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On Invariant Measures, Minimal Sets and a Lemma of Margulis

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Let G be a semisimple, or more generally a reductive, Lie group and let Γ be a lattice in G; i.e. G/Γ admits a finite G-invariant (Borel) measure. Let U be a horospherical subgroup of G; i.e. there exists $g \in G$ such that $U = \{x \in G \mid g^j x g^{-j} \to e \text{ as } j \to \infty\}$ where e is the identity element in G. The action of U on G/Γ is called a horospherical flow. In [3] the author obtained a classification of all *finite* invariant measures of a certain class of horospherical flows. In the present paper we show that if Γ is an 'arithmetic' lattice then every locally finite ergodic invariant measure of the action of any unipotent subgroup (a horospherical subgroup as above is always unipotent) is necessarily finite. The first step is the following theorem.

(0.1) **Theorem.** *Let $\{u_t\}_{t \in \mathbb{R}}$ be a one-parameter group of unipotent matrices in $SL(n, \mathbb{R})$. Then every locally finite, ergodic, $\{u_t\}$-invariant measure on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ is finite.*

Theorem 0.1 is closely related to the following result in [7] generally known as 'Margulis's lemma'.

(0.2) **Theorem.** *Let $\{u_t\}_{t \in \mathbb{R}}$ be as in Theorem 0.1. Then for any $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ the 'positive semi-orbit' $\{u_t x \mid t \geqq 0\}$ does not tend to infinity. That is, there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ such that $\{t \geqq 0 \mid u_t x \in K\}$ is unbounded.*

Certainly, in view of Theorem 0.1 for any $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ the positive semi-orbit and the negative semi-orbit cannot both tend to infinity. For otherwise the 'time' measure along the orbit would be an ergodic, locally finite measure, which is invariant under the flow but not finite.

On the other hand our proof of Theorem 0.1 involves finding a compact set K, for the given x, such that the set $\{t \geqq 0 \mid u_t x \in K\}$ has positive density (cf. Theorem 2.1). As we shall show in § 3 in view of the individual ergodic theorem the last fact implies Theorem 0.1 (cf. Theorem 3.2). Our proof of Theorem 2.1 is modelled over Margulis's proof of Theorem 0.2. However, besides the stronger formulation, there is also a technical difference in our approach. We do not introduce any condition

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analogus to A 2 in [7], which in our view makes that proof somewhat cumbersome and unnatural.

Later we generalise Theorem 0.1 to actions of unipotent subgroups firstly on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ and then more generally on any arithmetic homogeneous space of finite invariant measure (cf. Theorem 3.3 and Theorem 4.1). In §§ 5 and 6 we briefly recall the main results of [3] and reinterpret them in the light of the results in earlier sections. As an application we prove the following result (cf. Theorem 6.3).

(0.3) **Theorem.** *Consider the natural action of $\Gamma = SL(n, \mathbb{Z})$ on $\mathbb{R}^n$ via a basis $\{e_1, e_2, \ldots, e_n\}$. Then any locally finite, ergodic, Γ-invariant measure is a scalar multiple of either the Lebesgue measure on $\mathbb{R}^n$ or the counting measure on the discrete orbit of a point $x \in \mathbb{R}^n$ of the form $x = t \sum_{i=1}^{n} q_i e_i$ where $t \in \mathbb{R}$ and $q_i \in \mathbb{Q}$.*

It may be worthwhile to point out that the assumption in Theorem 0.3 about local finiteness is irredundant. Indeed in view of the results in [4] there exist uncountably many distinct, σ-finite, ergodic, Γ-invariant, continuous (non-atomic) measures which are not locally finite.

In § 7 we apply the study of invariant measures to the study of closed invariant and minimal sets of horospherical flows. In particular it is shown that for certain horospherical flows every minimal set arises from a closed double coset (cf. Theorem 7.2).

At this juncture I wish to express my gratitude to Professor S. Kakutani; Though he was not directly involved, the paper might not have been written but for my association with him.

§ 1. Preliminaries on Lattices

Let Λ be a lattice (a discrete co-compact subgroup) in $\mathbb{R}^n$. For any subgroup Δ of $\mathbb{R}^n$ let $\Delta_{\mathbb{R}}$ denote the $\mathbb{R}$-vector subspace generated by Δ. A subgroup Δ of Λ is said to be complete (in Λ) if $\Delta_{\mathbb{R}} \cap \Lambda = \Delta$. The set of all complete non-zero subgroups of Λ is denoted by $\mathscr{S}(\Lambda)$. Let S be a subset (possibly empty) of $\mathscr{S}(\Lambda)$ which is totally ordered with respect to inclusion as the partial order. We set

$$B(S) = \{\Delta \in \mathscr{S}(\Lambda) \mid \Delta \notin S \text{ and } S \cup \{\Delta\} \text{ is a totally ordered subset of } \mathscr{S}(\Lambda)\}.$$

On $\mathbb{R}^n$ we shall fix an inner product $\langle\, , \rangle$. For any subspace V let μ_V denote the Lebesgue measure on V which assigns unit measure to a parallelepiped whose vertices include an orthonormal basis and 0. For any non-zero discrete subgroup Δ let $d(\Delta)$ denote the number $\mu_{\Delta_{\mathbb{R}}}(F)$ where F is a (any) fundamental domain of Δ in $\Delta_{\mathbb{R}}$. As a convention we shall let $d(\{0\}) = 1$.

(1.1) **Lemma.** *Let $\{u_t\}_{t \in \mathbb{R}}$ be a one-parameter group of unipotent matrices in $SL(n, \mathbb{R})$. Let Δ be any discrete subgroup of $\mathbb{R}^n$. Then $d^2(u_t \Delta)$ is a polynomial in t of degree at most $2n^2$.*

Proof. Let $\{e_1, e_2, \ldots, e_k\}$ be a $\mathbb{Z}$-basis of Δ. Then

$$d^2(u_t \Delta) = |\det(a_{ij}(t))|$$

where $(a_{ij}(t))\, 1 \leqq i,j \leqq k$ is the matrix given by $a_{ij}(t) = \langle u_t e_i,\, u_t e_j \rangle$. The latter is a polynomial of degree at most $2n$. Hence $d^2(u_t \Delta)$ is a polynomial of degree at most $2n^2$.

(1.2) **Lemma.** *Let Λ be a lattice in $\mathbb{R}^n$. Let $c > 0$. Then the set of discrete subgroups Δ of Λ for which $d(\Delta) < c$ is finite.*

Proof is obvious.

(1.3) **Lemma.** *Let τ be a closed interval contained in $\mathbb{R}_+^* = (0, \infty)$ and let $\beta > 0$ and $n \in \mathbb{N}$ be given. Then there exists $\gamma = \gamma(\tau, \beta) > 0$ such that the following holds: If Λ is a lattice in $\mathbb{R}^n$ and S is a totally ordered subset of $\mathcal{S}(\Lambda)$ such that*

(i) *$d(\Delta) \in \tau$ for all $\Delta \in S$ and*

(ii) *$d(\Delta) > \beta$ for all $\Delta \in B(S)$*

then $\|z\| \geqq \gamma$ for all $z \in \Lambda - (0)$.

Proof. This follows easily from the following fact: Let Δ and Δ' be two discrete subgroups of $\mathbb{R}^n$ such that $\Delta \in \mathcal{S}(\Delta')$ and Δ'/Δ is cyclic. Then for any $z \in \Delta' - \Delta$, $\|z\| \geqq d(\Delta') \cdot d(\Delta)^{-1}$.

We also need the following alternative realisation of $d(\Delta)$. For $p > 0$ let E^p denote the p^{th} exterior power of $\mathbb{R}^n$ (where n is assumed to be fixed) and let $E = \sum_{p=0}^{n} E^p$ be the exterior algebra. Recall that E^1 may be identified with $\mathbb{R}^n$. We extend the norm $\|.\|$ on E^1 to E as follows. Let $\{e_1, e_2, \ldots, e_n\}$, be an orthonormal basis of E^1. On E^p, $p > 1$ choose the norm so that $\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \mid 1 \leqq i_1 < i_2 \ldots < i_p \leqq n\}$ is an orthonormal basis. It is easy to check that the above norm depends only on the norm on E^1 and not on the choice of the orthonormal basis. On $E^0 = \mathbb{R}$ choose any norm. The norms on E^p, $p \geqq 0$ extend uniquely to a (Hilbert) norm on E (which also we shall denote by $\|.\|$), such that $\{E^p\}_{p \geqq 0}$ are mutually orthogonal.

(1.4) **Lemma.** *Let Δ be a discrete subgroup of $\mathbb{R}^n$ and let $\{h_1, h_2, \ldots, h_k\}$ be a $\mathbb{Z}$-basis of Δ. Then $d(\Delta) = \|h_1 \wedge h_2 \wedge \ldots \wedge h_k\|$.*

Proof. Let $\{f_1, f_2, \ldots, f_k\}$ be an orthonormal basis of $\Delta_{\mathbb{R}}$. Then $d(\Delta)$ is $|\det A|$ where $A: \Delta_{\mathbb{R}} \to \Delta_{\mathbb{R}}$ is given by $Af_i = h_i$ for $i = 1, 2 \ldots k$. But by standard multilinear algebra we also have.

$$\|h_1 \wedge h_2 \wedge \ldots \wedge h_k\| = \|(\det A) f_1 \wedge f_2 \wedge \ldots \wedge f_k\| = |\det A|.$$

(1.5) **Lemma.** *Let $\{u_t\}_{t \in \mathbb{R}}$ be a one-parameter group in $SL(n, \mathbb{R})$. Then the function*

$$v(t) = \sup \left\{ \frac{d(u_t \Delta)}{d(\Delta)} \,\middle|\, \Delta \text{ any discrete subgroup of } \mathbb{R}^n \right\}$$

is continuous.

