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Invariant Measures and Minimal Sets of Horospherical Flows

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The class of all invariant measures of a transformation, or a flow, is an important aspect of its dynamics. Thus for instance if T is a homeomorphism of a locally compact space X then any compact minimal (non-empty) T-invariant set is the support of an invariant measure. Since $x \in X$ is an almost periodic point of T if and only if the closure of its orbit is a compact minimal set (cf. [11] Proposition 2.5), the class of invariant measures also determines the set of all almost periodic points. On the other hand, for any $x \in X$ the limit points of the sequence $n^{-1} \sum_{j=0}^{n-1} \delta_{T^j x}$ (where δ_y denotes the point measure based at $y \in X$) are T-invariant measures. In a certain intuitive sense these limiting measures describe how the orbit of x is "distributed" in X. Consider the particular case when T is a homeomorphism of a compact second countable space X such that T admits a unique invariant probability measure, say μ; such a transformation is said to be uniquely ergodic. If Y is the support of μ then for any $y \in Y$ the orbit is dense in Y and is uniformly distributed with respect to μ in the sense that for any continuous function φ the averages $n^{-1} \sum_{j=0}^{n-1} \varphi(T^j x)$ converge to $\int \varphi \, d\mu$. The classical theorem of H. Weyl, which asserts that if α is an irrational number then the sequence of fractional parts of $k\alpha$, $k=1,2\ldots$ is uniformly distributed, is indeed a simple consequence of the fact that the rotation of the circle by an angle $2\pi\alpha$, with α as above, is a uniquely ergodic transformation.

In [13], H. Furstenberg investigated which affine transformations of tori are uniquely ergodic. The study was later extended by W. Parry [20] to affine transformations of compact nilmanifolds. In each of these cases necessary and sufficient conditions for unique ergodicity have been obtained. It is also straightforward to extend the theory to translation flows induced by one-parameter subgroups.

A natural question arising at this stage is whether a similar phenomenon holds for transformations and flows on homogeneous spaces of semisimple or, more generally, reductive Lie groups. It may be noted that the geodesic and horocycle flows associated to surfaces of constant negative curvature fall in this class of flows.

Again, Furstenberg in [14] inaugurated the general area under consideration by proving that the horocycle flow associated to a *compact* surface of constant negative curvature is uniquely ergodic. A theorem of G.A. Hedlund asserting that the horocycle flow is minimal (i.e. every orbit is dense) falls out as an easy consequence of the above result.

Let G be a reductive Lie group; i.e. the adjoint representation is completely reducible. A discrete subgroup Γ of G is called a *lattice* in G, if G/Γ admits a finite G-invariant measure. We note that in general such a homogeneous space may or may not be compact. The lattice Γ is said to be *uniform* or *non-uniform* depending on whether G/Γ is compact or non-compact respectively. Let us now consider the special case $G = SL(2,\mathbb{R})$, the group of real 2×2 unimodular matrices. Let Γ be a lattice in $SL(2,\mathbb{R})$. Consider the action of the one-parameter subgroup N consisting of all upper triangular unipotent matrices, on $SL(2,\mathbb{R})/\Gamma$. Any horocycle flow associated to a surface of constant negative curvature and finite volume can be realised in this fashion for a suitable lattice Γ. If Γ is a uniform lattice then Furstenberg's result asserts that the action of N is uniquely ergodic. If $SL(2,\mathbb{R})/\Gamma$ is non-compact then, as a rule, the horocycle flow admits periodic trajectories and hence fails to be uniquely ergodic. However, it was proved by the present author in [8] that apart from the $SL(2,\mathbb{R})$-invariant measure the only other ergodic finite invariant measures for the horocycle flow are those based on the periodic trajectories. As a consequence, it can be deduced that the periodic trajectories are the only compact minimal (invariant, nonempty) sets. Alternatively, every almost periodic point of the horocycle flow is periodic.

In the light of these results for lattices in $SL(2,\mathbb{R})$ and other results discussed in the sequel it seems reasonable to expect the following:

I. Conjecture. *Let G be a reductive Lie group and Γ be a lattice in G. Let $\{u_t\}$ be a one-parameter subgroup in G such that for each t,* $\mathrm{Ad}\,u_t$ *is a unipotent matrix* ("Ad" *stands for the adjoint representation*). *Let σ be a $\{u_t\}$-invariant ergodic measure on G/Γ. Then there exist, a connected Lie subgroup L containing $\{u_t\}$ and $x\in G$, such that $Lx\Gamma/\Gamma$ is a closed subset of G/Γ and σ is a finite L-invariant measure supported on $Lx\Gamma/\Gamma$.*

The preceding discussion shows that the conjecture is satisfied in the case of $SL(2,\mathbb{R})$, provided σ is finite. In fact we only need to choose L to be either $SL(2,\mathbb{R})$ or N itself. It is not difficult to see that in general there could exist proper closed subgroups L, containing N as a proper subgroup, which indeed correspond to N-invariant ergodic measures as in the conjecture (cf. Corollary 2.4). In the case of $SL(2,\mathbb{R})$ this is ruled out by the fact that there exists no unimodular subgroup with desired proper inclusions.

Validity of the conjecture would also mean that every compact minimal $\{u_t\}$-invariant set is of the form $Lx\Gamma/\Gamma$ (in the notation as before). From the point of view of understanding the dynamics of these flows and certain applications it would be appropriate to prove the following stronger assertion.

II. Conjecture. *Let G,Γ and $\{u_t\}$ be as in Conjecture I. Then the closure of any orbit of $\{u_t\}$ in G/Γ is of the form $Lx\Gamma/\Gamma$ for a suitable connected Lie subgroup L of G.*

This conjecture is due to M.S. Raghunathan (oral communication), who also informed the author that its validity would yield a proof of a conjecture due to Davenport, on the density of the set of values of certain quadratic forms, at integral points.

In view of Hedlund's result quoted earlier Conjecture II is satisfied for uniform lattices in $SL(2,\mathbb{R})$. By a result of J.S. Dani [6] it is also true for the lattice $SL(2,\mathbb{Z})$ (cf. [10] for a more general result). These and certain easy consequences thereof are the only instances known to the author where the conjecture is satisfied for any one-parameter subgroup at all.

The main feature which distinguishes the one-parameter subgroup N of upper triangular unipotent matrices in $SL(2,\mathbb{R})$ is that it is a "horospherical subgroup". A subgroup U of a Lie group G is said to be *horospherical* if there exists $g\in G$ such that

$$U=\{u\in G\,|\,g^j u g^{-j}\to e \quad \text{as} \quad j\to\infty\}$$

where e is the identity element. There have been certain generalisations of Furstenberg's theorem from this point of view. W.A. Veech in [24] showed that if G is a semisimple Lie group without compact factors and Γ is a uniform lattice in G then the action of a maximal horospherical flow is uniquely ergodic. Subsequently R. Bowen [5] and R. Ellis and W. Perrizo [12] obtained similar results for the actions of all, not necessarily maximal, horospherical subgroups.

In [8] the author adapted the technique of Furstenberg [14] and Veech [24] to make it suitable for classification of ergodic invariant measures of maximal horospherical subgroups. Let G be a reductive Lie group and Γ be a lattice in G. Let N be a maximal horospherical subgroup of G. The main result in [8] consists of a characterisation of the G-invariant measures on G/N in the class of Γ-invariant ergodic measures (cf. §3 for details). Using the characterisation the author was, in particular, able to conclude that if G is a simple Lie group of $\mathbb{R}$-rank 1, then every finite N-invariant ergodic measure on G/Γ is either G-invariant or it is the measure supported on a compact orbit of N. In the present paper using the characterisation mentioned above we obtain a complete classification of N-invariant ergodic measures on G/Γ. We prove the following.

(9.1) **Theorem.** *Let G be a reductive Lie group and Γ be a lattice in G. Let N be a maximal horospherical subgroup of G. Let σ be a N-invariant ergodic probability measure. Then there exist, a connected Lie subgroup L of G containing N, and $x\in G$, such that* i) *$Lx\Gamma/\Gamma$ is a closed subset of G/Γ and* ii) *σ is L-invariant and is supported on $Lx\Gamma/\Gamma$.*

We note that conditions i) and ii) completely determine the measure σ. The theorem means that the analogue of Conjecture I is true for the actions of maximal horospherical subgroups, provided we restrict ourselves to *finite* N-invariant measures. For "arithmetic" lattices every N-invariant ergodic measure is finite [9]. We do not know whether this is true in general.

It is not difficult to show that if the group G as in Theorem 9.1 is also a semisimple group without compact factors and Γ is a uniform lattice then the subgroup L would have to be G itself. Actually in the sequel we obtain an explicit discription of the subgroup L from which also it is evident that in the above-mentioned special case L coincides with G (cf. Remark 8.4). Thus Theorem 9.1 extends Veech's theorem on unique ergodicity of maximal horospherical flows [24].

The general idea of the proof of Theorem 9.1 consists of the following. If the measure σ is not invariant under G^0, the connected component of the identity in G, then by the characterisation of Haar measures referred to above (cf. Theorem 3.4) σ is concentrated on a certain lower dimensional set. Using certain techniques from the theory of algebraic groups we show that it is possible to choose a subgroup with an orbit of positive measure such that the condition characterising the Haar measures is satisfied.

The contents of the paper are organised as follows. In Part I consisting of §§1 to 4 we discuss various preliminaries. In §1 we recall various results regarding measures on groups and homogeneous spaces. In §2 we give a description of a canonical class of invariant measures which are prototypes for the classification as in Theorem 9.1. §3 is devoted to recalling (cf. Theorem 3.4) the characterisation of the Haar measures from [8]. In §4 we obtain a generalisation (cf. Theorem 4.1) of W. Parry's result on unique ergodicity of flows on compact nilmanifolds. Though it is proved expressly for use in §6, the result also seems to be of independent interest. Part II, which is the crucial part in the paper, constitutes a proof of Theorem 9.1 for arithmetic lattices (cf. §2 for definition) in reductive groups satisfying the conditions of Theorem 3.4. In Part III we deduce the general result and discuss applications to minimal sets of horospherical flows, orbits of arithmetic groups etc.

The text also includes certain *corrections* to [8] (cf. Remarks 3.6 and 8.3).

Before concluding this introduction the author would like to thank Gopal Prasad for clarification of certain points involving the theory of algebraic groups. The author also thanks the referee for useful suggestions.

Part I: Preliminaries

§ 1. Measures on Homogeneous Spaces

(1.1) By a *measure* on a locally compact second countable space X, we man a positive locally finite Borel measure. The set of all measures on X is denoted by $\mathcal{M}(X)$. Let G be a locally compact (second countable, topological) group and let $\Phi: G\times X \to X$ be a measurable G-action. The subset of $\mathcal{M}(X)$ consisting of G-invariant measures under the action Φ is denoted by $\mathcal{M}(X, G)$, provided the action involved is clear from the context. Let G be a locally compact group and H and K be two closed subgroups of G. A measure on G is said to be (H, K)-invariant if it is invariant under the left action of H and also under the right action of K. The set of (H,K)-invariant measures on G is denoted by $\mathcal{M}(G, H, K)$. Let $\mathrm{d}k$ be a fixed Haar measure on K. We define a map $l_K: \mathcal{M}(G/K) \to \mathcal{M}(G)$ as follows. For any $\pi \in \mathcal{M}(G/K)$, $l_K(\pi)$ is defined to be the measure such that

(1.2) $$\int_G f(x)\,\mathrm{d}l_K(\pi)(x) = \int_{G/K} \int_K f(xk)\,\mathrm{d}k\,\mathrm{d}\pi(xK)$$

for all $f \in C_c(G)$, the space of continuous functions with compact support. The following fact was observed in [14].

(1.3) **Proposition.** *Suppose that K is a closed unimodular subgroup of a locally compact group G. Then for any $\pi \in \mathcal{M}(G/K)$, $l_K(\pi) \in \mathcal{M}(G, \{e\}, K)$, where $\{e\}$ is the trivial subgroup of G. Further $\pi \to l_K(\pi)$ is a one-one correspondence of $\mathcal{M}(G/K)$ onto $\mathcal{M}(G, \{e\}, K)$. Let H be any closed subgroup of G. Then for any $\pi \in \mathcal{M}(G/K, H)$ (action on the left) $l_K(\pi) \in \mathcal{M}(G, H, K)$ and $\pi \to l_K(\pi)$ induces a one-one correspondence of $\mathcal{M}(G/K, H)$ onto $\mathcal{M}(G, H, K)$.*

The inverse of the map l_K as above, defined from $\mathcal{M}(G,H,K)$ onto $\mathcal{M}(G/K,H)$, shall be denoted by p_K.

(1.4) Let H and K be two closed unimodular subgroups of a locally compact group G. We can now set up a natural one-one correspondence between $\mathcal{M}(G/K,H)$ and $\mathcal{M}(G/H,K)$ as follows: Let $i_G: G \to G$ be the map defined by $i_G(g) = g^{-1}$ for all $g \in G$. For any $v \in \mathcal{M}(G,H,K)$ the measure $i_G(v)$ (defined by $i_G(v)(E) = v(i_G^{-1}(E))$ for all Borel sets E) belongs to $\mathcal{M}(G,K,H)$. Thus for any $\pi \in \mathcal{M}(G/K,H)$ the measure $\sigma = p_H\, i_G\, l_K(\pi)$ belongs to $\mathcal{M}(G/H,K)$. Then $\pi \leftrightarrow \sigma$ is a one-one correspondence of $\mathcal{M}(G/K,H)$ onto $\mathcal{M}(G/H,K)$. In this case we shall say that π (respectively σ) is *dual* to σ (respectively π), provided the subgroups involved are clear from the context; clearly a dual is uniquely defined up to a scalar multiple.

