=-\frac{i}{2}Q_j,\quad [D,S_j]=\frac{i}{2}S_j. \tag{33}$$

Besides, apart from the parasusy algebra as described above (with $C_i=0$), we have

$$S_iS_j=0=Q_iS_j=S_iQ_j\quad\text{if}\quad j\neq i+1, \tag{34}$$
$$Q_iS_j^+=0=S_iS_j^+=Q_i^+S_j\quad\text{if}\quad i\neq j, \tag{35}$$

$$S_1 S_1^+ + \sum_{j=1}^{p} S_j^+ S_j = 2K, \tag{36}$$

$$S_1 Q_1^+ + Q_1 S_1^+ + \sum_{j=1}^{p} (S_j^+ Q_j + Q_j^+ S_j) = 4D, \tag{37}$$

It is quite remarkable that an identical algebra also follows in the case of the conformal ortho supersymmetric case [11].

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