Proof. By Lemma 1.4 for any $t \in \mathbb{R}$, $v(t)$ as above is the norm of the linear transformation $\wedge u_t$ of E obtained as the exterior power of u_t. Since $\{\wedge u_t\}_{t \in \mathbb{R}}$ is a one parameter group of linear transformations our contention is obvious.

(1.6) **Lemma.** *Let $\{\Delta_\alpha\}_{\alpha\in A}$ be a family of (discrete) subgroups of a lattice Λ, where A is some indexing set. Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$ and $\rho > 0$ be such that $d(u_{t_1}\Delta_\alpha) > \rho$ for all $\alpha \in A$ and $d(u_{t_2}\Delta_\alpha) \leqq \rho$ for some $\alpha\in A$. Let*

$$\bar{t} = \inf\{t\in[t_1, t_2] \mid d(u_t\Delta_\alpha) \leqq \rho \text{ for some } \alpha\in A\}.$$

Then there exists $\alpha\in A$ such that $d(u_{\bar{t}}\Delta_\alpha) = \rho$.

Proof. By Lemma 1.5 there exists $\varepsilon > 0$ such that $v(t) < 2$ whenever $|t| < \varepsilon$. By definition of $\bar{t}$ there exist sequences $\{t_j\}_{j=1}^{\infty}$ in $\mathbb{R}$ and $\{\alpha_j\}_{j=1}^{\infty}$ in A such that $t_j \searrow \bar{t}$ and $d(u_{t_j}\Delta_{\alpha_j}) = \rho$ for all $j\in\mathbb{N}$. We may assume $t_j < \bar{t} + \varepsilon$ for all $j\in\mathbb{N}$. Then $d(u_{\bar{t}}\Delta_{\alpha_j}) = d(u_{\bar{t}-t_j}\cdot u_{t_j}\Delta_{\alpha_j}) \leqq 2\rho$. Hence by Lemma 1.2 the set $\{\Delta_{\alpha_j} \mid j\in\mathbb{N}\}$ must be finite. Passing to a subsequence if necessary we may assume that $\Delta_{\alpha_j} = \Delta$ for all $j\in\mathbb{N}$. Now since $t_j \to \bar{t}$ and $d(u_{t_j}\Delta) = \rho$ we deduce that $d(u_{\bar{t}}\Delta) = \rho$.

§ 2. The Recurrence Lemma

The aim of this section is to prove the following.

(2.1) **Theorem.** *Let Λ be a lattice in $\mathbb{R}^n$ and let $\{u_t\}_{t\in\mathbb{R}}$ be a one parameter group of unipotent matrices in $SL(n, \mathbb{R})$. Let $k > 1$ be given. Then there exists $c > 0$ and $\rho > 0$ such that for any $t_0 > 0$.*

$$\lambda\{t\in[t_0, kt_0] \mid \|u_t z\| \geqq c \text{ for all } z\in\Lambda - (0)\} > \rho t_0$$

where λ is the Lebesgue measure on $\mathbb{R}$ such that $\lambda([a, b]) = b - a$ for all $a, b\in\mathbb{R}$, $a \leqq b$.

We need the following lemmas as in [7]. In the sequel let $\mathscr{P}_l$ denote the space of all non-negative polynomials on $\mathbb{R}$ of degree at most l.

(2.2) **Lemma.** *For any $k > 1$ and $l\in\mathbb{N}$ there exist constants $\varepsilon_1(k, l)$ and $\varepsilon_2(k, l)$ such that if $P\in\mathscr{P}_l$, $P(1) = 1$ and $P(t) \leqq 1$ for all $t\in[0, 1]$ then the values of P at all points of one of the intervals*

$$[k, k^2], [k^3, k^4], \ldots, [k^{2l+1}, k^{2l+2}]$$

belong to $[\varepsilon_1(k, l), \varepsilon_2(k, l)]$.

Proof. The set of polynomials in $\mathscr{P}_l$ which are uniformly bounded over a closed interval is compact. In particular they are uniformly bounded over the interval $[1, k^{2l+2}]$. Thus we only need to find the lower bound $\varepsilon_1(k, l)$. If such a bound did not exist then for any $j\in\mathbb{N}$ there exists $P_j\in\mathscr{P}_l$ such that $P_j(1) = 1$, $P_j(t) \leqq 1$ for all $t\in[0, 1]$ and each of the above intervals contains a point t such that $P_j(t) < 1/j$. Let $P\in\mathscr{P}_l$ be a limit point of $\{P_j\}_1^{\infty}$. Then $P(1) = 1$ and P has a zero in each of the $l+1$ intervals – this is a contradiction.

(2.3) **Lemma** *For any $k > 1$ and $l\in\mathbb{N}$ there exists a constant $\bar{\varepsilon}(k, l)$ such that if $P\in\mathscr{P}_l$, $P(t) = 1$ for some $t\in[0, 1]$ and $P(1) < \bar{\varepsilon}(k, l)$ then there exists $t\in[1, k]$ such that $P(t) = \bar{\varepsilon}(k, l)$.*

Proof. For otherwise for any $j \in \mathbb{N}$ there exists $P_j \in \mathscr{P}_l$ and $t_j \in [0, 1]$ such that $P_j(t_j) = 1$ and $P_j(t) < 1/j$ for all $t \in [1, k]$. Since $\{P_j\}$ is uniformly bounded on $[1, k]$ there exists $P \in \mathscr{P}_l$ which is a limit point of $\{P_j\}$. However $P(t) = 0$ for all $t \in [1, k]$ and $P(t) = 1$ for any limit point t of $\{t_j\}$, which is a contradiction.

The following two lemmas may be obtained from the preceding two by linear substitutions.

(2.4) **Lemma.** *For any $k > 1$ and $l \in \mathbb{N}$ there exist constants $\varepsilon_1(k, l)$ and $\varepsilon_2(k, l)$ such that the following holds: Let $c > 0$ and $0 \leqq t_1 \leqq t_2$. If $P \in \mathscr{P}_l$ is such that $P(t) \leqq c$ for all $t \in [t_1, t_2]$ and $P(t_2) = c$ then the values of P at all points of one of the intervals*

$$[t_1 + k(t_2 - t_1), t_1 + k^2(t_2 - t_1)], [t_1 + k^3(t_2 - t_1), t_1 + k^4(t_2 - t_1)] \\ \dots [t_1 + k^{2l+1}(t_2 - t_1), t_1 + k^{2l+2}(t_2 - t_1)]$$

lie in the range $[c\varepsilon_1(k, l), c\varepsilon_2(k, l)]$.

(2.5) **Lemma.** *For any $k > 1$ and $l \in \mathbb{N}$ there exists a constant $\bar{\varepsilon}(k, l)$ such that the following holds: Let $c > 0$ and $0 \leqq t_1 \leqq t_2$. If $P \in \mathscr{P}_l$ is such that $P(t) = c$ for some $t \in [t_1, t_2]$, $P(t_2) < c\bar{\varepsilon}(k, l)$ then there exists $t \in [t_2, t_1 + k(t_2 - t_1)]$ such that $P(t) = c\bar{\varepsilon}(k, l)$.*

Now let Λ be a fixed lattice in $\mathbb{R}^n$. Also let $k > 1$ be fixed throughout. Given a closed interval τ contained in $\mathbb{R}_+^* = (0, \infty)$, $\delta > 0$, $a \geqq 0$ and $t_0 \geqq 0$ we denote by $\mathscr{A}(\tau, \delta, a, t_0)$ the set of all totally ordered subsets S (possibly empty) of $\mathscr{S}(\Lambda)$ satisfying the following.

A1. For any $\Delta \in B(S)$ there exists $t \in [0, t_0]$ such that $d^2(u_{a+t}\Delta) \geqq \delta$ and

A2. For any $\Delta \in S$, $d^2(u_{a+t}\Delta) \in \tau$ for all $t \in [t_0, kt_0]$.

(2.6) *Remark.* The empty subset belongs to $\mathscr{A}(\tau, \delta, 0, t_0)$ for a suitable δ and arbitrary τ and t_0.

(2.7) **Proposition.** *Let Λ be a lattice in $\mathbb{R}^n$ and let $k > 1$ be fixed as before. Then for any closed interval τ in $(0, \infty)$ and $\delta > 0$ there exist constants $0 < c_1 < c_2$ and $\rho > 0$ such that the following holds: If S is a totally ordered subset of $\mathscr{S}(\Lambda)$ and $a \geqq 0$ and $t_0 \geqq 0$ are such that $S \in \mathscr{A}(\tau, \delta, a, t_0)$ and if X and Y are the sets defined by*

(2.8) $\quad X = \{t \in [t_0, kt_0] \mid \|u_{a+t}z\| \geqq c_1$ *for all* $z \in \Lambda - \{0\}\}$

and

(2.9) $\quad Y = \{t \in [t_0, kt_0] \mid$ *there exists a maximal totally ordered subset L of $B(S)$ such that $c_1 < d^2(u_{a+t}\Delta) < c_2$ for all $\Delta \in L$*$\}$

then

$$\lambda(X \cup Y) > \rho t_0.$$

Proof. Clearly it is enough to find the constants c_1, c_2 and ρ such that the contention holds for all totally ordered subsets S of a given cardinality, say p. We proceed by induction on $n - p$. If $p = n$, then the result is obvious. Now we shall assume the result for the class of totally ordered subsets of cardinality $\geqq p + 1$. Let $\tau \subset (0, \infty)$

and $\delta>0$ be given as in the statement of the Proposition. Let τ' be the smallest closed interval containing τ and $[\delta\bar{\varepsilon}(k_1)\varepsilon_1(k), \delta\bar{\varepsilon}(k_1)\varepsilon_2(k)]$ where $k_1=1+(\sqrt{k}-1)k^{-(4n^2+2)}$. Here and in the sequel for any $k'>1$, $\varepsilon_1(k')$, $\varepsilon_2(k')$ and $\bar{\varepsilon}(k')$ denote $\varepsilon_1(k',2n^2)$, $\varepsilon_2(k',2n^2)$ and $\bar{\varepsilon}(k',2n^2)$ respectively. Let $\delta'=(1/2)\delta\bar{\varepsilon}(k_1)$. By induction hypothesis there exist $c_1', c_2', 0<c_1'<c_2'$ and $\rho'>0$ such that the contention of the Proposition holds if $S'\in\mathscr{A}(\tau',\delta',a',t_0')$ for some $a'\geqq 0$ and $t_0'\geqq 0$, provided the cardinality of S' is at least $p+1$. Now put $\rho=(\sqrt{k}-1)\rho'/k$, $c_1=\min\{c_1', \delta\bar{\varepsilon}(k_1)\varepsilon_1(k), \gamma(\tau,\frac{1}{2}\delta\bar{\varepsilon}(k_1))\}$ (γ as in Lemma 1.3) and $c_2=\max\{c_2',\delta\bar{\varepsilon}(k_1)\varepsilon_2(k)\}$. We shall complete the proof of the Proposition by showing that with these values for the constants the contention of the Proposition holds for any $S\in\mathscr{A}(\tau,\delta,a,t_0)$ (for some $a\geqq 0$ and $t_0\geqq 0$) of cardinality p.