The above correspondences, which will be used crucially in the sequel, first appeared in [14]. Following are some of the properties of the correspondences which are easy to prove.

(1.5) Let G, H and K be as above. Let L be a closed unimodular normal subgroup of G contained in H. Then any (H,K)-invariant measure on G is also (H, LK)-invariant. Also for $v \in \mathcal{M}(G,H,K)$ the measure $p_L(v)$ is a $(H/L, KL/L)$-invariant measure on G/L, whenever $\overline{KL}$ is unimodular. Conversely if L is as above and $\eta: G \to G/L$ is the quotient homomorphism then the following holds: If $v \in \mathcal{M}(G, \{e\}, L)$ and $p_L(v) \in \mathcal{M}(G/L, H', K')$ for certain subgroups H' and K' of G/L and if K' and $\eta^{-1}(K')$ are unimodular, then $v \in \mathcal{M}(G, \eta^{-1}(H'), \eta^{-1}(K'))$.

(1.6) The simultaneous action on G of the subgroups H and K on the left and right hand sides can be viewed as an action of the direct product group $H \times K$. A measure $v \in \mathcal{M}(G,H,K)$ is said to be ergodic if it is ergodic as a $H \times K$-invariant measure. It is easy to verify that $\pi \in \mathcal{M}(G/K,H)$ is ergodic (as an H-invariant measure) if and only if $l_K(\pi)$ is an ergodic (H,K)-invariant measure in the above sense. In particular, it follows that under the one-one correspondence of $\mathcal{M}(G/H,K)$ onto $\mathcal{M}(G/K,H)$ as above, ergodic K-invariant measures correspond to ergodic H-invariant measures.

(1.7) Let G, H and K be as above. A measure $\pi \in \mathcal{M}(G/K,H)$ is said to be *H-finite* if the dual element in $\mathcal{M}(G/H,K)$, in the sense of §1.4, is a finite measure. If H is a discrete (countable) subgroup then $\pi \in \mathcal{M}(G/K,H)$ is H-finite if and only if for every Borel subset D of G such that $\{hD, h \in H\}$ are mutually disjoint, $l_K(\pi)(D)$ is finite.

(1.8) Direct Integral Decompositions and Ergodic Decompositions

Let X be a locally compact second countable space and let H be a locally compact second countable group acting continuously on X. Let $\pi \in \mathcal{M}(X,H)$. Let ξ be a countably separated (by Borel sets) partition of X and let X/ξ be the quotient space equipped with the quotient Borel structure. Let $\bar{\pi} \in \mathcal{M}(X/\xi)$ be the quotient measure of π. Assume that every element of ξ is invariant under the H-action. A *direct integral decomposition* of π with respect to ξ is a family $\{\pi_C \mid C \in X/\xi\}$ of measures on X satisfying the following conditions: a) there exists a measurable subset X' of X/ξ such that $\bar{\pi}(X/\xi - X') = 0$ and for each $C \in X'$, π_C is a H-invariant measure such that $\pi_C(X - C) = 0$ and b) for each Borel subset E of X, $\pi_C(E)$ is a measurable function of C and

$$\pi(E) = \int_{X/\xi} \pi_C(E)\, d\bar{\pi}(C).$$

(1.9) **Proposition.** *Let X, H and ξ be as above. Then any $\pi \in \mathcal{M}(X, H)$ admits a direct integral decomposition with respect to ξ.*

Proof. For finite measures this result is standard (cf. [23]). The general case can be deduced as follows: Let f be a continuous function on X such that $f(x) > 0$ for all $x \in X$ and $\int_X f \, d\pi = 1$. For $C \in X/\xi$ let π_C be the measure such that $d\pi_C = f^{-1} \cdot d\mu_C$ where $\{\mu_C \mid C \in X/\xi\}$ is a system of conditional measures with respect to ξ for the probability measure μ given by $d\mu = f d\pi$. Using the separability of H it is straightforward to verify that for a suitable subset X' of X/ξ such that $\bar{\pi}(X/\xi - X') = 0$ the measures $\{\pi_C, C \in X'\}$ have the desired properties.

Let X and H be as above and let $\pi \in \mathcal{M}(X, H)$. An *ergodic decomposition* of π is a direct integral decomposition $\{\pi_C\}$, with respect to a suitable H-invariant countably separated partition ξ, such that for almost all C, π_C is an ergodic H-invariant measure. It is well-known that a finite H-invariant measure admits an ergodic decomposition (cf. [23]). Using a similar construction and the ideas involved in [22] the same assertion can be upheld for any locally finite H-invariant measure. However we shall not need this in full generality and hence do not go into the details. We content ourselves with the following assertion.

(1.10) **Proposition.** *Let H and K be two closed unimodular subgroups of a locally compact second countable group G. Let $\pi \in \mathcal{M}(G/K, H)$ be H-finite. Then π admits an ergodic decomposition.*

Proof. Since the dual measure $\sigma \in \mathcal{M}(G/H, K)$ is finite and admits an ergodic decomposition, the same is true of π.

In the sequel we also need the following Lemma which can be easily proved using the map l introduced in Proposition 1.3.

(1.11) **Lemma.** *Let G be a locally compact second countable group and let H and K be closed unimodular subgroups of G. Let K_1 be a subgroup of finite index in K and let $\eta: G/K_1 \to G/K$ be the natural quotient map. If σ is an ergodic H-invariant measure on G/K, then there exists an ergodic H-invariant measure σ' on G/K_1 such that $\eta(\sigma') = \sigma$.*

(1.12) As the reader would have noted the maps l_H and p_H are defined only for (closed) unimodular subgroups. In the sequel while employing these maps, the unimodularity of the subgroups in question would follow (and we shall often not stop to prove it) from the fact that a Lie group which admits a lattice is necessarily unimodular. In this connection we also note the following.

(1.13) **Lemma.** (cf. Lemma 1.14, [8]). *Let G be a locally compact group and Γ be a lattice in G. Let H be a closed subgroup of G such that $H\Gamma$ is a closed subgroup of G. Then $H \cap \Gamma$ is a lattice in H.*

§2. Canonical Invariant Measures on Arithmetic Homogeneous Spaces

In this section we shall describe a class of ergodic invariant measures of maximal horospherical flows, which will be the prototypes for the classification coming up later.

Let k be a subfield of the field $\mathbb{C}$ of complex numbers and let $\mathbf{G}$ be an algebraic k-group; i.e. $\mathbf{G}$ is an algebraic group defined over k. We shall identify $\mathbf{G}$ with its group of $\mathbb{C}$-elements. For a proper subfield f of $\mathbb{C}$ containing k, we shall denote by $\mathbf{G}_f$ the group of all f-elements of $\mathbf{G}$. For various notions and results used in the sequel we refer the reader to [3].

(2.1) Let $\mathbf{G}$ be any algebraic $\mathbb{R}$-group. Then $\mathbf{G}_{\mathbb{R}}$ is a Lie group with finitely many connected components. As the reader will find later, the major task in proving Theorem 9.1 is to prove it in the special case of "arithmetic lattices" in $\mathbf{G}_{\mathbb{R}}$ where $\mathbf{G}$ is an algebraic $\mathbb{Q}$-group.

Let $\mathbf{G}$ be a Zariski connected algebraic $\mathbb{Q}$-group. Let $\mathbf{G}$ be viewed as a subgroup of $\mathbf{GL}(\mathbf{V})$ defined over $\mathbb{Q}$ where $\mathbf{V}$ is a vector space with a $\mathbb{Q}$-structure. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of $\mathbf{V}_{\mathbb{Q}}$. Let $\mathbf{G}_{\mathbb{Z}}$ be the subgroup consisting of elements which are represented (with respect to the basis) by integral matrices of determinant ± 1. A subgroup Γ of $\mathbf{G}_{\mathbb{Q}}$ is said to be *arithmetic* if it is commensurable with $\mathbf{G}_{\mathbb{Z}}$. It is well-known (cf. [1]) that the notion of an arithmetic subgroup depends only on the $\mathbb{Q}$-structure of $\mathbf{G}$ and not on the other choices involved. By a theorem of Borel and Harish-Chandra, for $\mathbf{G}$ as above, an arithmetic subgroup of $\mathbf{G}$ is a lattice in $\mathbf{G}_{\mathbb{R}}$, if and only if G does not admit non-trivial characters (algebraic homomorphisms into $\mathbf{GL}(1)$) defined over $\mathbb{Q}$; (cf. [2]). A lattice in $\mathbf{G}_{\mathbb{R}}$ is said to be an *arithmetic lattice* in $\mathbf{G}_{\mathbb{R}}$ if it is an arithmetic subgroup of $\mathbf{G}$.

Let $\mathbf{G}$ be an algebraic $\mathbb{Q}$-group. Then any horospherical subgroup (cf. Introduction for definition) of $\mathbf{G}_{\mathbb{R}}$ is a unipotent subgroup; that is, it consists entirely of unipotent elements. Further, if $\mathbf{G}$ is a reductive algebraic $\mathbb{R}$-group then a subgroup U of $\mathbf{G}_{\mathbb{R}}$ is a horospherical subgroup if and only if there exists a parabolic subgroup $\mathbf{P}$ such that $U = \mathbf{U}_{\mathbb{R}}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$.

Notation. Let G be a Lie group. For any subset E of G we denote by $\bar{E}$ the closure of E. For a closed subgroup H of G, H^0 denotes the connected component of the identity in H. When $G = \mathbf{G}_{\mathbb{R}}$, where $\mathbf{G}$ is an algebraic $\mathbb{R}$-group, these notations will be used only with respect to the Lie group topology (and not the Zariski topology).

The following simple fact about arithmetic subgroups is often useful.

(2.2) **Lemma.** *Let $\mathbf{G}$ be an algebraic $\mathbb{Q}$-group and Γ be an arithmetic subgroup of $\mathbf{G}$. Let $\mathbf{H}$ be an algebraic $\mathbb{Q}$-subgroup of $\mathbf{G}$ which admits no characters defined over $\mathbb{Q}$. Then $\Gamma\mathbf{H}_{\mathbb{R}}$ is closed.*

Proof. This can be deduced, for instance, from the fact that under these conditions there exists a $\mathbb{Q}$-rational representation of $\mathbf{G}$ such that $\mathbf{H}$ is the isotropy subgroup of a rational point (cf. [1], Proposition 7.8).

(2.3) **Proposition.** *Let $\mathbf{G}$ be a Zariski connected algebraic group defined over $\mathbb{Q}$. Let N be a maximal unipotent subgroup of $\mathbf{G}_{\mathbb{R}}$ and let V be the smallest normal subgroup of $\mathbf{G}_{\mathbb{R}}$ containing N. Let Γ be an arithmetic subgroup of $\mathbf{G}$ and let $\mathbf{H}$ be the smallest* algebraic $\mathbb{R}$-*subgroup of $\mathbf{G}$ containing $(\overline{V\Gamma})^0$. Then the following conditions are satisfied.*

i) *$\mathbf{H}$ is a normal subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$.*

ii) *$\mathbf{H}^0_{\mathbb{R}} = (\overline{V\Gamma})^0$ and $\mathbf{G}^0_{\mathbb{R}}/H^0_{\mathbb{R}}$ has no non-compact (semisimple) simple factors.*

iii) *$\mathbf{H}_{\mathbb{R}} \cap \Gamma$ is a lattice in $\mathbf{H}_{\mathbb{R}}$.*

iv) *The action of N on $\mathbf{H}^0_{\mathbb{R}}/H^0_{\mathbb{R}} \cap \Gamma$ is ergodic with respect to the $\mathbf{H}^0_{\mathbb{R}}$-invariant measure.*

v) *Every ergodic (N, Γ)-invariant measure on $\mathbf{G}_{\mathbb{R}}$ is concentrated on a single coset of the form $gH^0_{\mathbb{R}}\Gamma$, where $g \in \mathbf{G}_{\mathbb{R}}$.*