Let S be any totally ordered subset of $\mathscr{S}(\Delta)$ of cardinality p and let $a\geqq 0$ and $t_0\geqq 0$ be such that $S\in\mathscr{A}(\tau,\delta,a,t_0)$. Let $\sigma>0$ be such that the function $v(t)$ as defined in Lemma 1.5 satisfies $v(t)<\sqrt{2}$ for all t such that $|t|<\sigma$. Let X and Y be the sets defined by (2.8) and (2.9) respectively. Finally let $k_0=\sqrt{k}$.

Sublemma. Let the notation be as above. For any $s\in[t_0,k_0t_0]$ there exists $s'\in(s,k_0s)$ such that either $[s,s')\subset X$ or $s'\geqq s+\sigma$ and

$$\lambda((X\cup Y)\cap[s,s'])\geqq(k_0-1)^{-1}\rho(s'-s)=k^{-1}\rho'(s'-s).$$

We first show that validity of the sublemma implies that $\lambda(X\cup Y)>\rho t_0$, thus proving the Proposition. Inductively we construct a finite sequence $s_0,s_1,\ldots,s_r\in[t_0,kt_0]$ as follows. Choose $s_0=t_0$. Now suppose $s_0,s_1,\ldots,s_i$ have been chosen. If $s_i\notin[t_0,k_0t_0]$ we choose $r=i$ thus terminating the sequence. If $s_i\in[t_0k_0t_0]$ by Sublemma there exists s_{i+1} such that either a) $[s_i,s_{i+1})\subset X$ and $[s_i,s')\not\subset X$ for $s'>s_{i+1}$ or b) $s_{i+1}\geqq s_i+\sigma$ and $\lambda((X\cup Y)\cap[s_i,s_{i+1}])\geqq(k_0-1)^{-1}\rho(s_{i+1}-s_i)$. Observe that in view of the construction for any $i>0$ either $s_i-s_{i-1}\geqq\sigma$ or $s_{i+1}-s_i\geqq\sigma$. Hence there exists r such that $s_r\notin[t_0,k_0t_0]$ and the sequence terminates. Now

$$\begin{aligned}\lambda(X\cup Y)&\geqq\sum_{i=1}^{r}\lambda((X\cup Y)\cap[s_{i-1},s_i])\\&\geqq(k_0-1)^{-1}\rho\sum_{i=1}^{r}(s_i-s_{i-1})\\&=(k_0-1)^{-1}\rho(s_r-s_0)\\&>\rho t_0\end{aligned}$$

since $s_r\notin[t_0,k_0t_0]$.

Proof of the Sublemma. Let $s\in[t_0,k_0t_0]$ be given. Put

$$\mathscr{C}=\{\Delta\in B(S)\,|\,d^2(u_{a+s}\Delta)\leqq(1/2)\delta\bar{\varepsilon}(k_1)\}.$$

We consider two cases separately.

Case i) Assume that $\mathscr{C}$ is non-empty.

By Lemma 2.5 for every $\Delta\in\mathscr{C}$ the set $H(\Delta)=\{t\in[s,k_1s]\,|\,d^2(u_{a+s}\Delta)=\delta\bar{\varepsilon}(k_1)\}$ is non-empty. For $\Delta\in\mathscr{C}$ put $t(\Delta)=\inf\{t\in H(\Delta)\}$ and $y=\sup\{t(\Delta)\,|\,\Delta\in\mathscr{C}\}$. Since by Lemma 1.1 $\mathscr{C}$ is finite there exists $\bar{\Delta}$ such that $t(\bar{\Delta})=y$.

By Lemma 2.4 there exists an interval $[s_1, s_2]$ contained in $[s, k_0 s]$ such that

(a) there exists j, $0 \leqq j \leqq 2n^2$ such that $s_1 = s + k^{2j+1}(y-s)$ and $s_2 = s + k^{2j+2}(y-s)$ and

(b) for all $t \in [s_1, s_2]$, $d^2(u_{a+t}\bar{\Delta}) \in [\delta\bar{\varepsilon}(k_1)\varepsilon_1(k), \delta\bar{\varepsilon}(k_1)\varepsilon_2(k)]$.

Let $S' = S \cup \{\bar{\Delta}\}$. It is easy to verify that $S' \in \mathscr{A}(\tau', \delta', a+s, s_1 - s)$. Here A 1 follows from the fact that $s_1 \geqq s + k(y-s) \geqq y$ and A 2 follows from (b), since $s_2 - s = k(s_1 - s)$. Since S' has cardinality $p+1$ by induction hypothesis if

$$X' = \{t \in [s_1 - s, k(s_1 - s)] \mid \|u_{a+s+t} z\| > c_1' \text{ for all } z \in \Lambda - (0)\}$$

and

$$Y' = \left\{\begin{matrix} t \in [s_1 - s, k(s_1 - s)] \mid \text{there exists a maximal totally ordered} \\ \text{subset } L' \text{ of } B(S') \text{ such that } c_1' < d^2(u_{a+s+t}\Delta) < c_2' \text{ for all } \Delta \in L' \end{matrix}\right\}$$

then $\lambda(X' \cup Y') > \rho'(s_1 - s)$.

It is obvious from the definition that the set $X' + s (= \{t + s \mid t \in X'\})$ is contained in $X \cap [s, s_2]$. Also, if L' is a maximal totally ordered subset of $B(S')$ then $L = L' \cup \{\bar{\Delta}\}$ is a maximal totally ordered subset of $B(S)$. Since $k(s_1 - s) = s_2 - s$ and $d^2(u_{a+t}\bar{\Delta}) \in [\delta\bar{\varepsilon}(k_1)\varepsilon_1(k), \delta\bar{\varepsilon}(k_1)\varepsilon_2(k)]$ for all $t \in [s_1, s_2]$ it follows that $Y' + s$ is contained in $Y \cap [s, s_2]$. Hence

$$\begin{aligned} \lambda((X \cup Y) \cap [s, s_2]) &\geqq \lambda((X' + s) \cup (Y' + s)) = \lambda(X' \cup Y') \geqq \rho'(s_1 - s) \\ &= k^{-1}\rho'(s_2 - s). \end{aligned}$$

Recall that $s_2 \in [s, k_0 s]$. Also since $d^2(u_{a+s}\bar{\Delta}) \leqq \frac{1}{2}\delta\bar{\varepsilon}(k_1)$ and $d^2(u_{a+y}\bar{\Delta}) = \delta\bar{\varepsilon}(k_1)$ it follows that $v(y-s) \geqq \sqrt{2}$. Hence $(y-s) \geqq \sigma$. But by definition $s_2 > s_1 \geqq y$. Hence $s_2 > s + \sigma$. Thus the sublemma holds with $s' = s_2$.

Case ii) Assume $\mathscr{C}$ is empty.

Consider the set

$$E = \{t \in [s, k_0 s] \mid d^2(u_{a+t}\Delta) > \tfrac{1}{2}\delta\bar{\varepsilon}(k_1) \text{ for all } \Delta \in B(S)\}.$$

Then by assumption $s \in E$. By Lemma 1.3 for every $t \in E$, $\|u_{a+t} z\| \geqq \gamma(\tau, \frac{1}{2}\delta\bar{\varepsilon}(k_1))$ for all $z \in \Lambda - (0)$. Hence $E \subset X$. If $E = [s, k_0 s)$ then we are through. Otherwise let

$$s' = \inf\{t \in [s, k_0 s] \mid t \notin E\}.$$

By Lemma 1.6 there exists $\Delta \in B(S)$ such that $d^2(u_{a+s'}\Delta) = \frac{1}{2}\delta\bar{\varepsilon}(k_1)$. Hence $s' \notin E$. In particular $s' > s$. Also $[s, s') \subset E \subset X$, which proves the sublemma.

Proof of Theorem 2.1. Recall that there exists $\delta > 0$ such that the empty set Φ belongs to $\mathscr{A}(\tau, \delta, 0, t_0)$ for any closed interval $\tau \subset (0, \infty)$ and $t_0 \geqq 0$. Let c_1, c_2 and ρ be the constants as in Proposition 2.7 corresponding to δ as above and an arbitrarily chosen τ. Let $t_0 \geqq 0$ be arbitrary. Let X and Y be as defined in (2.8) and (2.9) respectively, for the special case being considered. Consider any $t \in Y$. There exists a maximal totally ordered subset L of $B(\Phi) = \mathscr{S}(\Lambda)$ such that $c_1 < d^2(u_t\Delta) < c_2$. Applying Lemma 1.3 to $S = L$ we conclude that for $t \in Y$, $\|u_t z\| \geqq \gamma = \gamma([c_1, c_2], c')$ for all $z \in \Lambda - (0)$ where c' may be chosen arbitrarily. Hence for any $t \in X \cup Y$, $\|u_t z\| \geqq c = \min\{c_1, \gamma\}$ since $\lambda(X \cup Y) > \rho t_0$ the theorem is proved.

Theorem 2.1 can be reinterpreted in terms of the action of the one parameter group $\{u_t\}_{t \in \mathbb{R}}$ on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ as follows.

(2.10) **Theorem.** *Let $\{u_t\}_{t \in \mathbb{R}}$ be a one-parameter group of unipotent matrices in $SL(n, \mathbb{R})$. Let $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Then there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ and $\rho > 0$ such that for any $t_0 > 0$*

$$\lambda\{t \in [0, t_0] \mid u_t x \in K\} > \rho t_0.$$

Proof. Consider the action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n$ with respect to a basis $\{e_1, e_2, \ldots, e_n\}$. This induces an action of $SL(n, \mathbb{R})$ on the set of lattices in $\mathbb{R}^n$. Via this action $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ can be identified with the set of lattices Λ in $\mathbb{R}^n$ such that $d(\Lambda) = d(\Lambda_0)$ where Λ_0 is the lattice with $\{e_1, e_2, \ldots, e_n\}$ as a $\mathbb{Z}$-basis. By the well-known Mahler criterion (cf. [8], Corollary 10.9) for any $c > 0$ the set of lattices Λ such that $d(\Lambda) = d(\Lambda_0)$ and $\|z\| \geqq c$ for all $z \in \Lambda - (0)$ corresponds, under the above identification, to a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Thus by Theorem 2.1 given $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ there exists a compact set K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ and $\rho' > 0$ such that for any $t_0 > 0$, $\lambda\{t \in [t_0/2, t_0] \mid u_t x \in K\} > \rho' t_0/2$. Hence putting $\rho = \rho'/2$ we get $\lambda\{t \in [0, t_0] \mid u_t x \in K\} > \rho t_0$.

We conclude this section with a similar recurrence property for the action of (iterates of) a single unipotent matrix $u \in SL(n, \mathbb{R})$.

(2.11) **Theorem.** *Let $u \in SL(n, \mathbb{R})$ be a unipotent matrix. Then for any $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ there exists a compact subset K of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ and $\rho > 0$ such that*

$$\frac{1}{m} \sum_{j=0}^{m-1} \chi_K(u^j x) \geqq \rho$$

for all $m \in \mathbb{N}$, where χ_K denotes the characteristic function of K.

Proof. Given any unipotent matrix u there exists a one-parameter group $\{u_t\}_{t \in \mathbb{R}}$ of unipotent matrices such that $u_1 = u$. Indeed since $(u - I)^m = 0$ for some $m \in \mathbb{N}$, $\xi = \log u = \sum_{l=0}^{\infty} (-1)^l (u - I)^l / l$ is defined and consequntly $u_t = \exp t\xi$ defines a one parameter group having the above properties. Let $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ be given. By Theorem 2.10 there exists a compact set K' and $\rho' > 0$ such that

$$\lambda\{t \in [0, t_0] \mid u_t x \in K'\} > \rho t_0.$$

Put

$$K = \{u_t y \mid y \in K' \text{ and } -1 \leqq t \leqq 0\}.$$

Then K is a compact subset of $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Clearly for $j \in \mathbb{N}$, $u^j x \in K$ whenever there exists $t \in [j, j+1]$ such that $u_t x \in K'$. Hence the Theorem.

§ 3. Invariant Measures on Lattice Spaces

Using the results of the preceding section we now prove Theorem 0.1 (and also a discrete analogue). We need the following theorem.

(3.1) **Theorem.** *(Individual ergodic theorem.) Let (X,μ) be a σ-finite (finite or infinite) measure space and let T(respectively $\{\varphi_t\}_{t\in\mathbb{R}}$) be a (jointly) measurable μ-preserving transformation (resp. flow). Then for any $f\in L^1(X,\mu)$*

$$\frac{1}{m}\sum_{j=0}^{m-1} f(T^j x)\ \left(\text{resp.}\ \frac{1}{s}\int_0^s f(\varphi_t x)\,dt\right)$$

converges a.e. as $m\to\infty$ (resp. $s\to\infty$). The limit function f^* is contained in $L^1(X,\mu)$. Also there exists a measurable T-(resp. $\{\varphi_t\}_{t\in\mathbb{R}}$) invariant set N with $\mu(N)=0$ such that for all $x\in X-N$, $f^*(Tx)=f^*(x)$ (resp. $f^*(\varphi_t x)=f^*(x)$ for all $t\in\mathbb{R}$). Further if $\mu(X)<\infty$ then $\int_X f^*\,d\mu=\int_X f\,d\mu$.

For a transformation this is usually the first theorem that one learns in ergodic theory. Somehow it is usually not stated for a flow in most standard books. However the theorem for a (measurable) flow $\{\varphi_t\}_{t\in\mathbb{R}}$ can be deduced by applying the theorem for the transformation φ_1 to the function $\int_0^1 f(\varphi_t x)\,dt$.

(3.2) **Theorem.** *Let $X=SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. Let U be a cyclic or a one-parameter group of unipotent matrices in $SL(n,\mathbb{R})$. Consider the left action of U on X. Let π be a locally finite, U-invariant (Borel) measure on X. Then there exists a sequence $\{X_i\}_{i=1}^{\infty}$ of measurable U-invariant subsets of X such that $\pi(X_i)<\infty$ for all $i\in\mathbb{N}$ and $\pi\left(X-\bigcup_1^{\infty} X_i\right)=0$. In particular every locally finite ergodic U-invariant measure is finite.*

Proof. Let f be a continuous function on X such that $f(x)>0$ for all $x\in X$ and $\int_X f\,d\pi=1$. Let $f^*(x)=\lim_{m\to\infty}\frac{1}{m}\sum_{j=0}^{m-1} f(u^j x)$ if U is the cyclic subgroup generated by u and $f^*(x)=\lim_{s\to\infty}\frac{1}{s}\int_0^s f(u_t x)\,dt$ if $U=\{u_t\}_{t\in\mathbb{R}}$. By Theorem 3.1, $f^*(x)$ is defined π a.e. and is contained in $L^1(X,\pi)$. We show that $f^*(x)>0$ π a.e. Assume that $U=\{u_t\}_{t\in\mathbb{R}}$. Let $x\in X$ be such that $f^*(x)$ is defined (by convergence of the integral). By Theorem 2.10 there exists a compact set K and $\rho>0$ such that for any $t_0>0$

$$\lambda\{t\in[0,t_0]\mid u_t x\in K\}>\rho t_0$$

Put

$$\theta=\inf\{f(y)\mid y\in K\}.$$

Since f is continuous and positive $\theta > 0$. Hence

$$f^*(x) = \lim_{s\to\infty} \frac{1}{s}\int_0^s f(u_t x)\,dt > \theta\rho > 0.$$

If U is cyclic we can produce a similar argument using Theorem 2.11.

Recall that by Theorem 3.1 there exists a measurable U-invariant subset N of X such that $\pi(N)=0$ and for all $x \in X-N, f^*(ux)=f^*(x)$ for all $u \in U$. Now for $i \in \mathbb{N}$ put

$$X_i = \left\{ x \in X - N \,|\, f^*(x) > \frac{1}{i} \right\}.$$

Then each X_i is U-invariant. Also since $f^* \in L^1(X,\pi)$, $\pi(X_i) < \infty$. Finally since $f^*(x) > 0$ π-a.e., $\pi\left(X - \bigcup_1^\infty X_i\right) = 0$.

Our next aim is to generalise Theorem 3.2 to action of an arbitrary unipotent subgroup of $SL(n,\mathbb{R})$. We recall that a subgroup U is said to be unipotent if every element of U is unipotent. A unipotent subgroup is necessarily nilpotent.

(3.3) **Theorem.** *Let U be a unipotent subgroup of $SL(n,\mathbb{R})$. Consider the action (on the left) of U on $X = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$. Let π be a locally finite, U-invariant (Borel) measure on X. Then there exists a sequence $\{X_i\}_1^\infty$ of measurable U-invariant subsets of X such that $\pi(X_i) < \infty$ for all $i \in N$ and $\pi\left(X - \bigcup_1^\infty X_i\right) = 0$. In particular every locally finite ergodic U-invariant measure is finite.*

(3.4) *Remark.* There is no loss of generality in assuming U to be a closed subgroup. This is because the subgroup of elements which preserve a locally finite measure is closed.

Proof. Any closed unipotent subgroup U admits a normal series

$$(e) = U_0 \subset U_1 \subset \ldots \subset U_{m-1} \subset U_m = U$$

such that for each $j = 1, 2 \ldots m$, U_j is a closed subgroup and U_j/U_{j-1} is either cyclic or isomorphic to $\mathbb{R}$. Further the length m of such a normal series depends only on U; we shall call it the *rank* of U.

We shall prove by induction on the rank that if U is a closed unipotent subgroup and π is a U-invariant measure on X then there exists $f \in L^1(X,\pi)$, $f \geqq 0$ and $f \neq 0$, which is U-invariant. If the rank is 1 then U is either cyclic or a one-parameter group. In this case the assertion follows from Theorem 3.2. Now assume that the above assertion holds for all closed unipotent subgroups of rank $< m$ and let U be a closed unipotent subgroup of rank m. It is obvious from the definition of the rank that U admits a closed unipotent normal subgroup V of rank $m-1$ such that U/V is isomorphic to either $\mathbb{Z}$ or $\mathbb{R}$. Let π be a U-invariant measure on X. By induction hypothesis there exists $f \in L^1(X,\pi)$, $f \geqq 0$ and $f \not\equiv 0$ such that f is V-invariant.