Proof of i), ii) *and* iii). These statements follow from standard facts from the theory of algebraic groups. We only outline the proof. Firstly, since evidently the unipotent radical of $\mathbf{G}$ is contained in $\mathbf{H}$, by passing to the quotient we may, without loss of generality, assume $\mathbf{G}$ to be reductive. In this case there exists a normal semisimple algebraic subgroup $\mathbf{G}'$ defined over $\mathbb{Q}$ such that N is contained in $\mathbf{G}'_{\mathbb{R}}$. Then by Lemma 2.2, $G'_{\mathbb{R}}\Gamma$ is closed and therefore $(\overline{V\Gamma})^0$ is contained in $\mathbf{G}'_{\mathbb{R}}$. For any $g \in \mathbf{G}_{\mathbb{Q}}$, $g\Gamma g^{-1}$ and Γ are commensurable. This implies that $(\overline{V\Gamma})^0$ is normalised by $\mathbf{G}_{\mathbb{Q}}$. But $\mathbf{G}_{\mathbb{Q}}$ is Zariski dense in $\mathbf{G}$ (cf. [3], Theorem 2.14) and therefore considering the adjoint representation of $\mathbf{G}$, we can conclude that $(\overline{V\Gamma})^0$ is normal in $\mathbf{G}_{\mathbb{R}}$. Hence it is a semisimple normal subgroup of $\mathbf{G}'_{\mathbb{R}}$. But a connected semisimple Lie subgroup is always the connected component of the identity in the group of $\mathbb{R}$-elements of an algebraic $\mathbb{R}$-subgroup. Thus $\mathbf{H}^0_{\mathbb{R}} = (\overline{V\Gamma})^0$. Since $\mathbf{H}^0_{\mathbb{R}}$ contains N, the group $\mathbf{G}^0_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}}$ is a reductive Lie group with no non-trivial horospherical subgroup. Hence it has no non-compact simple factors. Being the smallest algebraic $\mathbb{R}$-subgroup containing the normal subgroup $(\overline{V\Gamma})^0$ of $\mathbf{G}_{\mathbb{R}}$, $\mathbf{H}$ is normal in $\mathbf{G}$. Observe that by the theorem of Borel and Harish-Chandra mentioned earlier $\mathbf{G}'_{\mathbb{R}} \cap \Gamma$ is a lattice in $\mathbf{G}'_{\mathbb{R}}$. Since $\mathbf{H}_{\mathbb{R}}$ is contained in $\mathbf{G}'_{\mathbb{R}}$, $\mathbf{G}'_{\mathbb{R}}/\mathbf{H}_{\mathbb{R}}$ is compact and $\mathbf{H}_{\mathbb{R}}\Gamma$ is closed it follows that $\mathbf{H}_{\mathbb{R}} \cap \Gamma$ is a lattice in $\mathbf{H}_{\mathbb{R}}$. In particular, by Borel's density theorem (cf. [21], Chapter V) $\mathbf{H}$ coincides with the Zariski closure of $\mathbf{H}_{\mathbb{R}} \cap \Gamma$ in $\mathbf{G}$. Since $\Gamma \subset \mathbf{G}_{\mathbb{Q}}$, we deduce that $\mathbf{H}$ is defined over $\mathbb{Q}$.

Proof of iv). As in the earlier part we may assume $\mathbf{G}$ to be reductive. Recall that in this case $\mathbf{H}^0_{\mathbb{R}}$ is a connected semisimple Lie group and $\mathbf{H}^0_{\mathbb{R}} \cap \Gamma$ is a lattice in $\mathbf{H}^0_{\mathbb{R}}$. It is also easy to verify that V is a connected semisimple normal Lie subgroup of $\mathbf{H}^0_{\mathbb{R}}$. Also V has no compact factors and N is a totally non-compact subgroup of V; i.e. the image of N under any non-trivial homomorphism of V has non-compact closure. Under this condition C.C. Moore's ergodicity theorem implies that any N-invariant function $f \in L^2(\mathbf{H}^0_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}} \cap \Gamma)$is V-invariant (cf. [19], Theorem 2). On the other hand $V(\mathbf{H}^0_{\mathbb{R}} \cap \Gamma)$ is dense in $\mathbf{H}^0_{\mathbb{R}}$. The subgroup V being normal in $\mathbf{H}^0_{\mathbb{R}}$, the last two statements imply that every N-invariant function in $L^2(\mathbf{H}^0_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}} \cap \Gamma)$ is $\mathbf{H}^0_{\mathbb{R}}$-invariant (cf. [8], Lemma 8.2). Therefore the action of N on $\mathbf{H}^0_{\mathbb{R}}/\mathbf{H}^0_{\mathbb{R}} \cap \Gamma$ is ergodic.

Proof of v). Clearly the partition of $\mathbf{G}_{\mathbb{R}}$ into cosets of the form $g\mathbf{H}^0_{\mathbb{R}}\Gamma, g \in \mathbf{G}_{\mathbb{R}}$ is countably separated (by Borel sets). On the other hand since $\mathbf{H}^0_{\mathbb{R}}$ is normal in $\mathbf{G}_{\mathbb{R}}$ each of the cosets is invariant under the left action of N and also under the right action of Γ. Hence every ergodic (N, Γ)-invariant measure is concentrated on a single coset as above.

(2.4) **Corollary.** *Let $\mathbf{G}$ be a reductive algebraic group defined over $\mathbb{Q}$ and let Γ be an arithmetic lattice in $\mathbf{G}_{\mathbb{R}}$. Let N be a maximal horospherical subgroup of $\mathbf{G}_{\mathbb{R}}$. Let $\mathbf{Q}$ be an algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ containing N. Let V be the smallest normal subgroup of $\mathbf{Q}_{\mathbb{R}}$ containing N and let $H = (\overline{V(\mathbf{Q}_{\mathbb{R}} \cap \Gamma)})^0$. Then for any $y = zq$, where $z \in \mathbf{G}_{\mathbb{R}}$ is such that $zHz^{-1} = H$ and $q \in \mathbf{G}_{\mathbb{Q}}$, $Hy\Gamma/\Gamma$ is a closed subset of $\mathbf{G}_{\mathbb{R}}/\Gamma$ and there* *exists a finite H-invariant measure supported on $Hy\Gamma/\Gamma$, which is also an ergodic N-invariant measure.*

(2.5) *Remark.* We note that if $\mathbf{G}_{\mathbb{R}}$ and Γ are as above and if $\mathbf{G}_{\mathbb{R}}/\Gamma$ is not compact then there exist proper parabolic subgroups which are defined over $\mathbb{Q}$. Since any parabolic subgroup contains a conjugate of any horospherical subgroup, by the above corollary we obtain ergodic invariant measures for maximal horospherical flows other than the $\mathbf{G}_{\mathbb{R}}$-invariant measure. A similar remark holds in the general case, as we will observe later. We emphasize that the ergodic invariant measures obtained as above are supported on *a single orbit* of a closed connected subgroup of $\mathbf{G}_{\mathbb{R}}$ and are invariant under that subgroup.

§3. A Characterisation of Haar Measures

The classification of ergodic invariant measures of horospherical flows in later sections depends on a characterisation of Haar measures proved in [8]. Since the presentation of the theorem involves rather elaborate notation we devote this section to describe the result and generalise it slightly.

In this section let G be a connected reductive Lie group and let L be the smallest normal subgroup of G such that G/L is a semisimple Lie group with trivial center and no non-trivial compact factors.

Let Γ be a lattice in G. A horospherical subgroup V of G is said to be *Γ-rational* if $VL\cap\Gamma$ is a (necessarily uniform) lattice in VL. Let U and U^- be two maximal Γ-rational horospherical subgroups and let P and P^- be their respective normalisers in G. Then U and U^- are said to be *opposite* (to each other) if $P\cap P^-$ is a reductive Levi subgroup in both P and P^-; i.e. $P=(P\cap P^-)\cdot U$ and $P^-=(P\cap P^-)\cdot U^-$. Now let (U, U^-)be a pair of opposite maximal Γ-rational horospherical subgroups and let P and P^- be their respective normalisers. Let S^0 be the unique maximum vector subgroup contained in the center of $(P\cap P^-)^0$, such that the adjoint action of S^0 on the Lie algebra $\mathfrak{g}$ of G is simultaneously diagonalisable over $\mathbb{R}$. Thus

$$\mathfrak{g}=\mathfrak{z}+\sum_{\lambda\in\Lambda}\mathfrak{g}_\lambda$$

where $\mathfrak{z}$ is the centraliser of S^0, Λ is a set of non-trivial characters of S^0 into $\mathbb{R}^+$ and $\mathfrak{g}_\lambda$ for $\lambda\in\Lambda$ is the corresponding eigenspace. There exists a unique subset Λ^+ of Λ such that $\sum_{\lambda\in\Lambda^+}\mathfrak{g}_\lambda$ is the Lie subalgebra corresponding to U. Now for any $t\in\mathbb{R}^+$ put

$$S_t=\{s\in S^0\,|\,\lambda(s)\leqq t\quad\text{for all}\quad\lambda\in\Lambda^+\}.$$

(3.1) **Proposition.** *Let the notations be as above. Let K be a maximal compact subgroup of G. Then there exist, a finite subset J of G, a compact subset C of $(P^-)^0$ and $t\in\mathbb{R}^+$ such that*

a) *$G=\Gamma JCS_tK$ and*
b) *for all $j\in J, j^{-1}\Gamma j\cap U^-$ is a lattice in U^-.*

Proof. This is a consequence of the results on construction of fundamental domains, due to Borel for arithmetic groups [1], and Garland and Raghunathan [15] for lattices in $\mathbb{R}$-rank 1 groups, put together using Margulis's arithmeticity theorem [18]. Details of the deduction for the case when G is a connected semisimple Lie group with trivial center and without compact factors can be found in [8] (cf. Proposition 2.3, [8]). The general case can be derived from this using the fact that $L \cap \Gamma$ is a (necessarily uniform) lattice in L (cf. [8], Lemma 9.1).

(3.2) **Definition.** A finite subset J of G for which Proposition 3.1 holds, for a suitable choice of the compact set C and $t > 0$, is called a *sufficient set of cusp elements* for Γ with respect to the triple (P^-, U, K).

(3.3) *Remark.* If $G = \mathbf{G}_{\mathbb{R}}^0$ (notation as in §2), where $\mathbf{G}$ is a reductive algebraic $\mathbb{Q}$-group and Γ is an arithmetic lattice in $\mathbf{G}_{\mathbb{R}}$ then $\mathbf{G}_{\mathbb{Q}} \cap G$ contains a sufficient set of cusp elements for Γ.

Proof. If $G = \mathbf{G}_{\mathbb{R}}$ (that is, if $\mathbf{G}_{\mathbb{R}}$ is connected) then the remark follows from Theorem 13.1 in [1] (cf. the proof of Proposition 2.3 in [8]). The general case can be deduced as follows. Since the center of $\mathbf{G}_{\mathbb{R}}$ intersects Γ in a (necessarily uniform) lattice, the remark needs a proof only when $\mathbf{G}$ is a semisimple algebraic $\mathbb{Q}$-group. Further, by passing to a covering (isogeny) defined over $\mathbb{Q}$, $\mathbf{G}$ may be assumed to be simply connected. Under these conditions $\mathbf{G}_{\mathbb{R}}$ is connected and hence the remark is justified.

(3.4) **Theorem.** *Let G be a connected reductive Lie group and let Γ be a lattice in G. Let F be the smallest normal subgroup of G containing every connected non-compact simple Lie subgroup of G. Suppose that $G = \overline{F\Gamma}$. Let U and U^- be a pair of maximal Γ-rational horospherical subgroups opposite to each other and let P and P^- be their respective normalisers. Let N be a maximal horospherical subgroup such that $U \subset N \subset P$. Let J be a sufficient set of cusp elements for Γ with respect to (P^-, U, K) where K is a maximal compact subgroup of G. Let π be a Γ-invariant Γ-finite measure on G/N. Then π is G-invariant if and only if*

$$\pi(G/N - jP^- N/N) = 0 \quad \text{for all} \quad j \in J. \tag{3.5}$$

Proof. By a well-known consequence of the Bruhat decomposition (cf. [25], Proposition 1.2.4.10) $P^- N$ is an open dense subset of G and $G - P^- N$ has zero Haar measure. Hence we only need to prove the converse statement. Further since π as in the hypothesis admits an ergodic decomposition (cf. Proposition 1.10) we need to prove the theorem only in the case when π is also ergodic.

For a connected semisimple Lie group with trivial center and without compact factors the converse statement is just essentially Theorem 2.4 in [8] except for the following:

(3.6) *Remark.* While setting up the notation for the statement of Theorem 2.4 in [8], the set X_0 was inadvertently defined to be $\bigcup_{j \in J} jP^- N/N$, instead of $\bigcap_{j \in J} jP^- N/N$ as actually required.

(3.7) *Remark.* The phrase "sufficient set of cusp elements" was not introduced in [8]. There we had fixed a set J satisfying certain conditions (as in Proposition 2.3, [8]). The proof of Theorem 2.4 depends only on the conditions included in the definition of a sufficient set of cusp elements.

Now, as before let L be the smallest closed normal subgroup of G such that G/L is a semisimple Lie group with trivial center and without compact factors. Let $G' = G/L$ and let $\eta: G \to G'$ be the quotient homomorphism. Then $\Gamma' = \eta(\Gamma)$ is a lattice in G' and the quotient map $\bar{\eta}: G/\Gamma \to G'/\Gamma'$ is proper (cf. [8] Lemma 9.1). Let $\pi \in \mathcal{M}(G/N, \Gamma)$ be a Γ-finite ergodic measure satisfying (3.5). Let $\sigma \in \mathcal{M}(G/\Gamma, N)$ be the (finite) measure dual to π and let $\pi' \in \mathcal{M}(G'/\eta(N'), \Gamma')$ be the measure dual to $\bar{\eta}(\sigma) \in \mathcal{M}(G'/\Gamma', \eta(N'))$. It is easy to verify that π' satisfies the conditions of Theorem 3.4 for suitable (canonical) choices of the subgroups and the set of cusp elements. By the special case of the theorem considered above, it follows that π' is G'-invariant. Therefore $\bar{\eta}(\sigma)$ is G'-invariant. Hence by Proposition 9.3 in [8], there exists a closed analytic subgroup H of G such that σ is H-invariant and is concentrated on a single closed orbit of H. Since $\bar{\eta}(\sigma)$ is G'-invariant H must also be such that $\eta(H) = G/L$; that is, $HL = G$. Since G is a reductive Lie group, this condition implies that H contains every non-compact connected simple Lie subgroup of G. Hence $F \subset H$. Let $y \in G$ be such that σ is concentrated on $Hy\Gamma/\Gamma$, where $Hy\Gamma$ is a closed set. We have $Hy\Gamma = HFy\Gamma = HyF\Gamma$. But since $Hy\Gamma$ is closed and $F\Gamma$ is dense in G we must have $H = G$; that is, σ is G-invariant. Equivalently, π is G-invariant.