Since U/V is either cyclic or a one-parameter group there exists a subgroup H of U, such that $U = H \cdot V$ (semi-direct product). By Theorem 3.2 there exists a sequence

$\{X_i\}_1^\infty$ of measurable H-invariant subsets such that $\pi(X_i) < \infty$ for all $i \in \mathbb{N}$ and $\pi\left(X - \bigcup_1^\infty X_i\right) = 0$. Clearly X_i may be assumed to be pairwise disjoint. Now let f^* be the function defined by

$$f^*(x) = \lim_{r \to \infty} \frac{1}{r} \sum_{j=0}^{r-1} f(u^j x)$$

if $H = \{u^n | j \in \mathbb{Z}\}$ and

$$f^*(x) = \lim_{s \to \infty} \frac{1}{s} \int_0^s f(u_t x)\, dt$$

if $H = \{u_t\}_{t \in \mathbb{R}}$. Note that by Theorem 3.1 there exists a H-invariant subset N of X such that $\pi(N) = 0$ and for all $x \in X - N$ $f^*(x)$ is defined and $f^*(hx) = f(x)$ for all $h \in H$. Now

$$\int_X f^* d\pi = \sum_{i=1}^{\infty} \int_{X_i} f^* d\pi = \sum_{i=1}^{\infty} \int_{X_i} f d\pi = \int_X f d\pi.$$

Here the middle step follows from Theorem 3.1 since X_i is a H-invariant set of finite measure. Hence $f^* \in L^1(X, \pi)$ and $f^* \neq 0$. Since H normalises V, f^* is V-invariant. Since $U = H \cdot V$ it now follows that f^* is U-invariant. This proves the claim.

To complete the proof we proceed as follows. Let ν be a probability measure equivalent to π. Let $\mathscr{C}$ be the class of measurable sets E such that $E = \bigcup_1^\infty E_i$ where each E_i, $i = 1, 2 \dots$ is a measurable U-invariant set and $\pi(E_i) < \infty$. Put $\beta = \operatorname{Sup}\{\nu(E) | E \in \mathscr{C}\}$. A routine argument shows that there exists $E_0 \in \mathscr{C}$ such that $\nu(E_0) = \beta$. If $\beta = 1$ then clearly the theorem stands proved. Suppose $\beta < 1$. Consider the measure π' on $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ defined by $\pi'(E) = \pi(E \cap E_0^\sim)$ where $E_0^\sim$ is the complement of E_0. Clearly π' is a locally finite U-invariant (non-zero) measure. Hence there exists an integrable U-invariant function f such that $\int f d\pi' > 0$. It is obvious that the set $E_1 = \{x \in E_0^\sim | f(x) > 0\}$ belongs to $\mathscr{C}$ and $\nu(E_1) > 0$. This contradicts the definition of β since $E_0 \cup E_1 \in \mathscr{C}$ and $\nu(E_0 \cup E_1) > \beta$.

§ 4. Invariant Measures on Arithmetic Homogeneous Spaces

Theorem 3.3 can be readily generalised to more general arithmetic homogeneous spaces. Let G be a (connected) Lie group. A subgroup Γ of G is said to be *arithmetic* if there exists a linear $\mathbb{R}$-algebraic group L (the group of $\mathbb{R}$-rational elements) defined over $\mathbb{Q}$ and a surjective homomorphism φ of L^0, the connected component of the identity in L, onto G satisfying the following two conditions.

i) The kernel of φ is compact.

ii) Let $L_{\mathbb{Z}}$ be the subgroup consisting of all integral elements (with respect to the given $\mathbb{Q}$-structure) in L^0 with determinant ± 1. Then Γ is commensurable with $\varphi(L_{\mathbb{Z}})$ i.e. $\Gamma \cap \varphi(L_{\mathbb{Z}})$ is a subgroup of finite index in both Γ and $\varphi(L_{\mathbb{Z}})$.

A discrete subgroup Γ of a Lie group G is said to be a *lattice* if the homogeneous space G/Γ admits a finite G-invariant (Borel) measure. An arithmetic group which is also a lattice is called an *arithmetic lattice.*

Let G be a Lie group. A subgroup V of G is said to be *horospherical* if there exists $g \in G$ such that

$$V=\{u \in G \mid g^j u g^{-j} \to e \text{ as } j \to \infty\}$$

where e is the identity element in G. Any subgroup U of a horospherical subgroup V is said to be *horocyclic.*

We note that any horospherical subgroup is a nilpotent analytic subgroup. Further if G is an $\mathbb{R}$-algebraic group then any horospherical subgroup is a unipotent algebraic subgroup of G. A partial converse is also true; viz. any unipotent subgroup in a reducivte $\mathbb{R}$-algebraic group is horocyclic. These results are wellknown (cf. for instance, [5], §1 for a general idea of the proofs). The generalisation of Theorem 3.3 sought after is the following:

(4.1) **Theorem.** *Let G be a Lie group and Γ be an arithmetic lattice in G. Let U be a horocyclic subgroup of G. Let π be a locally finite U-invariant measure on G/Γ. Then there exists a seauence $\{X_i\}_{i=1}^{\infty}$ of measurable U-invariant subsets of G/Γ such that $\pi(X_i)<\infty$ for all $i=1,2,\ldots$ and $\pi\left(G/\Gamma-\bigcup_1^{\infty} X_i\right)=0$. In particular an ergodic, locally finite, U-invariant measure is finite.*

In the proof of the theorem we need the following lemma.

(4.2) **Lemma.** *Let $\varphi\colon G_1 \to G_2$ be a surjective homomorphism of Lie groups. Let U_2 be a horocyclic subgroup of G_2. Then there exists a horocyclic subgroup U_1 of G_1 such that $\varphi(U_1)=U_2$.*

Proof. Let V_2 be a horospherical subgroup containing U_2. Thus there exists $g_2 \in G_2$ such that

$$V_2=\{u_2 \in G_2 \mid g_2^j u_2 g_2^{-j} \to e_2 \text{ as } j \to \infty\}$$

e_2 being the identity element in G_2. Let $g_1 \in G_1$ be such that $\varphi(g_1)=g_2$ and let

$$V_1=\{u_1 \in G_1 \mid g_1^j u_1 g_1^{-j} \to e_1 \text{ as } j \to \infty\}$$

e_1 being the identity element in G_1. Indeed V_i, $i=1,2$ are analytic subgroups of G_i, $i=1,2$ respectively and the Lie subalgebra of V_i is the maximal $\operatorname{Ad} g_i$ invariant subspace $\mathscr{I}_i$ of the Lie algebra of G_i, on which all the eigenvalues of $\operatorname{Ad} g_i$ are of absolute value less than 1. Hence we must have $\varphi(V_1)=V_2$. Now choose $U_1=V_1 \cap \varphi^{-1}(U_2)$. Then $\varphi(U_1)=U_2$.

Proof of the Theorem. Firstly consider the case when G is a $\mathbb{R}$-algebraic group defined over $\mathbb{Q}$ and Γ is a subgroup of finite index in $G_{\mathbb{Z}}$. Since G admits a lattice there exists no non-trivial character (algebraic homomorphism into $GL(1)$) of G defined over $\mathbb{Q}$. Hence there exists a homomorphism ρ of G into $SL(n,\mathbb{R})$, for some n, defined over $\mathbb{Q}$ such that $\rho(\Gamma)$ is contained in $SL(n,\mathbb{Z})$ and the natural map $\bar{\rho}\colon G/\Gamma \to SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is proper (cf. [8], Proposition 10.15). Let π be a locally finite U-invariant measure on G/Γ. Then the measure $\bar{\rho}\pi$ on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, defined by $\bar{\rho}\pi(E)=\pi(\bar{\rho}^{-1}E)$ for any Borel set E, is locally finite and $\rho(U)$-invariant. Recall that U is a unipotent subgroup of G. Hence $\rho(U)$ is a unipotent subgroup of $SL(n,\mathbb{R})$. For the particular case at hand Theorem 4.1 follows immediately from Theorem 3.3.

We now consider the general case. By hypothesis there exists an $\mathbb{R}$-algebraic group L defined over $\mathbb{Q}$ and a surjective homomorphism φ: $L^0\to G$, with compact kernel such that $\varphi(L_{\mathbb{Z}})$ and Γ are commensurable. Hence there exists a subgroup Δ of finite index in $L_{\mathbb{Z}}$ such that $\varphi(\Delta)$ is a normal subgroup of Γ of finite index. Now let π be a locally finite U-invariant measure on G/Γ. We lift π to a measure on L^0/Δ as follows. Set $K=\ker\varphi$ and let m be the normalised Haar measure on K. Also let $\gamma_1,\gamma_2,\ldots,\gamma_p$ be a set of representatives of $\Gamma/\varphi(\Delta)$. For any bounded measurable function ψ on $G/\varphi(\Delta)$ let ψ_l be the function defined by $\psi_l(g\varphi(\Delta))=\psi(g\gamma_l\varphi(\Delta))$. Observe that since $\varphi(\Delta)$ is normal in Γ, ψ_l is a well-defined function on $G/\varphi(\Delta)$. Further $\sum_{l=1}^{p}\psi_l$ is constant over fibers of the natural map of $G/\varphi(\Delta)$ onto G/Γ. Now let $\theta\in C_c(L^0/\Delta)$. Put

$$\int_{L^0/\Delta}\theta(x\Delta)\,d\sigma=\int_{G/\Gamma}\left[\sum_{l=1}^{p}\left\{\int_K\theta(kx\Delta)\,dm(k)\right\}_l\right]d\pi.$$