(3.8) *Remark.* If $G = \mathbf{G}^0_{\mathbb{R}}$, where $\mathbf{G}$ is an algebraic $\mathbb{Q}$-group and Γ is an arithmetic lattice in G then in Theorem 3.4 the condition of Γ-finiteness of π is redundant. This is because for G and Γ as above, every invariant measure of any horospherical subgroup N (– indeed of any unipotent subgroup) is the limit of an increasing sequence of finite N-invariant measures (cf. Theorem 4.1 [9]). In other words, every Γ-invariant measure on G/N is the limit of an increasing sequence of Γ-finite Γ-invariant measures.

§ 4. Invariance of Lifted Measures

The aim of this section is to prove the following theorem which may be viewed as a generalisation of W. Parry's result [20] on unique ergodicity of translations of compact nilmanifolds.

(4.1) **Theorem.** *Let G be a connected Lie group and Γ be a lattice in G. Let U be an analytic subgroup of G and let V be the smallest normal subgroup of G containing U.* Suppose that $\mathrm{H} = (\overline{V\Gamma})^0$ *is a nilpotent Lie group. Let μ be a U-invariant measure on G/Γ such that the projection of μ on $G/\overline{V\Gamma}$ is G-invariant. Then μ is G-invariant.*

Proof. In view of Lemma 1.13 $H \cap \Gamma$ is a lattice in H. Since H is a nilpotent Lie group this implies that $H/H \cap \Gamma$ is compact (cf. [21], Theorem 2.1). Consequently the canonical quotient map of G/Γ onto $G/H\Gamma$ is proper. Hence the projection of μ on $G/H\Gamma$ is indeed a locally finite Borel measure.

Now let $X = G/\Gamma$ and let ξ be the partition of X into cosets of the form $gH\Gamma/\Gamma$. The quotient space X/ξ is canonically isomorphic to $G/H\Gamma$. In particular ξ is a measurable partition of X. Let $\bar{\mu}$ be the quotient measure on X/ξ and let $\{\mu_C \mid C \in X/\xi\}$ be a system of conditional measures. We recall that such a system is defined $\bar{\mu}$ a.e. and is unique $\bar{\mu}$ a.e.. Since U is a separable locally compact group we can deduce that for C in the complement of a set of zero $\bar{\mu}$-measure, μ_C is U-invariant. Set

(4.2) $$X' = \{C \in X/\xi \mid l_\Gamma(\mu_C) \text{ is } (U, H\Gamma)\text{-invariant}\}.$$

We shall show that $\bar{\mu}(X/\xi - X') = 0$. This would imply that $l_\Gamma(\mu) = l_{H\Gamma}(\bar{\mu})$ and since by hypothesis $\bar{\mu}$ is G-invariant, in view of Proposition 1.3 μ would be G-invariant. To prove the previous statement we need the following result, which is a reinterpretation of W. Parry's theorem referred above.

(4.3) **Lemma.** *Let H be a connected nilpotent Lie group and let Δ be a lattice in H. Then there exist countably many proper normal analytic subgroups $\{H_i\}_{i \in \mathbb{N}}$ of H such that the following conditions hold*:

i) *For any $i \in \mathbb{N}$, $H_i \Delta$ is closed and*

ii) *For any analytic subgroup U of H either $U \subset H_i$ for some i or every U-invariant measure on H/Δ is H-invariant.*

Proof. Firstly we note that there is no loss of generality in assuming, as we do, that H is simply connected. Then $H/[H,H]$ is naturally isomorphic to $\mathbb{R}^k$ for some k.It is well-known that $[H,H]\Gamma/[H,H]$ is a lattice in $H/[H,H]$. Thus the image of Δ in $\mathbb{R}^k$ is a lattice and hence for a suitable choice of the basis it can be identified with $\mathbb{Z}^k$. Let $\{H_i'\}_{i \in \mathbb{N}}$ denote the (countable) family of all subspaces of $\mathbb{R}^k$ which are defined by linear equations with rational coefficients. Let $\{H_i\}_{i \in \mathbb{N}}$ be the subgroups of H which are inverse images of H_i' under the quotient map of H onto $H/[H,H] = \mathbb{R}^k$. It is evident that for any $i \in \mathbb{N}$, $H_i \Delta$ is closed. We only need to verify condition (ii) which is done as follows. Let U be any analytic subgroup of H which is not contained in H_i for any i. We can choose a one-parameter subgroup $\{u_t\}_{t \in \mathbb{R}}$ of U which is not contained in H_i for any i. The image of $\{u_t\}_{t \in \mathbb{R}}$ in $\mathbb{R}^k$ under the quotient map is a line $D = \{t\alpha \mid t \in \mathbb{R}\}$, where $\alpha \in \mathbb{R}^k$, which is not contained in H_i' for any i. It is well-known that under this condition the flow on $\mathbb{R}^k/\mathbb{Z}^k$ induced by D is uniquely ergodic; that is, the Haar measure is the only D-invariant probability measure on $\mathbb{R}^k/\mathbb{Z}^k$. But by Parry's theorem (cf. [20], Theorem 5) this implies that any $\{u_t\}$-invariant measure on H/Δ is H-invariant, which proves the lemma.

We now return to the proof of Theorem 4.1. The following proof involves essentially the same idea as the author has used in [7], Lemma 4.2, for lifting ergodicity under a similar situation. Let $g \in G$ be such that the measure μ_C, where $C = gH\Gamma/\Gamma$, is U-invariant. Put $\Delta = H \cap \Gamma$. We define a measure ν_g on H/Δ as follows. The map $\varphi_g \colon H/\Delta \to G/\Gamma$ defined by $\varphi_g(h\Delta) = gh\Gamma$ for all $h \in H$ is a homeomorphism onto C. Set $\nu_g = \varphi_g^{-1}(\mu_C)$. Then ν_g is a $g^{-1}Ug$-invariant measure on H/Δ.

Recall that H is a connected nilpotent Lie group and that Δ is a lattice in H. Now let H_i, $i \in \mathbb{N}$ be the proper closed normal subgroups of H as in Lemma 4.3. For $i \in \mathbb{N}$ put

$$X_i = \left\{ C \in X/\xi \;\middle|\; \begin{array}{l} \text{there exists a } g \in G \text{ such that} \\ C = gH\Gamma/\Gamma \text{ and } g^{-1}Ug \subset H_i \end{array} \right\}.$$

We would like to show that $\bar{\mu}(X_i) = 0$ for all i. Since $\bar{\mu}$ is G-invariant and the subgroups H_i are normal in H evidently it is enough to show that the sets

$$E_i = \{g \in G \mid gUg^{-1} \subset H_i\}$$

are sets of zero Haar measure in G.

Let if possible $i \in \mathbb{N}$ be an index such that E_i has positive Haar measure. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{u}$ and $\mathfrak{h}_i$ denote the Lie subalgebras corresponding to U and H_i respectively. Let r be the dimension of $\mathfrak{u}$ and consider the representation ϱ of G on $\overset{r}{\bigwedge}\mathfrak{g}$ obtained as the rth exterior power of the adjoint representation. Let $\mathfrak{O}_i = \overset{r}{\bigwedge}\mathfrak{h}_i \subset \overset{r}{\bigwedge}\mathfrak{g}$ and let f be a non-zero element in the one-dimensional subspace $\overset{r}{\bigwedge}\mathfrak{u}$. Clearly $g \in E_i$ if and only if $\varrho(g)f \in \mathfrak{O}_i$. Let ψ be any linear functional on $\overset{r}{\bigwedge}\mathfrak{g}$ such that $\psi(\mathfrak{O}_i) = 0$. Then $\psi(\varrho(g)f)$ is a real analytic function on G which vanishes on E_i. Since E_i has positive Haar measure and G is connected this implies that $\psi(\varrho(g)f) = 0$ for all $g \in G$. Since ψ is arbitrary we must have $\varrho(g)f \in \mathfrak{O}_i$ for all $g \in G$. However this is equivalent to $gUg^{-1} \subset H_i$ for all $g \in G$. Therefore V, the smallest normal subgroup containing U, must be contained in H_i. Since $H_i\Gamma$ is closed this implies that $H = (\overline{V\Gamma})^0$ is contained in H_i. Since H_i is a proper subgroup of H this is a contradiction. Hence $\bar{\mu}(X_i) = 0$ for all i.

Since μ_C is U-invariant for $\bar{\mu}$-almost all C and $\bar{\mu}(X_i) = 0$ for all $i \in \mathbb{N}$, by Lemma 4.3 for $\bar{\mu}$-almost all $C = gH\Gamma/\Gamma$ the measure ν_g on H/Δ is H-invariant. Let $g \in G$ be such that this is satisfied. Then $l_\Delta(\nu_g)$ is the Haar measure on H. Since $\mu_C = \varphi_g(\nu_g)$, for $C = gH\Gamma/\Gamma$ the last assertion implies that $l_\Gamma(\mu_C)$ is invariant under the action of H on the *right*. Hence $l_\Gamma(\mu_C)$ is $(U, H\Gamma)$-invariant. Thus the complement of the set X' as defined in (4.2) is of zero $\bar{\mu}$-measure, which proves the theorem.

Part II: A Special Case

§5. More About Algebraic Groups and Their Subgroups

The notations introduced in this section will be adhered to throughout Part II.

(5.1) Let $\mathbf{G}$ be a reductive Zariski connected algebraic $\mathbb{Q}$-group. Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus in $\mathbf{G}$. Let $\mathbf{A}$ be a maximal $\mathbb{R}$-split torus containing $\mathbf{S}$ and defined over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $\mathfrak{G}$ be the Lie algebra of $\mathbf{G}$ and consider the root space decomposition of $\mathfrak{G}$ with respect to $\mathbf{A}$; thus

$$\mathfrak{G} = \mathfrak{Z} + \sum_{\alpha \in \Phi} \mathfrak{G}^\alpha$$

where $\mathfrak{Z}$ is the Lie subalgebra corresponding to the centraliser of $\mathbf{A}$ in $\mathbf{G}$, Φ is a root system of characters on $\mathbf{A}$ and each $\mathfrak{G}^\alpha$ is the root space corresponding to the root α, given by

$$\mathfrak{G}^\alpha = \{\xi \in \mathfrak{G} \mid (Ad\, a)\xi = \alpha(a)\xi \quad \text{for all} \quad a \in \mathbf{A}\},$$

where Ad denotes the adjoint representation. We fix an order on Φ and let Φ^+ and Δ respectively denote the set of positive roots and the set of simple roots with respect to the order. For any α let $\mathbf{G}_\alpha$ denote the subgroup generated by $\exp \mathfrak{G}^\alpha$. For any subset Ψ of Φ let $\mathbf{G}(\Psi)$ denote the subgroup generated by $\{\mathbf{G}_\alpha \mid \alpha \in \Psi\}$ together with $\mathbf{Z}$, the latter being the centraliser of $\mathbf{A}$ in $\mathbf{G}$. For any subset Θ of Δ let $\langle\Theta\rangle$ denote the set of roots in Φ which are integral combinations of elements of Θ and let $[\Theta] = \langle\Theta\rangle \cup \Phi^+$. For any subset Θ of Δ let $\mathbf{P}_\Theta$ and $\mathbf{P}_\Theta^-$ denote the subgroups $\mathbf{G}([\Theta])$

and $\mathbf{G}(-[\Theta])$ respectively and let $\mathbf{U}_\Theta$ and $\mathbf{U}_\Theta^-$ denote the unipotent radicals of $\mathbf{P}_\Theta$ and $\mathbf{P}_\Theta^-$ respectively. The subgroup $\mathbf{P}_\Theta$, is called the *standard parabolic* $\mathbb{R}$*-subgroup* of $\mathbf{G}$ associated to Θ. Any parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ is conjugate to a unique standard parabolic $\mathbb{R}$-subgroup. For any subset Θ of Δ let $\mathbf{Z}_\Theta$ denote the subgroup generated by $\{\mathbf{G}_\alpha \mid \alpha \in \langle\Theta\rangle\}$ together with $\mathbf{Z}$. Then $\mathbf{Z}_\Theta$ is a reductive algebraic $\mathbb{R}$-subgroup. Further $\mathbf{P}_\Theta = \mathbf{Z}_\Theta \mathbf{U}_\Theta$ (semi-direct product) is a Levi decomposition called the *standard Levi decomposition.*

Let Δ_0 denote the subset of Δ consisting of those roots whose restriction to $\mathbf{S}$ is the trivial character on $\mathbf{S}$. We set $\mathbf{P} = \mathbf{P}_{\Delta_0}$ and $\mathbf{P}^- = \mathbf{P}_{\Delta_0}^-$. Then $\mathbf{P}$ and $\mathbf{P}^-$ are minimal parabolic $\mathbb{Q}$-subgroups (i.e. parabolic subgroups defined over $\mathbb{Q}$).