(Functions constant over the fibers of a quotient map are viewed as functions on the quotient space in a natural way.) This defines a locally finite measure σ on L^0/Δ. It is straightforward to verify that σ is $\varphi^{-1}(U)$-invariant. By Lemma 4.2 there exists a horocyclic subgroup V of L^0 such that $\varphi(V)=U$. Since σ is V-invariant, by the special case of the theorem considered earlier it follows that there exists a sequence $\{Y_i\}_{i=1}^{\infty}$ of measurable U-invariant subsets such that $\sigma(Y_i)<\infty$ for all $i\in\mathbb{N}$ and $\sigma\left(L^0/\Delta-\bigcup_1^{\infty}Y_i\right)=0$. For any $i,j\in\mathbb{N}$ put

$$X_{ij}=\left\{\varphi(x)\Gamma\,\middle|\,\sum_{l=1}^{p}\left\{\int_K\chi_i(kx\Delta)\,dm(k)\right\}_l>\frac{1}{j}\right\}$$

where χ_i denotes the characteristic function of Y_i. Let $u\in U$ be given. Choose $v\in V$ such that $\varphi(v)=u$. Then

$$u\varphi(x)\Gamma=\varphi(vx)\Gamma$$

and

$$\begin{aligned}\int_K\chi_i(kvx\Delta)\,dm(k)&=\int_K\chi_i(v(v^{-1}kv)x\Delta)\,dm(k)\\&=\int_K\chi_i(vkx\Delta)\,dm(k)\\&=\int_K\chi_i(kx\Delta)\,dm(k).\end{aligned}$$

The last step follows since Y_i is V-invariant. Thus we deduce that for each i,j, X_{ij} is U-invariant. Further clearly for any i,j, $\pi(Y_i)>\pi(X_{ij})(1/j)$. Hence for each

$i, j, \pi(X_{ij}) < \infty$. Lastly since $\sigma\left(L^0/\Delta - \bigcup_1^{\infty} Y_i\right) = 0$ by Fubini's theorem for π-almost every x, $\sum_{l=1}^{p} \{\int_K \chi_i(kx\Delta)\, dm(k)\}_l > 0$ for some i; In other words, $\pi\left(G/\Gamma - \bigcup_{i,j=1}^{\infty} X_{ij}\right) = 0$, which completes the proof.

(4.3) *Remark.* The part of the assertion in Theorem 4.1 pertaining to ergodic invariant measures is also true for an arbitrary arithmetic subgroup which is not a lattice. This can be deduced as follows. Given an arithmetic subgroup Γ of a Lie group G there exists a closed normal subgroup H of G such that i) $H \cap \Gamma$ is of finite index in Γ, ii) $H \cap \Gamma$ is an arithmetic lattice in H and iii) H contains every horospherical subgroup of G. Indeed if G is a $\mathbb{R}$-algebraic group defined over $\mathbb{Q}$ and $\Gamma = G_{\mathbb{Z}}$ then the intersection of the kernels of all characters of G which are defined over $\mathbb{Q}$ has the above-mentioned properties. In general, the image under φ, the homomorphism as in the definition of arithmeticity, of the corresponding group has the requisite properties. Now let U be a horocyclic subgroup of G and let π be a locally finite, ergodic, U-invariant measure on G/Γ. The sets $xH\Gamma/\Gamma$, $x \in G$ form a measurable partition of G/Γ and each $xH\Gamma/\Gamma = Hx\Gamma/\Gamma$ is U-invariant. Hence by ergodicity of π there exists $x_0 \in G$ such that π is concentrated on $Hx_0\Gamma/\Gamma$; i.e. the complement has zero π-measure. But the restriction of the U-action on $Hx_0\Gamma/\Gamma$ is equivalent, in a natural way, to the action of U on $H/(x_0\Gamma x_0^{-1}) \cap H$. Also $(x_0\Gamma x_0^{-1}) \cap H = x_0(\Gamma \cap H)x_0^{-1}$ is an arithmetic lattice in H. Viewing π as a U-invariant measure on $H/x_0(\Gamma \cap H)x_0^{-1}$ and applying Theorem 4.1 we conclude that π is finite.

Presumably, the assertion in Theorem 4.1 for a non-ergodic U-invariant measure π is also true for any arithmetic subgroup and may be proved using a form of direct integral decomposition of π into ergodic measures (cf. [10]). However the author does not wish to go into the details.

(4.4) *Remark.* It may be pointed out that the analogue of Theorem 4.1 is generally not true for subgroups which are not horocyclic. We offer the following example. Let $G = SL(2, \mathbb{R})/\{\pm I\}$, $\Gamma = SL(2, \mathbb{Z})/\{\pm I\}$ and

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{R},\ a > 0 \right\}.$$

We first show that ΓH is closed. Let $\mathscr{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half plane and let S denote the space of all line elements (a point together with a unit tangent direction) over $\mathscr{H}$. The standard action of G on $\mathscr{H}$ as the group of non-Euclidian motions induces an action of G on S. Let s_0 be the line element at i in the direction of the imaginary axis. It is well-known that $g \to gs_0$ is a diffeomorphism of G onto S. Let Ω denote the set of all line elements over points in the set $\{z \in \mathbb{C} \mid |\operatorname{Re} z| \leqq \frac{1}{2}$ and $|z| \geqq 1\}$. It is easy to verify that Hs_0 is a closed subset of $\Omega \cup \omega\Omega$ where $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is well-known that (*cf.* [9], for instance) $S = \Gamma\Omega$ and that for any non-trivial element $\gamma \in \Gamma$, $\gamma\Omega \cap \Omega$ is contained in the boundary of Ω. Further there exist only finitely many elements in Γ such that $\gamma\Omega \cap \Omega$ is non-empty. Using these properties of Ω it is straightforward to verify that $\Gamma H s_0$ is closed. Hence ΓH must be a closed subset of G.

Now consider the action of H on G/Γ (on the left). In view of the above $H\Gamma/\Gamma$ is a closed H-orbit, which is obviously not periodic. Let λ' be a (non-zero) H-invariant measure on $H\Gamma/\Gamma$ and let λ be the measure on G/Γ supported on $H\Gamma/\Gamma$, whose restriction to the latter equal $s\lambda'$. Then λ is clearly an infinite, locally finite, ergodic, H-invariant measure on G/Γ.

Before concluding the section the author would like to thank the referee, whose suggestions motivated the present general form of the results in this section. In an earlier manuscript the author had employed a more restrictive definition of arithmeticity.

§ 5. Invariant Measures of Horospherical Flows

We now relate the results of the preceding sections to the work in [3]. Let G be a reductive Lie group and Γ be a lattice in G; i.e. G/Γ admits a finite G-invariant measure. In [3] we obtained a classification of *finite* invariant measures of maximal horospherical flows (see introduction for definitions) on G/Γ when G has no simple factors of $\mathbb{R}-\text{rank} \geq 2$. In view of Theorem 4.1 when Γ is arithmetic (with respect to some $\mathbb{Q}$-structure) then the assumption about finiteness may be omitted. Thus we have the following.

(5.1) **Theorem.** *Let G be a reductive $\mathbb{R}$-algebraic group defined over $\mathbb{Q}$ and let Γ be an arithmetic lattice in G. Assume that every simple non-compact factor of G is of $\mathbb{R}-\text{rank}\ 1$. Let U be a maximal horospherical subgroup of G and let π be an ergodic U-invariant (locally finite) measure on G/Γ. Then there exists a closed subgroup L and $g \in G$ such that*

i) *$Lg\Gamma$ is closed and $\pi(G/\Gamma - Lg\Gamma/\Gamma) = 0$ and*

ii) *L contains U, $g\Gamma g^{-1} \cap L$ is a lattice in L and π is the L-invariant measure on $Lg\Gamma/\Gamma \simeq L/g\Gamma g^{-1} \cap L$.*

(5.2) *Remark.* The proof in [3] also shows that the subgroup L above has the following property: there exists a (closed) normal subgroup V of G such that $V \cap \Gamma$ is a lattice in V and $L = UV$.

§ 6. Invariant Measures of Arithmetic Groups

The usefulness of Theorem 4.1 is more apparent when we consider Γ-invariant masures on G/U where Γ is an arithmetic lattice in G and U a maximal horospherical subgroup. We need the following duality principle due to H. Furstenberg. (cf. [3] for details)

(6.1) **Proposition.** *Let G be a Lie group and let U and Γ be any closed unimodular subgroups of G. Then there exists a canonical one-one correspondence $\pi \leftrightarrow \sigma$ of U-invariant measures on G/Γ and Γ-invariant measures on G/U such that for any $\varphi \in C_c(G)$*

$$\int_{G/\Gamma} d\pi(x\Gamma) \int_{\Gamma} \varphi(x\gamma)\,d\gamma = \int_{G/U} d\sigma(xU) \int_{U} \check{\varphi}(xu)\,du$$

where $d\gamma$ and du denote (fixed) Haar measures on Γ and U respectively and $\check{\varphi}$ denotes the function defined by $\check{\varphi}(g)=\varphi(g^{-1})$.

Under the above correspondence ergodic U-invariant measures on G/Γ correspond to ergodic Γ-invariant measures on G/U. Unfortunately the condition of finiteness of U-invariant measures on G/Γ does not correspond to any intrinsic property of Γ-invariant measures on G/U. In [3] we introduced the following definition.

Definition. Let G be a Lie group and let U and Γ be closed unimodular subgroups of G. A Γ-invariant measure σ on G/U is said to be Γ-finite if the U-invariant measure π on G/Γ corresponding to σ under the one-one correspondence as in Proposition 6.1 is finite.

Now let G be a reductive $\mathbb{R}$-algebraic group defined over $\mathbb{Q}$, and let Γ be an arithmetic lattice in G. Let U be a maximal horospherical subgroup of G. Assume that U contains a subgroup V which is a maximal horospherical subgroup defined over $\mathbb{Q}$. (This assumption amounts to choosing a suitable base point on the homogeneous space G/U.) By a $\mathbb{Q}$-rational horospherical subgroup opposite to V we mean a subgroup of the form

$$V^{-}=\{h\in G \mid g^{-j}hg^{j}\to e \text{ as } j\to\infty\}$$

where $g\in G$ is such that

$$V=\{h\in G \mid g^{j}hg^{-j}\to e \text{ as } j\to\infty\}$$

e being the identity element. We note that such a subgroup exists but not in general unique. By a $\mathbb{Q}$-rational $\mathbb{R}$-parabolic subgroup P^{-} opposite to V we mean the normaliser of a subgroup of the form V^{-} as above.