(5.2) We now describe a criterion to determine which of the standard parabolic subgroups are defined over $\mathbb{Q}$.

Let $\mathbf{k}$ be the field of algebraic numbers and let $\mathbf{D}$ be a maximal torus in $\mathbf{G}$ containing $\mathbf{A}$ and defined over $\mathbf{k}$. Let $\Delta_{\mathbb{C}}$ be the system of simple roots with respect to $\mathbf{D}$ such that the simple roots in Δ are restrictions of elements of $\Delta_{\mathbb{C}}$ to $\mathbf{A}$. It is well-known that $\Delta_{\mathbb{C}}$ can also be regarded as a system of simple roots with respect to $\mathbf{D}_\mathbf{k}$, as the restrictions of distinct roots are distinct. Let Σ be the Galois group of $\mathbf{k}$ over $\mathbb{Q}$. Then there exists an action of Σ on $\Delta_{\mathbb{C}}$ defined as follows. Let $\sigma \in \Sigma$. Then there exists an element $g \in \mathbf{G}_\mathbf{k}$ such that the corresponding inner automorphism c_g transforms $\sigma(\mathbf{D}_\mathbf{k})$ into $\mathbf{D}_\mathbf{k}$ and the system of simple roots $\sigma(\Delta_{\mathbb{C}})$ relative to $\sigma(\mathbf{D}_\mathbf{k})$ into the system $\Delta_{\mathbb{C}}$ relative to $\mathbf{D}_\mathbf{k}$. The action of $c_g \cdot \sigma$ on the set of characters on $\mathbf{D}_\mathbf{k}$ is independent of the choice of g and leaves $\Delta_{\mathbb{C}}$ invariant. Further, associating to σ the automorphism of $\Delta_{\mathbb{C}}$ thus induced, yields an action of Σ on $\Delta_{\mathbb{C}}$ (cf. [3], §6). Now let $\eta: \Delta_{\mathbb{C}} \to \Delta$ be the map which associates to each $\alpha \in \Delta_{\mathbb{C}}$ its restriction to $\mathbf{A}$. Also let

$$\mathfrak{C} = \{\Psi \subset \Delta \mid \eta^{-1}(\Psi) \text{ is invariant under the above-mentioned action of } \Sigma \text{ on } \Delta_{\mathbb{C}}.\}$$

A subset of Δ which belongs to $\mathfrak{C}$ is said to be $\mathbb{Q}$*-saturated.* The rationality criterion sought after is the following:

(5.3) **Proposition.** (cf. [3], §6). *For any $\Psi \subset \Delta$ the standard parabolic $\mathbb{R}$-subgroup $\mathbf{P}_\psi$ is defined over $\mathbb{Q}$ if and only if $\Delta_0 \subset \Psi$ and Ψ is $\mathbb{Q}$-saturated.*

In the sequel we also need the following observation.

(5.4) **Lemma.** *Let $\mathbf{L} \subset \mathbf{U}_{\Delta_0}$ be an algebraic subgroup defined over $\mathbb{Q}$. Suppose that $\mathbf{L}$ is normalised by $\mathbf{Z}_{\Delta_0}$. Then the set $\{\alpha \in \Delta \mid \mathbf{G}_\alpha \subset \mathbf{L}\}$ is $\mathbb{Q}$-saturated.*

Proof. We note that the maximal torus $\mathbf{D}$ involved in the definition of $\mathfrak{C}$ as above can be chosen to be contained in $\mathbf{Z}_{\Delta_0}$ and that in this case the element g, transforming $\sigma(\mathbf{D}_\mathbf{k})$ into $\mathbf{D}_\mathbf{k}$ and $\sigma(\Delta_{\mathbb{C}})$ into $\Delta_{\mathbb{C}}$ (where $\sigma \in \Sigma$), can be chosen to be in $(\mathbf{Z}_{\Delta_0})_\mathbf{k}$. From this it is straightforward to deduce the lemma. We omit the details.

(5.5) Retaining the notations as in §5.1 we shall now introduce certain Lie subgroups, through which we shall be able to use the results recalled in §3.

Set $G = \mathbf{G}_\mathbb{R}$, $Z = \mathbf{Z}_\mathbb{R}$, $P = \mathbf{P}_\mathbb{R}$ and $P^- = \mathbf{P}_\mathbb{R}^-$. Next, let $\square$ be the empty subset of Δ and put $R = (\mathbf{P}_\square)_\mathbb{R}$ and $N = (\mathbf{U}_\square)_\mathbb{R}$. Observe that $\mathbf{P}_\square$ is the minimal standard $\mathbb{R}$-parabolic subgroup of $\mathbf{G}$ and $\mathbf{U}_\square$ is a maximal unipotent $\mathbb{R}$-subgroup of $\mathbf{G}$. Thus N is a maximal horospherical subgroup of G (cf. §2.1). The standard Levi decomposition of $\mathbf{P}_\square$ yields $R = Z \cdot N$ (semi-direct product, with N as a normal subgroup).

Now let Γ be an arithmetic lattice in $G=\mathbf{G}_{\mathbb{R}}$. Let $\mathbf{W}$ be the unipotent radical of a parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$. Then the horospherical subgroup $\mathbf{W}_{\mathbb{R}}$ is Γ-rational in the sense of §3 if and only if $\mathbf{W}$ is defined over $\mathbb{Q}$. Recall also that $\mathbf{P}=\mathbf{P}_{\Delta_0}$ and $\mathbf{P}^-=\mathbf{P}^-_{\Delta_0}$ are minimal parabolic $\mathbb{Q}$-subgroups and that $\mathbf{U}_{\Delta_0}$ and $\mathbf{U}^-_{\Delta_0}$ are their respective unipotent radicals. It follows that the subgroups $U=(\mathbf{U}_{\Delta_0})_{\mathbb{R}}$ and $U^-=(\mathbf{U}^-_{\Delta_0})_{\mathbb{R}}$, which are contained in G, are maximal Γ-rational horospherical subgroups of G^0. Further $P\cap G^0$ and $P^-\cap G^0$ are the normalisers of U and U^- in G^0. In particular, (U,U^-) form a pair of maximal Γ-rational horospherical subgroups opposite to each other. It may be remarked that because of our choices as above we automatically have $U\subset N\subset P$.

Lastly, in the sequel, the subgroup $\mathbf{G}_{\mathbb{Q}}$, which is contained in G, will be denoted by $G_{\mathbb{Q}}$.

(5.6) Let $N(\mathbf{A})$ and $N(\mathbf{S})$ denote the normalisers of $\mathbf{A}$ and $\mathbf{S}$ respectively in $\mathbf{G}$. Then $W=N(\mathbf{A})_{\mathbb{R}}/Z$ is the $\mathbb{R}$-Weyl group of G. By the Bruhat decomposition (cf. [3], Theorem 5.15) we have

$$G=\bigcup_{w\in W} Rx_wR \qquad \text{(disjoint union)}$$

where $\{x_w\}_{w\in W}$ is any set of representatives in $N(\mathbf{A})_{\mathbb{R}}$ for W.

We fix a set of representatives and, by abuse of notation, write w for x_w. It is well-known that there exists a unique element $w_0\in W$ such that Rw_0R is an open dense subset of G and that for $w\neq w_0$, RwR is a lower dimensional submanifold of G. Indeed $\mathbf{R}w_0\mathbf{R}$, where $\mathbf{R}=\mathbf{P}_{\square}$, is Zariski open in $\mathbf{G}$ and $Rw_0R=\mathbf{R}w_0\mathbf{R}\cap G$. Similarly using the fact that any parabolic subgroup of $\mathbf{G}$ is Zariski connected it is easy to prove the following.

(5.7) **Proposition.** *Let $\mathbf{Q}$ be a standard parabolic $\mathbb{R}$-subgroup. Then for any $g\in G$ there exists a unique $w\in W$ such that RwR is open in $\mathbf{Q}_{\mathbb{R}}gR$. Further the complement of RwR in $\mathbf{Q}_{\mathbb{R}}gR$ is a finite union of lower dimensional submanifolds.*

(5.8) **Proposition.** *There exists $\varrho\in N(\mathbf{S})_{\mathbb{Q}}$ such that $\varrho^{-1}P\varrho=P^-$ and $P\varrho R=Pw_0R$.*

Proof. There exists $\varrho\in \mathrm{N}(\mathbf{S})_{\mathbb{Q}}$ such that $\varrho^{-1}\mathbf{P}\varrho=\mathbf{P}^-$ and $\mathbf{P}\varrho\mathbf{U}_{\Delta_0}$ is Zariski open in $\mathbf{G}$ (cf. [3]). By the uniqueness property of w_0 recalled earlier we get that $Rw_0R\subset P\varrho R$. Hence $P\varrho R=Pw_0R$.

Let W_0 be the subset of W consisting of those elements $w\in W$ such that $w\neq w_0$ and RwR is open (and dense) in PwR. Then by Propositions 5.7 and 5.8 and the Bruhat decomposition we have

$$G-P\varrho R=\bigcup_{w\in W_0} PwR. \tag{5.9}$$

(5.10) The Weyl group W can also be viewed in a natural way as a group of linear transformations of $V=X(\mathbf{A})\otimes\mathbb{R}$ where $X(\mathbf{A})$ is the group of characters on $\mathbf{A}$ (cf. [3] §5). There exists on V a W-invariant inner product such that $\{s_\alpha\,|\,\alpha\in\Delta\}$, where s_α is the reflexion in the hyperplane orthogonal to α, is a set of generators for W. For $w\in W$ the minimum possible word length of any expression for w in terms of $\{s_\alpha\,|\,\alpha\in\Delta\}$ is called the *length* of w and is denoted by $l(w)$. In the sequel we need the following results involving $l(w)$.

(5.11) **Proposition.** *Let $\alpha \in \Delta$ and $w \in W$. Then $l(s_\alpha w) = l(w) \pm 1$ and*

a) *$l(s_\alpha w) > l(w)$ implies that $R s_\alpha R w R = R s_\alpha w R$, and*

b) *$l(s_\alpha w) < l(w)$ implies that $R s_\alpha R w R$ is contained in $RwR \cup R s_\alpha w R$ and intersects RwR in a non-empty set.*

The Proposition is a consequence of the fact that $(G, R, N(\mathbf{A}), \{s_\alpha \mid \alpha \in \Delta\})$ is a Tits system (cf. [25] §1.2.3 and Lemma A, §29.3, [17]).

(5.12) **Lemma.** *Let $\alpha \in \Delta$ and $w \in W$. Then $l(s_\alpha w) > l(w)$ if and only if $\alpha \in w(\Phi^+)$.*

This is well-known; it can be deduced from the observations in §10.2 of [16].

(5.13) **Theorem.** *Let $w \in W$ and let $w = s_1 s_2 \ldots s_l$, where $l = l(w)$ and $s_i \in \{s_\alpha \mid \alpha \in \Delta\}$, be a reduced decomposition for w. Then the set $A(w) = \{s_{i_1} \cdot s_{i_2} \cdots s_{i_k} \mid k \in \mathbb{N},\ 1 \leqq i_1 < i_2 \cdots < i_k \leqq l\}$ is independent of the choice of the reduced decomposition and*

$$\overline{RwR} = \bigcup_{w' \in A(w)} Rw'R.$$

This is just Corollaire 3.15 in [4] for the field of real numbers with respect to the euclidean topology. Noting that the equality also holds with respect to the (restriction of the) Zariski topology (cf. Theorem 3.13, [4]) one deduces the following.

(5.14) *Remark.* For any $w \in W$, $\overline{RwR}$ is the set of $\mathbb{R}$-points of an algebraic variety defined over $\mathbb{R}$.

(5.15) **Corollary.** *Let $w \in W$ and $\Psi = \{\alpha \in \Delta \mid l(s_\alpha w) < l(w)\}$. Then*

$$(\mathbf{P}_\Psi)_{\mathbb{R}} = \{g \in G \mid g\overline{RwR} = \overline{RwR}\}.$$

Proof. Indeed, the right hand side is a subgroup containing R and hence it is necessarily of the form $(\mathbf{P}_\Theta)_{\mathbb{R}}$, with $\Theta \subset \Delta$ (cf. [25]). Evidently Θ must be the set $\{\alpha \in \Delta \mid s_\alpha \overline{RwR} = \overline{RwR}\}$. Let $\alpha \in \Psi$. Then by Proposition 5.11 and Theorem 5.13, $s_\alpha RwR \subset RwR \cup Rs_\alpha wR \subset \overline{RwR}$. Hence $\alpha \in \Theta$. Therefore $\Psi \subset \Theta$. On the other hand if $\alpha \notin \Psi$, then $Rs_\alpha RwR = Rs_\alpha wR$, which by Theorem 5.13 is not contained in $\overline{RwR}$, and consequently $\alpha \notin \Theta$. Therefore $\Psi = \Theta$.

In the sequel we also need the following decomposition Lemma.

(5.16) **Lemma.** *Let $\mathbf{Q} = \mathbf{P}_\Theta$ be a standard parabolic $\mathbb{R}$-subgroup and $w \in W$. Then there exist analytic subgroups N^+ and N^- of G such that the following conditions hold.*

a) *N^+ and N^- are normalised by every element of Z.*

b) *$N = N^+ N^-$ and $N^+ \cap N^-$ is the trivial subgroup.*

c) *For any $g \in \mathbf{Q}_{\mathbb{R}} wN$ there exist uniquely defined elements $x \in \mathbf{Q}_{\mathbb{R}}$ and $n \in N^-$ such that $g = xwn$.*

Proof. Put

$$\Psi^+ = \{\alpha \in \Phi^+ \mid w(\alpha) \in \langle \Theta \rangle \cup \Phi^+\}$$

where $\langle \Theta \rangle$ is the set of roots which are integral combinations of elements of Θ. Also let $\Psi^- = \Phi^+ - \Psi^+$. It is straightforward to verify that $\mathfrak{N}^+ = \sum_{\alpha \in \Psi^+} \mathfrak{G}^\alpha$ and

$\mathfrak{N}^- = \sum_{\alpha \in \Psi^-} \mathfrak{G}^\alpha$ are Lie subalgebras of $\mathfrak{G}$ defined over $\mathbb{R}$. Hence there exist (uniquely defined) analytic subgroups N^+ and N^- such that the corresponding Lie subalgebras of $\mathfrak{G}$, over $\mathbb{C}$, coincide with $\mathfrak{N}^+$ and $\mathfrak{N}^-$ respectively. It is then easy to check that assertions a) and b) are satisfied. (cf. [25], Proposition 1.1.4.5, for an idea of the proof of (b)). Also evidently $wN^+w^{-1} \subset \mathbf{Q}_\mathbb{R}$ and $wN^-w^{-1} \cap \mathbf{Q}_\mathbb{R}$ is the trivial subgroup. Thus $\mathbf{Q}_\mathbb{R} wN = \mathbf{Q}_\mathbb{R}(wN^+w^{-1})(wN^-) = \mathbf{Q}_\mathbb{R} wN^-$. Now if $g \in \mathbf{Q}_\mathbb{R} wN^-$ then $gw^{-1} \in \mathbf{Q}_\mathbb{R}(wN^-w^{-1})$ and because of the above condition there exist uniquely defined elements $x \in \mathbf{Q}_\mathbb{R}$ and $y \in wN^-w^{-1}$ such that $gw^{-1} = xy$; If $n \in N^-$ is the element such that $y = wnw^{-1}$ then we have $g = xwn$, the elements being uniquely defined.

§ 6. Invariant Measures on Arithmetic Homogeneous Spaces

We now obtain a classification of ergodic invariant measures of maximal horospherical flows on homogeneous spaces corresponding to certain arithmetic lattices.

(6.1) **Theorem.** *Let the subgroups G, N, Z, $G_\mathbb{Q}$ etc. be as in § 5.5. Further let $\Gamma \subset G_\mathbb{Q}$ be an arithmetic lattice in G such that $(\overline{F\Gamma})^0 = G^0$, where F is the smallest normal subgroup of G containing every connected non-compact simple Lie subgroup of G. Let σ be an N-invariant ergodic measure on G/Γ. Then there exist, a closed subgroup L of G and a $y \in ZG_\mathbb{Q}$, such that the following conditions are satisfied.*

i) *$L^0 y\Gamma/\Gamma$ is a closed orbit which admits a finite L^0-invariant measure (unique upto a scalar multiple).*
ii) *$\sigma(G/\Gamma - L^0 y\Gamma/\Gamma) = 0$ and σ* is *L^0-invariant.*

Further, in the notation and terminology as in § 5.1, we have the following:

iii) *There exists a standard parabolic $\mathbb{R}$-subgroup $\mathbf{Q}$ of the ambient group $\mathbf{G}$ such that the following holds: L is a normal subgroup of $\mathbf{Q}_\mathbb{R}$ and further, if V is the smallest normal subgroup of $\mathbf{Q}_\mathbb{R}$ containing N, then $L^0 = (\overline{V\Gamma})^0 = (\overline{V(\mathbf{Q}_\mathbb{R} \cap \Gamma)})^0$.*

iv) *L is the group of $\mathbb{R}$-elements of an algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$.*

Note. Γ being an arithmetic lattice, the condition $(\overline{F\Gamma})^0 = G^0$ as above, forces G to be a semisimple Lie group. We shall however not need this information.

Proof. In view of Lemma 1.11, for proving the theorem, we may without loss of generality assume that $\Gamma \subset G^0$. Now let π be the (Γ, N)-invariant measure on G defined by $\pi = i_G l_\Gamma(\sigma)$ (cf. § 1). We note that $G = ZG^0$ (cf. [3], § 14). Hence there exists $\xi \in Z$ such that $\pi(G^0\xi) > 0$. Since ξ normalises N, the theorem is true for σ if and only if it is true for its right translate by ξ. Hence in the proof we may assume $\pi(G^0) > 0$. By ergodicity of σ this already implies that $\pi(G - G^0) = 0$.

Now, if $\pi(G^0 - jP^-N) = \pi(G^0 - j(P^- \cap G^0)N) = 0$ for all $j \in G_\mathbb{Q} \cap G^0$ (notations as in § 5.5) then in view of Remark 3.3 and 3.8, Theorem 3.4 implies that π is G^0-invariant. In this case evidently the theorem holds if we choose $L = G$ and y to be the identity element. Next suppose otherwise. Thus there exists $j \in G_\mathbb{Q} \cap G^0$ such that $\pi(G - jP^-N) > 0$. Recall that

$$\begin{aligned} G - jP^- N &= G - jP^- R = G - j\varrho^{-1} P \varrho R \\ &= j\varrho^{-1}(G - P\varrho R) \\ &= j\varrho^{-1}\Big(\bigcup_{w \in W_0} PwR\Big) \end{aligned}$$

where $\varrho \in G_{\mathbb{Q}}$ and $W_0 \subset W$ are as in (5.9). Let $w \in W_0$ be chosen so that i) there exists $q \in G_{\mathbb{Q}}$ such that $\pi(qPwR) > 0$ and ii) $\pi(r(\overline{PwR} - PwR)) = 0$ for all $r \in G_{\mathbb{Q}}$. We note that since $\pi(G - jP^- N) > 0$ and $\varrho \in G_{\mathbb{Q}}$, such an element w exists.

Now let $Y = \overline{PwR}$ and $Q' = \{g \in G \mid gY = Y\}$. Since $w \in W_0$, $Y = \overline{RwR}$. Hence by Corollary 5.15 $Q' = (\mathbf{P}_{\Theta'})_{\mathbb{R}}$ where $\Theta' = \{\alpha \in \Delta \mid l(s_\alpha w) < l(w)\}$. Next let $\mathbf{Q}$ be the largest standard parabolic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ and contained in $\mathbf{P}_{\Theta'}$. Since Q' contains P such a subgroup exists. Further, in view of Proposition 5.3, $\mathbf{Q} = \mathbf{P}_\Theta$ where Θ is the largest $\mathbb{Q}$-saturated subset of Θ'. Now let V be the smallest normal subgroup of $\mathbf{Q}_{\mathbb{R}}$ containing a maximal unipotent subgroup. Let $\mathbf{L}$ be the smallest algebraic $\mathbb{R}$-subgroup of $\mathbf{G}$ containing $(\overline{V\Gamma})^0$. Then by Proposition 2.3, $\mathbf{L}$ is a normal subgroup of $\mathbf{Q}$ and it is defined over $\mathbb{Q}$. Let $\mathbf{Q} = \mathbf{Z}_\Theta \cdot \mathbf{U}_\Theta$ be the standard Levi decomposition of $\mathbf{Q}$. Then clearly $\mathbf{L} = (\mathbf{L} \cap \mathbf{Z}_\Theta) \cdot \mathbf{U}_\Theta$ is a Levi decomposition for $\mathbf{L}$. Since $\mathbf{Z}_\Theta$ is a Zariski connected reductive algebraic group defined over $\mathbb{Q}$ and $\mathbf{L} \cap \mathbf{Z}_\Theta$ is a normal subgroup of $\mathbf{Z}_\Theta$ defined over $\mathbb{Q}$ there exists a normal algebraic subgroup $\mathbf{Z}'$ of $\mathbf{Z}_\Theta$ defined over $\mathbb{Q}$ such that $\mathbf{Z}' \cap \mathbf{L}$ is finite and $\mathbf{Z}_\Theta = (\mathbf{L} \cap \mathbf{Z}_\Theta)\mathbf{Z}'$. Since $\mathbf{L} \cap \mathbf{Z}_\Theta$ contains a maximal unipotent subgroup of $\mathbf{Z}_\Theta$ it is evident from the root space decomposition that $\mathbf{Z}'$ is contained in $\mathbf{Z}$. In particular

$$\mathbf{Q}_{\mathbb{R}}^0 = (\mathbf{Z}_\Theta)_{\mathbb{R}}^0 (\mathbf{U}_\Theta)_{\mathbb{R}}^0 \subset (\mathbf{L} \cap \mathbf{Z}_\Theta)_{\mathbb{R}}^0 \cdot \mathbf{Z}'_{\mathbb{R}} (\mathbf{U}_\Theta)_{\mathbb{R}} \subset \mathbf{L}_{\mathbb{R}} \mathbf{Z}.$$

We shall now put $Q = \mathbf{Q}_{\mathbb{R}}$, and $L = \mathbf{L}_{\mathbb{R}}$. Since Q^0 is connected, the last inclusion relation implies that $Q^0 \subset L^0 Z$. Since $Q = Q^0 Z$ (cf. [3], §14) we can also conclude that $Q = L^0 Z$.

The proof of the theorem will be completed by showing that there exists $z_0 \in Z$ such that π is a (Γ, L^0)-invariant measure supported on $\Gamma q z_0 L^0$, the latter being a closed set. The proof depends on the following.

(6.2) **Proposition.** *Let π' be the measure on G defined by $\pi'(E) = \pi(q(E \cap Y))$ for any Borel subset E of G. Then π' is (L^0, N)-invariant.*

We defer the proof of the Proposition until the next section and first complete the proof of the theorem assuming the validity of the Proposition. This will be achieved in several steps.

Step i) π' is ergodic as a $(q^{-1}\Gamma q \cap Q, N)$-invariant measure.

Proof. Consider the family

$$\mathscr{F} = \left\{ \begin{array}{l} E \mid \pi'(E) > 0 \text{ and there exist } k \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_k \in q^{-1}\Gamma q \\ \text{such that } E = Y \cap \gamma_1 Y \ldots \cap \gamma_k Y \end{array} \right\}.$$

Since Y is the set of real points of an algebraic variety (cf. Remark 5.14), $\mathscr{F}$ admits a minimal element, say E_0. Then evidently for any $\gamma \in q^{-1}\Gamma q$ either $\gamma E_0 = E_0$ or $\pi'(\gamma E_0 \cap E_0) = 0$. We shall show that $E_0 = Y$ and then deduce the above claim.

Since by Proposition 6.2, π' is invariant under the left action of L^0, there exists a measure $\bar{\pi}'$ on $L^0\backslash G$ such that for any $\varphi\in C_c(G)$

$$\int \varphi(x)\mathrm{d}\pi'(x)=\int \mathrm{d}\bar{\pi}'(L^0 x)\int \varphi(yx)\mathrm{d}\lambda(y) \tag{6.3}$$

where λ is a Haar measure on L^0. It is straightforward to verify that for any $x\in G$ the set $L\cap E_0 x^{-1}$ is the set of real points of an algebraic variety defined over $\mathbb{R}$. We deduce that for any $x\in G$ either $L^0 x$ is contained in E_0 or $\lambda(L^0\cap E_0 x^{-1})=0$. Hence, in view of (6.3) it follows that there exists a L^0-invariant Borel subset E_1 of E_0 such that $\pi'(E_0-E_1)=0$. In particular, $\pi'(E_1)>0$.

Now let $x\in E_1$ and let $\xi\in W$ be such that $x\in R\xi R=N\xi R$. Since E_1 is L^0-invariant and L^0 contains N we get that $L^0\xi R$ is contained in $E_1 R$. But $L^0\xi R=L^0\xi ZR=L^0 Z\xi R=Q\xi R$. Hence $E_1 R$ is a (finite) union of double cosets of the form $Q\xi R$ where $\xi\in W$. But recall that $\pi'(E_1 R)>0$ and by the choice of w, for any $\xi\in W$, $\pi'(Q\xi R)>0$ implies that $w\in Q\xi R$. Hence QwR is contained in $E_1 R$. But clearly $E_1 R$ is contained in E_0. Hence $QwR\subset E_0$. However QwR is an open dense subset of Y whereas E_0 is a closed subset of Y. Hence we must have $E_0=Y$. In other words, for any $\gamma\in q^{-1}\Gamma q$ either $\gamma Y=Y$ or $\pi'(\gamma Y\cap Y)=0$. It may be observed that π' is the restriction to Y of the measure $q^{-1}\pi$ (defined by $(q^{-1}\pi)(E)=\pi(qE)$ for any Borel set E). Since $q^{-1}\pi$ is clearly an ergodic $(q^{-1}\Gamma q,N)$-invariant measure, the above condition implies that π' is ergodic as a $(q^{-1}\Gamma q\cap Q,N)$-invariant measure.