Now fix a minimal $\mathbb{Q}$-rational $\mathbb{R}$-parabolic subgroup P^{-} opposite to V. Let K be a maximal compact subgroup of G. Since Γ is a lattice in G there exists a finite subset J of G and a 'Siegel set Ω corresponding to the triple (K, V, P^{-})' (i.e. a set of the form $WS_{\delta}K$ as in Proposition 2.3 in [3]) such that $G=\Gamma J\Omega$. A finite subset J of G for which the above is satisfied shall be called a *sufficient set of cusp elements* for Γ with respect to (K, V, P^{-}).

It is well-known that $P^{-}V$ is an open subset of G and that $G-P^{-}V$ is a finite union of lower dimensional submanifolds. Hence the G-invariant measure on G/U assigns zero measure to $G/U-gP^{-}U/U$ for any $g\in G$. In [3] we proved the following converse statement (cf. Theorem 2.4, [3]) for Γ-finite, Γ-invariant measures σ; in view of Theorem 4.1 the assumption of Γ-finiteness is redundant for arithmetic groups.

(6.2) **Theorem.** *Let the notations G, Γ, U, V, K and P^{-} be as above. Let J be a sufficient set of cusp elements for Γ with respect to (K, V, P^{-}). Let σ be an ergodic Γ-invariant (locally finite) measure on G/U. If $\sigma(G/U-jP^{-}U/U)=0$ for all $j\in J$ then σ is G-invariant.*

We now deduce the following result.

(6.3) **Theorem.** *Consider the natural action of $\Gamma = SL(n,\mathbb{Z})$ on $\mathbb{R}^n$ via a basis $\{e_1, e_2, \ldots, e_n\}$. Then any ergodic Γ-invariant (locally finite) measure σ on $\mathbb{R}^n$ is a scalar multiple of either the Lebesgue measure or the counting mesire on the discrete orbit of a point $x \in \mathbb{R}^n$ of the form $x = t \sum_{i=1}^{n} q_i e_i$ where $t \in \mathbb{R}$ and $q_i \in \mathbb{Q}$ for $i = 1, 2, \ldots, n$.*

Proof. We shall actually prove the following slightly stronger assertion: Let σ be an ergodic Γ-invariant locally finite measure on $\mathbb{R}^n - (0)$ (i.e. a priori the measure of a neighbourhood of 0 may not be finite). Then σ satisfies the contention of the theorem.

We proceed by induction on n. If $n = 1$ the assertion is obvious. Now consider the action of $\Gamma = SL(n,\mathbb{Z})$ on $\mathbb{R}^n$ via a basis $\{e_1, e_2, \ldots, e_n\}$. Let G denote the group $SL(n,\mathbb{R})$. Let Q be the isotropy subgroup at e_1 of the action of G on $\mathbb{R}^n - (0)$ via the basis $\{e_1, e_2, \ldots, e_n\}$. Since the action of G on $\mathbb{R}^n - (0)$ is transitive $\mathbb{R}^n - (0)$ may be identified with G/Q. Let U be the subgroup of G consisting of all upper triangular unipotent matrices. Then U is a maximal horospherical subgroup of G and it is contained in Q. The measure σ, considered on G/Q via the identification, is Γ-invariant and by Proposition 6.1 corresponds canonically to a Q-invariant measure π on G/Γ. In particular π is a U-invariant and hence corresponds to a Γ-invariant measure on $\tilde{\sigma}$ on G/U.

Now let P^- denote the subgroup consisting of all lower triangular matrices. Then it is easy to verify that P^- is a $\mathbb{Q}$-rational $\mathbb{R}$ parabolic subgroup opposite to U. Also note that Γ is a lattice in G and that $\{e\}$ is a sufficient set of cusp elements for Γ; (cf. [8], ch. X). Hence by Theorem 6.2 $\tilde{\sigma}$ is G-invariant if and only if $\tilde{\sigma}(G/U - P^- U/U) = 0$. It is also easy to verify that $\tilde{\sigma}(G/U - P^- U/U) = 0$ if and only if $\sigma(V - (0)) = 0$ where V is the subspace of $\mathbb{R}^n$ generated by $\{e_2, e_3, \ldots, e_n\}$. Thus if $\sigma(V - (0)) = 0$ then $\tilde{\sigma}$ is G-invariant; that is, σ is the Lebesgue measure (up to a scalar multiple).

Next suppose that $\sigma(V - (0)) > 0$. Let $\mathscr{W}$ denote the class of all proper subspaces W of $\mathbb{R}^n$ such that W is generated (as $\mathbb{R}$-subspace) by certain elements of the form $\sum_{i=1}^{n} q_i e_i$, where $q_i \in \mathbb{Q}$, and $\sigma(W) > 0$. Then $\mathscr{W}$ is non-empty since $V \in \mathscr{W}$. Let $W \in \mathscr{W}$ be an element of minimum possible dimension. Let $\Gamma' = \{\gamma \in \Gamma \mid \gamma(W) = W\}$ and $\Gamma'' = \{\gamma \in \Gamma \mid \gamma/W = Id\}$. Then Γ'/Γ'' is isomorphic to $SL(m,\mathbb{Z})$ where m is the dimension of W. Also the action of Γ' on W corresponds to the natural action via a basis $\{f_1, f_2, \ldots, f_m\}$ where each f_j, $j = 1, 2, \ldots, m$ is of the form $\sum_{i=1}^{m} q_i e_i$ where $q_i \in \mathbb{Q}$.

Consider the measure σ' on $W - (0)$ obtained by restriction of σ (i.e. $\sigma'(E) = \sigma(E)$ for any Borel subset E of $W - (0)$). Then σ' is Γ'-invariant. We claim that σ' is ergodic under the Γ'-action. Let E be a measurable Γ' invariant set such that $\sigma'(E) > 0$. Let $\gamma \in \Gamma - \Gamma'$ be arbitrary. Then $\gamma E \cap W \subset \gamma W \cap W \neq W$. Since $W \in \mathscr{W}$ is of minimum possible dimension $\gamma W \cap W \notin \mathscr{W}$. This is possible only if $\sigma(\gamma W \cap W) = 0$. Now consider $E' = \bigcup_{\gamma \in \Gamma} \gamma E$. Since σ is ergodic under the Γ-action and $\sigma(E')\geqq\sigma(E)>0$ we must have $\sigma(\mathbb{R}^n-(0)-E')=0$. In particular $\sigma'(W-(0)-E')=0$. Hence $\sigma'(W-(0)-E)=\sigma(W-(0)-E')+\sigma(W\cap(E'-E))=0$. Hence σ' is ergodic with respect to the Γ'-action. By induction hypothesis (upto a scalar multiple) σ' is either the (restriction of the) Lebesgue measure on W or the counting measure on the discrete orbit of a point $x=t\sum_{j=1}^{m} q'_j f_j=t\sum_{i=1}^{n} q_i e_i$ where $q_i, q'_j\in\mathbb{Q}$ and $t\in\mathbb{R}$. In the latter case clearly σ is the counting measure (upto a scalar) on the (discrete) Γ-orbit of x. We complete the proof by showing that the former is impossible.

Suppose that σ' is the (restriction of the) Lebesgue measure on W. We fix a norm $\|.\|$ on $\mathbb{R}^n$. For any subspace L of $\mathbb{R}^n$ let λ_L denote the Lebesgue measure on L such that a parallelepiped whose vertices include an orthonormal basis and 0 is assigned unit measure. By normalising σ if necessary we may assume $\sigma'=\lambda_W$. Now let

$$A=\{x\in\mathbb{R}^n \mid 1\leqq\|x\|\leqq 2\}$$

It is easy to verify that there exists $\gamma\in\Gamma$ such that $\gamma^j\notin\Gamma'$ for any $j\in\mathbb{Z}-(0)$. As seen before since $W\in\mathscr{W}$ is of minimum possible dimension, $\sigma(W\cap\gamma^j W)=0$ for all $j\in\mathbb{Z}-(0)$. Hence

$$\begin{aligned}\sigma(A)&\geqq\sum_{-\infty}^{\infty}\sigma(\gamma^j W\cap A)=\sum_{-\infty}^{\infty}\sigma'(W\cap\gamma^{-j}A)\\&=\sum_{-\infty}^{\infty}\lambda_W(W\cap\gamma^{-j}A)\\&=\sum_{-\infty}^{\infty}\lambda_{\gamma^j W}(\gamma^j W\cap A)\cdot J(\gamma^{-j})\end{aligned}$$

where $J(\gamma^{-j})$ is the Jacobian of the linear transformation $\gamma^{-j}/\gamma^j W\colon \gamma^j W\to W$, equipped with norms obtained by restriction of $\|.\|$. Now let $\{h_1, h_2, \ldots, h_m\}$ be an orthonormal basis of W and $h=h_1\wedge h_2\wedge\ldots\wedge h_m\in\bigwedge^m W\subset\bigwedge^m\mathbb{R}^n$. Then by Lemma 1.4 $J(\gamma^{-j})=\|(\bigwedge^m\gamma^{-j})h\|$. Also note that by choice of A for any subspaces W', W'' of dimension m, $\lambda_{W'}(W'\cap A)=\lambda_{W''}(W''\cap A)=C$ say. Hence

$$\sigma(A)\geqq\sum_{-\infty}^{\infty}C\|(\bigwedge^m\gamma^{-j})h\|=C\sum_{-\infty}^{\infty}\|(\bigwedge^m\gamma)^j h\|.$$

Since σ is locally finite $\sigma(A)<\infty$. Hence $(\bigwedge^m\gamma)^j h\to 0$ both as $j\to+\infty$ and as $j\to-\infty$. But since $\bigwedge^m\gamma$ is a non-singular linear transformation of $\bigwedge^m\mathbb{R}^n$ and $h\neq 0$ this is impossible.

6.4 *Remark.* The hypothesis in Theorem 6.3 that σ be locally finite is irredundant. In view of the results in [4] there exist uncountably many distinct ergodic, Γ-invariant, σ-finite, non-atomic measures on $\mathbb{R}^n$ which are not locally finite.