Step ii) There exists $z\in Z$ such that $\pi'(L^0 wzN)>0$.

Proof. Put $\Gamma_1=(q^{-1}\Gamma q)\cap Q$. Evidently, $L_1=\Gamma_1 L^0$ is a closed subgroup of Q. Further since $Q=L^0 Z$ we also get $Q=L_1 Z$. Let Z_1 be a Borel subset of Z such that for any $x\in Q$ there exists a unique element $z\in Z_1$ such that $x\in L_1 z$; that is, Z_1 is a Borel section for $L_1\backslash Q$. Now $QwN=L_1 Z_1 wN$ and using Lemma 5.16 it is easy to verify that $\{L_1 zwN\mid z\in Z_1\}$ is a countably separated partition of QwN. Each element of the partition is invariant under the left action of Γ_1 and the right action of N. Since $\pi'(QwN)>0$ and π' is ergodic as a (Γ_1,N)-invariant measure it follows that π' is concentrated on a single element of the above partition. Observe that since w normalises Z, each element of the partition also has the form $L_1 wzN$. Thus there exists $z\in Z$ such that $\pi'(L_1 wzN)>0$. Since $L_1=\Gamma_1 L^0$ and π' is (Γ_1,N)-invariant we conclude that $\pi'(L^0 wzN)>0$.

Step iii). wNw^{-1} is contained in L^0.

Proof. Let N^+ and N^- be the subgroups of N as in Lemma 5.16 corresponding to $\mathbf{Q}$ and w as above. Observe that since wN^+w^{-1} consists only of unipotent elements and is contained in Q, it must be contained in L^0. Therefore we have $L^0 wzN=L^0 wNz=L^0(wN^+w^{-1})(wN^-z)=L^0 wzN^-$. Further, the map $\tau: L^0\times N^-\to L^0 wzN^-$, sending $(x,n)\in L^0\times N^-$ to $xwzn$, is a Borel isomorphism of $L^0\times N^-$ onto $L^0 wzN$. Since by Proposition 6.2, π' is (L^0,N)-invariant it follows that the restriction of π' to $L^0 wzN$ is the image under τ of the product of the Haar measures λ and ν on L^0 and N^- respectively. By Proposition 2.3 $L^0\cap\Gamma_1$ is a lattice in L^0. Let Ω be a fundamental domain in L^0 for the left action of $L^0\cap\Gamma_1$; that is, Ω is a Borel subset of L^0 such that for any $x\in L^0$ there exists a unique element $a\in\Omega$ such that $x\in(L^0\cap\Gamma_1)a$. Put $D=\tau(\Omega\times N^-)$. Then for any $\gamma\in L^0\cap\Gamma_1$, $\pi'(\gamma D\cap D)=0$. Since $\Gamma_1\subset Q=L^0 Z$, for any $\gamma\in\Gamma_1$ the sets $\gamma L^0 wzN$ and $L^0 wzN$ are disjoint unless

$\gamma\in L^0$. Hence we have $\pi'(\gamma D\cap D)=0$ for all $\gamma\in\Gamma_1$. Finally, for $\gamma\in q^{-1}\Gamma q-\Gamma_1 = q^{-1}\Gamma q - Q$ we know that $\pi'(\gamma Y\cap Y)=0$ and therefore we conclude that $\pi'(\gamma D\cap D)=0$ for all $\gamma\in q^{-1}\Gamma q$. Recall that π' is the restriction of the measure $q^{-1}\pi$ and that $q^{-1}\pi$ is a $(q^{-1}\Gamma q, N)$-invariant ergodic measure on G. Since $q^{-1}\Gamma q$ is an arithmetic lattice in G, by Theorem 4.1 in [9], $q^{-1}\pi$ is $q^{-1}\Gamma q$-finite. Hence $(q^{-1}\pi)(D)$ is finite (cf. §1.7). Since $D\subset Y$ this means that $\pi'(D)$ is finite. But recall that $\pi'(D) = \lambda(\Omega)\times\nu(N^-)$. Hence $\nu(N^-)$ is finite. However, being an analytic subgroup of N, N^- is a simply connected nilpotent Lie group and hence $\nu(N^-)$ can be finite only if N^- is the trivial subgroup. Thus finally we have $N=N^+N^-=N^+$ and therefore $wNw^{-1}=wN^+w^{-1}\subset L^0$.

Step iv) $w\in Q$.

Proof. Since $wNw^{-1}\subset L\subset Q$, wNw^{-1} is a maximal horospherical subgroup contained in Q. But as wRw^{-1} is the unique minimal parabolic subgroup of G with wNw^{-1} as the unipotent radical, we conclude that wRw^{-1} is contained in Q. Since Q and R are parabolic subgroups and $R\subset Q$, the above is possible only if $w\in Q$ (cf. [25], §1.2.3 and [17], §29.3, Lemma D).

The author is grateful to the referee for suggesting the above simpler proof of step iv).

Step v). Completion of the proof.

Recall that $\pi'(L^0wzN)>0$. Since $w\in Q=L^0Z$ there exists $z'\in Z$ such that $L^0w = L^0z'$. Thus $L^0wzN=L^0z'zN=L^0Nz'z=L^0z'z=L^0z_0$, where $z_0=z'z\in Z$. Thus we have $\pi'(L^0z_0)>0$. Observe that since L is normal in Q, $L^0z_0=z_0L^0$ and in particular, it is invariant under the action of L^0 on the *right* hand side. Further, since the restriction of the measure π' to L^0z_0 is L^0-invariant under the left action and L^0 is unimodular, we deduce that the restriction is invariant under the action on the right; i.e. $\pi'(Et)=\pi'(E)$ for any Borel subset E of L^0z_0 and any $t\in L^0$. Since $q^{-1}\pi$ is a $(q^{-1}\Gamma q, N)$-invariant ergodic measure, whose restriction to Y is π', the last assertion implies that $q^{-1}\pi$ is the (unique upto a scalar multiple) $(q^{-1}\Gamma q, L^0)$-invariant measure concentrated on $q^{-1}\Gamma q z_0 L^0$. Therefore π is the (Γ, L^0)-invariant measure concentrated on $\Gamma q z_0 L^0$. Hence σ is the L^0-invariant measure concentrated on the L^0-orbit of $y=(qz_0)^{-1}$.

Since $\mathbf{L}$ is an algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{Q}$ and admits no characters defined over $\mathbb{Q}$, by Lemma 2.2 $q^{-1}\Gamma q L^0$ is closed. Hence so are $\Gamma q z_0 L^0$ and $L^0 y\Gamma$. Therefore $L^0y\Gamma/\Gamma$ is a closed orbit of L^0. Also as noted earlier the isotropy subgroup of the action of L^0 is clearly a lattice in L^0. Hence the orbit admits a finite L^0-invariant measure.

Thus the proof of i) and ii) as in the statement of Theorem 6.1 is complete. Assertions iii) and iv) are evident from Proposition 2.3 and the choice of L.

6.4 *Remark.* In the proof of Theorem 6.1, it may be of some interest to note that a posteriori, $Q=Q'$. This may be proved as follows. Since $wNw^{-1}\subset L\subset Q$, $w(\Phi^+)\subset\langle\Theta\rangle\cup\Phi^+$ (notations as before). On the other hand, for any $\alpha\in\Theta'$, $l(s_\alpha w)<l(w)$. Therefore by Lemma 5.12, $\langle\Theta'\rangle\cap\Phi^-\subset w(\Phi^+)$, where $\Phi^- = \Phi-\Phi^+$. Hence $\langle\Theta'\rangle\cap\Phi^-\subset\langle\Theta\rangle\cup\Phi^+$. This implies that $\langle\Theta'\rangle\cap\Phi^+\subset\langle\Theta\rangle$ and therefore, $\Theta'\subset\Theta$. But as $\Theta\subset\Theta'$, we conclude that $\Theta'=\Theta$. Hence $Q'=Q$.

§ 7. *Proof of Proposition* 6.2

We now proceed to prove Proposition 6.2 keeping the same notations as before. Recall that π' is a (Γ_1, N)-invariant measure concentrated on Y, where $\Gamma_1 = (q^{-1}\Gamma q)\cap Q$. Further by the choice of the element $w\in W_0$ we have $\pi'(j(Y-PwR)) = 0$ for all $j\in G_{\mathbb{Q}}$. Since $P\subset Q$, in particular we have $\pi'(Y-QwN)=0$.

Recall that $Q=L^0 Z$. Let Z_0 be a Borel subset of Z such that for every $x\in Q$ there exists a unique element $z\in Z_0$ such that $x\in L^0 z$. It is easy to verify, using Lemma 5.16, that the partition of QwN into $\{L^0 zwN \mid z\in Z_0\}$ is a countably separated partition. Hence π' admits a direct integral decomposition, say $\{\pi'_z\}_{z\in Z_0}$, with respect to this partition (cf. Proposition 1.9) where each $\pi'_z, z\in Z_0$ is $(\Gamma_1\cap L^0, N)$-invariant measure on G and $\pi'_z(G\text{–}L^0 zwN)=0$. Since $L_{\mathbb{Q}}$ is countable and $\pi'(Y-jPwR)=\pi'(j(Y-PwR))=0$ for all $j\in L_{\mathbb{Q}}$ we can choose each $\pi'_z, z\in Z_0$ to be such that $\pi'_z(Y-jPwR)=0$ for all $j\in L_{\mathbb{Q}}$. Also to prove Proposition 6.2 it is enough to prove that each π'_z as above is L^0-invariant; equivalently, it is enough to prove that $p_N(\pi'_z)$ is L^0-invariant (cf. Proposition 1.3). Let $z\in Z_0$ be fixed arbitrarily. Let $N'=L\cap wNw^{-1}$ and let $i\colon L/N'\to LzwN/N$ be the map defined by $i(gN')=gzwN$. Then i is a homeomorphism. Let $\beta=i^{-1}(p_N(\pi'_z))$ where $p_N(\pi'_z)$ is viewed as a measure on $L^0 zwN/N$, by restriction. Let $\Gamma'=\Gamma_1\cap L^0$ and $P'=P\cap L$. It is straightforward to verify that β is a Γ'-invariant measure on L^0/N' and $\beta(L^0/N'-jP'N'/N')=0$ for all $j\in L_{\mathbb{Q}}\cap L^0$. Therefore to prove Proposition 6.2 it is enough to prove the following.

(7.1) **Proposition.** *Let $\beta\in\mathcal{M}(L^0/N',\Gamma')$ and $\beta(L^0/N'-jP'N'/N')=0$ for all $j\in L_{\mathbb{Q}}\cap L^0$. Then β is L^0-invariant.*

Proof. Recall that $\mathbf{L}=\mathbf{T}\cdot\mathbf{U}_\Theta$ (semi-direct product) where $\mathbf{T}=\mathbf{L}\cap\mathbf{Z}_\Theta$. $\mathbf{T}$ is a reductive Zariski connected algebraic group defined over $\mathbb{Q}$. Put $T=\mathbf{T}_{\mathbb{R}}$. Then T_0 is a connected reductive Lie group. Further T_0 satisfies the condition of Theorem 3.4; viz. if F is the smallest (normal) subgroup of T_0 containing every non-compact simple Lie subgroup of T_0 then $(\overline{F(T\cap\Gamma)})^0=T^0$ (cf. Proposition 2.3). Let $\eta\colon\mathbf{L}\to\mathbf{T}$ be the canonical projection homomorphism and let $\Gamma^*=\eta(\Gamma')$, $N^*=\eta(N')$ and $P^*=\eta(P')$. Then Γ^* is an arithmetic lattice in T. Further the natural quotient map $\bar\eta\colon L^0/\Gamma'\to T^0/\Gamma^*$ is proper.

Let β_1 be the N'-invariant measure on L^0/Γ' dual to β (cf. §1). Then $\bar\eta(\beta_1)$ is a N^*-invariant measure on T^0/Γ^*. Let β^* be the Γ^*-invariant measure on T^0/N^* dual to $\bar\eta(\beta_1)$. Then the hypothesis implies that

$$\beta^*(T^0/N^*-jP^*N^*/N^*)=0 \text{ for all } j\in T_{\mathbb{Q}}\cap T^0. \tag{7.2}$$

Now we shall first prove, using Theorem 3.4, that β^* is T^0-invariant and then use Theorem 4.1 to deduce that β' is L^0-invariant, thus proving Proposition 7.1.