§ 7. Closed Invariant Sets and Minimal Sets

One of the authors schief motivation (or justification!) in classifying invariant measures of unipotent subgroups has been that such a study would reflect on the structure of invariant subsets under the action. For instance, since any compact minimal subset of the action of a nilpotent group is the support of an ergodic invariant measure for the action, classification of the latter as in Theorem 5.1 automatically determines all compact minimal subsets for the particular action. The results of preceding sections throw more light in that direction.

(7.1) **Proposition.** *Let G be a Lie group and let Γ be an arithmetic lattice in G. Let U be a horospherical subgroup of G. Let C be a non-empty closed U-invariant subset of G/Γ. Then there exists a U-invariant probability measure π on G/Γ such that the support of π is contained in C;* i.e. $\pi(G/\Gamma - C)=0$.

Remark. If C is compact this result follows from general results in the line of Krylov-Bogoliubov theorem which follow from fixed point theorems. Such general results, however, are not true for actions on non-compact spaces.

Proof of the Proposition. For simplicity we only consider the case when $G = SL(n,\mathbb{R})$ and $\Gamma = SL(n,\mathbb{Z})$. The general result can be deduced using the same techniques as in § 4. Clearly we may assume U to be a closed subgroup. As in the proof of Theorem 3.3 we proceed by induction on the rank of U. If the rank is 0 then U is the identity subgroup and the contention is obvious. Now assume that the result holds for all closed unipotent subgroups of rank $\leqq m-1$ and let U be a closed unipotent subgroup of rank m. As noted in the proof of Theorem 3.3 U contains a closed normal subgroup V of rank $m-1$ such that U/V is either cyclic or a one parameter group. Also there exists a subgroup H of U such that $U = H\cdot V$ (semi-direct product). By induction hypothesis there exists a V-invariant probability measure π on G/Γ such that support of π is contained in C.

Now let $\{K_q\}_{q=1}^{\infty}$ be a sequence of compact sets such that for each $q\in\mathbb{N}$, K_q is contained in the interior of K_{q+1} and $\bigcup_{q=1}^{\infty} K_q = G/\Gamma$. Let $u\in H$ be chosen so that $u \neq e$ and so that u generates H if H is cyclic. For any $q\in\mathbb{N}$ put

$$C_q = \left\{ x\in C \;\middle|\; \frac{1}{r}\sum_{j=0}^{r-1} \chi_q(u^j x) \geqq \frac{1}{q} \text{ for all } r\in\mathbb{N} \right\}$$

where χ_q denotes the characteristic function of K_q. Clearly C_q, $q\in\mathbb{N}$ are Borel subsets of C and in view of Theorem 2.11 $C = \bigcup_{q=1}^{\infty} C_q$. In particular we conclude that there exists $\bar{q}\in\mathbb{N}$ such that $\pi(C_{\bar{q}})>0$.

We now define a measure π' as follows. Let $\{\varphi_i\}_{i=1}^{\infty}$ be an everywhere dense sequence in $C_c(G/\Gamma)$ (in the 'supremum norm' topology). Put

$$S(i,r) = \frac{1}{r}\sum_{j=0}^{r-1} \int \varphi_i(u^j x)\, d\pi(x).$$

For a fixed $i \in \mathbb{N}$ the set $\{S(i,r) | r \in \mathbb{N}\}$ is bounded. Hence by the standard procedure of passing to subsequences we get a sequence $\{r_l\}_{l=1}^{\infty}$ such that for any $i \in \mathbb{N}$, $S(i, r_l)$ converges as $l \to \infty$. Since $\{\varphi_i\}_{i=1}^{\infty}$ is everywhere dense it follows that for any $\varphi \in C_c(G/\Gamma)$ the sequence

$$\frac{1}{r_l} \sum_{j=0}^{r_l-1} \int \varphi(u^j x)\, d\pi(x)$$

converges to a limit as $l \to \infty$. Thus we can define a measure π' on G/Γ by setting the above limit to be $\int \varphi\, d\pi'$. Let $\psi \in C_c(G/\Gamma)$ be such that $\psi(y) \geqq 0$ for all $y \in G/\Gamma$ and $\psi(x) = 1$ for all $x \in K_{\bar{q}}$. Then

$$\begin{aligned}\int \psi\, d\pi' &= \lim_{l\to\infty} \frac{1}{r_l} \sum_{j=0}^{r_l-1} \int \psi(u^j x)\, d\pi(x) \\ &= \lim_{l\to\infty} \int \frac{1}{r_l} \sum_{j=0}^{r_l-1} \psi(u^j x)\, d\pi(x) \\ &\geqq \frac{1}{\bar{q}}\, \pi(C_{\bar{q}}) > 0.\end{aligned}$$

This shows that π' is indeed a non-zero measure. Also clearly $\pi'(G/\Gamma) \leqq 1$. By normalising π' we get a probability measure which also we denote by π'. It is obvious that support of π' is contained in C. Since u normalises V and π is V-invariant it follows that π' is V-invariant. Also it is easy to verify that π' is u-invariant. Since $U = H \cdot V$ this completes the proof if H is cyclic. If $H = \{u_t\}_{t \in \mathbb{R}}$ and $u = u_1$ then we define a measure π'' as follows. For any $\varphi \in C_c(G/\Gamma)$ put

$$\int \varphi(x)\, d\pi''(x) = \int \left[\int_0^1 \varphi(u_t x)\, dt\right] d\pi(x).$$

Then it is easy to see that π'' defines a $\{u_t\}_{t\in\mathbb{R}}$-invariant probability measure whose support is contained in C. Also since H normalises V, π'' is V-invariant. Finally since $U = H \cdot V$, π'' is U-invariant.

(7.2) **Corollary.** *Let G, Γ and U be as in Theorem* 5.1. *Let C be a minimal (closed non-empty) U-invariant subset of G/Γ. Then there exists a closed subgroup L of G containing U and $g \in G$ such that $C = Lg\Gamma/\Gamma$. The subgroup L also satisfies the condition in Remark* 5.2.

Proof. By Proposition 7.1 there exists a U-invariant probability measure π on G/Γ such that the support of π is contained in C. By ergodic decomposition of π (cf. [10]) we obtain an ergodic U-invariant measure π_1 on G/Γ whose support is contained in C. Hence by Theorem 5.1 there exists a closed subgroup L containing U and $g \in G$ such that $Lg\Gamma$ is closed and the support of π_1 equals $Lg\Gamma/\Gamma$. Hence $Lg\Gamma/\Gamma$ is contained in C. Since C is minimal and $Lg\Gamma/\Gamma$ is closed and U-invariant we conclude that $C = Lg\Gamma/\Gamma$.

Observe that there is a canonical one-one correspondence $C \leftrightarrow D$ between closed U-variant subsets of G/Γ and closed Γ-invariant subsets of G/U given by

$$D = \{gU | g^{-1}\Gamma \in C\}.$$

Hence the Corollary also determines minimal Γ-invariant closed subsets of G/U.

We can derive a similar corollary from Theorem 6.2. Let G, Γ, U and P^- be as in Theorem 6.2. Recall that $G/U - P^- U/U$ is a finite union of lower dimensional manifolds. We prove:

(7.3) **Corollary.** *Let G, Γ, U and P^- be as in Theorem 6.2. Let J be a sufficient set of cusp elements for Γ. Let Y be a non-empty proper closed Γ-invariant subset of G/U. Then there exists $j \in J$ such that $Y \cap (G/U - jP^- U/U)$ is non-empty.*

Proof. Let C be the closed U-invariant subset of G/Γ corresponding to the Γ-invariant set Y, under the one-one correspondence noted above. By Proposition 7.1 there exists a U-invariant probability measure π on G/Γ whose support is contained in C. In view of ergodic decomposition the measure may be assumed to be ergodic. Let σ be the ergodic Γ-invariant measure on G/U corresponding to π under the one-one correspondence as in Proposition 6.1. Then support of σ is contained in Y. Now if $\sigma(G/U - jP^- U/U) = 0$ for all $j \in J$ then by Theorem 6.2 σ must be G-invariant. Since Y is proper there exists $j \in J$ such that $\sigma(G/U - jP^- U/U) > 0$; In particular $Y \cap (G/U - jP^- U/U)$ is non-empty.

Let G be a Lie group and Γ be a (not necessarily arithmetic) uniform lattice in G; i.e. G/Γ is compact. Then it is well-known that every orbit of a horospherical subgroup U on G/Γ (and equivalently every Γ-orbit on G/U) is dense. There are several proofs for this in literature, including one of the present author's (cf. [2] Proposition 4.6). The proof in [6] is perhaps the most elegent. On the other hand in the notation as in Corollary 7.3, for an arithmetic lattice $G/U - P^- U/U$ is non-empty if and only if Γ is non-uniform. Thus Corollary 7.3 may be viewed as a generalisation of the above result for uniform lattices to a not necessarily uniform arithmetic lattice. We also note here for the benefit of the uninitiated that in a certain sense majority of the lattices are arithmetic.

Arguments along the lines of Corollary 7.3 using Theorem 6.3 yield that under the natural action of $\Gamma = SL(n, \mathbb{Z})$ on $\mathbb{R}^n$ via a basis $\{e_1, e_2, \ldots, e_n\}$ every closed Γ-invariant subset of $\mathbb{R}^n$ contains a point of the form $t \sum_{i=1}^{n} q_i e_i$ where $t \in \mathbb{R}$ and $q_i \in \mathbb{Q}$. In this case, however, this result is weaker than the following result in [1] (af. Theorem 5.2 in [1]): The Γ-orbit of $x \in \mathbb{R}^n$ is not dense in $\mathbb{R}^n$ if an only if x is of the form $t \sum_{i=1}^{n} q_i e_i$ where $t \in \mathbb{R}$ and $q_i \in \mathbb{Q}$.

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