Recall that $\mathbf{Z}_\Theta$ is a reductive Zariski connected algebraic group defined over $\mathbb{Q}$ and $\mathbf{T}$ is a normal subgroup of $\mathbf{Z}_\Theta$ defined over $\mathbb{Q}$. Hence there exists a normal algebraic subgroup $\mathbf{C}$ defined over $\mathbb{Q}$ such that $\mathbf{C}\cap\mathbf{T}$ is finite, $\mathbf{C}\cdot\mathbf{T}$ has finite index in $\mathbf{Z}_\Theta$ and every element of $\mathbf{C}$ commutes with every element of $\mathbf{T}$. This in particular implies that the subgroups $\mathbf{S}_1$ and $\mathbf{A}_1$ defined to be the Zariski-connected components of the identity in $\mathbf{S}\cap\mathbf{T}$ and $\mathbf{A}\cap\mathbf{T}$ respectively are maximal $\mathbb{Q}$-split and maximal $\mathbb{R}$-split tori (in $\mathbf{T}$) respectively. Let Φ_1 be the set of those *non-trivial* characters on $\mathbf{A}_1$ which are restrictions of elements of $\langle\Theta\rangle$. Then Φ_1 is the root system of $\mathbf{T}$ with respect to $\mathbf{A}_1$ and the root space corresponding to $\alpha_0\in\Phi_1$ is $\sum_\alpha \mathfrak{G}^\alpha$, where the summation is taken over all those roots in $\langle\Theta\rangle$ whose restriction to $\mathbf{A}_1$ is α_0. On Φ_1 we choose the ordering which is *opposite* to the ordering induced by Φ; that is, $\alpha_0\in\Phi_1$ is taken to be positive if it is the restriction of a negative root with respect to the ordering on Φ. We show that with respect to this ordering the subgroup N^* is indeed the standard maximal horospherical subgroup of T. Since $\Theta\subset\{\alpha\in\Delta\,|\,l(s_\alpha w)<l(w)\}$ by Lemma 5.12 for any $\alpha\in\Theta$ we get that $-\alpha\in w(\Phi^+)$. Since the restrictions of $\{-\alpha\,|\,\alpha\in\Theta\}$ form the system of simple roots in Φ_1^+ the last assertion implies that the standard maximal horospherical subgroup of T is contained in wNw^{-1}, and hence in N^*. But then since N^* consists only of unipotent elements it must be the standard maximal horospherical subgroup.

Let $\mathbf{P}_1^-$ be the minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{T}$ which contains the centraliser of $\mathbf{S}_1$ in $\mathbf{T}$ and the maximal horospherical subgroup corresponding to all the negative roots in Φ_1. We show that P^* is contained in $\mathbf{P}_1^-$. Since $\mathbf{P}=\mathbf{Z}_{\Delta_0}\cdot\mathbf{U}_{\Delta_0}$ we have $P'=L\cap P\subset(\mathbf{L}\cap\mathbf{Z}_{\Delta_0})\cdot\mathbf{U}_{\Delta_0}=(\mathbf{T}\cap\mathbf{Z}_{\Delta_0})\cdot\mathbf{U}_{\Delta_0}$. Hence $P^*=\eta(P')\subset(\mathbf{T}\cap\mathbf{Z}_{\Delta_0})\cdot\eta(\mathbf{U}_{\Delta_0})$. Evidently $\mathbf{T}\cap\mathbf{Z}_{\Delta_0}$ centralises $\mathbf{S}_1$ and hence it is contained in $\mathbf{P}_1^-$. On the other hand the Lie subalgebra corresponding to $\eta(\mathbf{U}_{\Delta_0})$ is a certain sum of root spaces of the form $\mathfrak{G}^\alpha$, $\alpha\in\Phi^+$. Since the restriction of any such root to $\mathbf{A}_1$ does not belong to Φ_1^+ it follows that $\eta(\mathbf{U}_{\Delta_0})$ is contained in $\mathbf{P}_1^-$. Hence $P^*\subset\mathbf{P}_1^-$. Put $P_1^-=(\mathbf{P}_1^-)_{\mathbb{R}}$. Then $P^*\subset P_1^-$ and by (7.2) for any $j\in T_{\mathbb{Q}}\cap T^0$

$$\beta^*(T^0/N^*-jP_1^-N^*/N^*)\leqq\beta^*(T^0/N^*-jP^*N^*/N^*)=0.$$

Since Γ^* is an arithmetic lattice in T and β^* is a Γ^*-invariant measure, in view of the above verifications and Remarks 3.3 and 3.8, Theorem 3.4 implies that β^* is T^0-invariant.

Next, using Theorem 4.1 we deduce that β is L^0-invariant. We need to verify the following

(7.3) **Lemma.** *Let V be the smallest normal subgroup of L^0 containing $(\mathbf{U}_\Theta)_{\mathbb{R}}\cap wNw^{-1}$. Then $(\overline{\Gamma'V})^0=(\mathbf{U}_\Theta)_{\mathbb{R}}$*

Proof. Recall that Γ' is an arithmetic lattice in $L=T(\mathbf{U}_\Theta)_{\mathbb{R}}$ and that $\mathbf{U}_\Theta$ is defined over $\mathbb{Q}$. Hence $\Gamma'(\mathbf{U}_\Theta)_{\mathbb{R}}$ is closed. Therefore $\overline{\Gamma'V^0}$ is contained in $(\mathbf{U}_\Theta)_{\mathbb{R}}$.

Observe that the subgroups $(\mathbf{U}_\Theta)_{\mathbb{R}}\cap wNw^{-1}$ and L^0 are normalised by every element of Z. Hence it follows that V is a normal subgroup of Q. For any $g\in\mathbf{Q}_{\mathbb{Q}}, g\Gamma'g^{-1}$ is commensurable with Γ' and consequently $g(\overline{\Gamma'V})^0g^{-1}=(\overline{\Gamma'V})^0$. Hence the Lie subalgebra of $\overline{\Gamma'V^0}$ is invariant under the adjoint action of every $g\in\mathbf{Q}_{\mathbb{Q}}$. Since $\mathbf{Q}_{\mathbb{Q}}$ is Zariski dense in $\mathbf{Q}$ (cf. [3], Corollary 3.20) the same holds for every element of Q. Hence $(\overline{\Gamma'V})^0$ is a normal subgroup of Q. Being an analytic subgroup consisting only of unipotent elements, it coincides with the group of $\mathbb{R}$-elements of a (unique) Zariski connected algebraic $\mathbb{R}$-subgroup $\mathbf{U}'$ of $\mathbf{Q}$. In view of the above, $\mathbf{U}'$ is a normal (algebraic) subgroup of $\mathbf{Q}$. Further since $(\overline{\Gamma'V})^0$ contains an arithmetic lattice, $\mathbf{U}'$ must be defined over $\mathbb{Q}$. Thus by Lemma 5.4 it follows that the set, say Δ', of simple roots α such that $\mathbf{G}_\alpha$ is contained in $\mathbf{U}'$, is $\mathbb{Q}$-saturated. Since $\Theta' = \{\alpha \in \Delta \mid l(s_\alpha w) < l(w)\}$, by Lemma 5.12 $\Delta - \Theta' \subset w(\Phi^+)$. But clearly $(\Delta - \Theta) \cap w(\Phi^+)$ is contained in Δ'. Hence $(\Delta - \Theta') \subset (\Delta - \Theta) \cap w(\Phi^+) \subset \Delta'$. Since Θ is the largest $\mathbb{Q}$-saturated subset of Θ' we deduce that $\Delta' = \Delta - \Theta$. Thus $\mathbf{U}'$ contains the subgroups $\mathbf{G}_\alpha, \alpha \in \Delta - \Theta$. But $\mathbf{U}_\Theta$ has no proper subgroup which contains all $\mathbf{G}_\alpha, \alpha \in \Delta - \Theta$ and is normalised by $\mathbf{Z}_\Theta$. Hence $\mathbf{U}' = \mathbf{U}_\Theta$ which proves the Lemma.

Recall that $\beta_1 \in \mathcal{M}(L^0/\Gamma', N')$ is dual to β and that $\bar{\eta}(\beta_1)$ is dual to $\beta^* \in \mathcal{M}(T^0/N^*, \Gamma^*)$. Since β^* is T^0-invariant $\bar{\eta}(\beta_1)$ is T^0-invariant (cf. (1.5)). Since β_1 is N'-invariant in view of Lemma 7.3 and Theorem 4.1 it follows that β_1 is L^0-invariant. Hence the same is true of β, which completes the proof of Proposition 7.1.

Part III: Conclusions

In the next two sections we shall deduce the main theorem from Theorem 6.1.

§ 8. Lattices in Semisimple Lie Groups

In this section let G be a connected semisimple Lie group with trivial center and without compact factors and let Γ be a lattice in G.

A lattice Λ in a semisimple Lie group H is said to be *irreducible* if the only positive dimensional normal subgroup F for which $F\Lambda$ is closed is H itself.

The lattice Γ can be "decomposed" into irreducible lattices in normal subgroups of G. More precisely we have the following

(8.1) **Proposition** (cf. [21], Theorem 5.22). *Let G and Γ be as above. Then there exist normal (semisimple) Lie subgroups $G_i, i \in I$ (a suitable indexing set) such that*

i) $G = \prod_{i \in I} G_i$ *(direct product)*

ii) *For each $i \in I$, $\Gamma_i = \Gamma \cap G_i$ is an irreducible lattice in G_i, and*

iii) $\Gamma' = \prod_{i \in I} \Gamma_i$ *is a (normal) subgroup of finite index in Γ.*

The decomposition is unique upto reindexing.

(8.2) **Theorem.** *Let G and Γ be as above and let N be a maximal horospherical subgroup of G. Let σ be a finite N-invariant ergodic measure on G/Γ. Then there exist a connected Lie subgroup L and a $y \in G$ such that the following conditions are satisfied*

i) *$Ly\Gamma/\Gamma$ is a closed orbit of L, which admits a finite L-invariant measure*

ii) *σ is L-invariant and is supported on $Ly\Gamma/\Gamma$.*

iii) *$N \subset L$ and there exists a normal subgroup N' of L such that $N' \subset N$ and L/N' is reductive.*

Proof. Firstly consider the special case when Γ is an irreducible lattice in G. In this case by Margulis's arithmeticity theorem (cf. [18]) there are the following three (not necessarily mutually exclusive) cases possible: i) G/Γ is compact, ii) $\mathbb{R}$-rank of G is 1 or iii) there exists an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ such that $G = \mathbf{G}^0_{\mathbb{R}}$ (via a topological isomorphism) and Γ corresponds to an arithmetic lattice in G, with respect to the $\mathbb{Q}$-structure on $\mathbf{G}$. Let us consider each case separately. In case i), G admits no non-trivial Γ-rational horospherical subgroup. Consequently the conditions of Theorem 3.4 are trivially satisfied and we obtain that σ is G-invariant. (This result is also proved earlier by several authors as mentioned in the introduction.) Thus $L=G$ answers the theorem. In case iii) the desired result follows directly from Theorem 6.1; we only need to note that if Theorem 8.2 is true for a suitably chosen maximal horospherical subgroup then it is true for all of them, since any two are conjugate to each other. Lastly consider case ii). We may assume G/Γ to be non-compact. Since $\mathbb{R}$-rank of G is 1, G/Γ being non-compact implies that there exists a maximal horospherical subgroup which is Γ-rational. Since, as seen above, we have the freedom to choose the maximal horospherical subgroup, we may assume N to be Γ-rational. Now let N^- be a maximal Γ-rational horospherical subgroup opposite to N. Let P and P^- be the normalisers of N and N^- respectively. Then by the Bruhat decomposition (cf. [25], Theorem 1.2.3.1) there exists $w\in G$ such that $G=PwN\cup P$ and $w^{-1}Pw=P^-$. Let J be a sufficient set of cusp elements for Γ with respect to (P^-,N,K) where K is a maximal compact subgroup of G. As before let π be the (Γ,N)-invariant measure on G associated to σ. Clearly the Γ-invariant measure $p_N(\pi)$ is Γ-finite. Now if $\pi(G-jP^-N)=0$ for all $j\in J$ then by Theorem 3.4 π is G-invariant and hence so is σ. Next suppose otherwise and let $j\in J$ be such that $\pi(G-jP^-N)>0$. Let $x=jw^{-1}$ and let π' be the measure defined by $\pi'(E)=\pi(xE)$ for any Borel subset E of G. Then π' is a $(x^{-1}\Gamma x,N)$-invariant ergodic measure on G and $\pi'(P)=\pi(jw^{-1}P)=\pi(G-jP^-N)>0$. Let $P=Z\cdot N$ be the Langlands decomposition of P. Put $\Gamma'=x^{-1}\Gamma x=wj^{-1}\Gamma jw^{-1}$. Then $\Gamma'N=w(j^{-1}\Gamma j)(w^{-1}Nw)w^{-1}=w(j^{-1}\Gamma j)\cdot N^-w^{-1}$. But since $j^{-1}\Gamma j\cap N^-$ is a lattice in N^-, $(j^{-1}\Gamma j)N^-$ is closed and hence so is $\Gamma'N$. Since every element of Z normalises N it follows that each $\Gamma'zN$, $z\in Z$ is a closed set and $\{\Gamma'zN\,|\,z\in Z\}$ yields a countably separated partition of $\Gamma'P$. Since π' is a (Γ',N)-invariant ergodic measure and $\pi'(P)>0$ we deduce that there exists $z\in Z$ such that π' is concentrated on $\Gamma'zN$. Hence π is concentrated on ΓxzN. Therefore, σ is concentrated on the N-orbit of $y\Gamma/\Gamma$ where $y=(xz)^{-1}$. Also $y\Gamma y^{-1}\cap N=z^{-1}w(j^{-1}\Gamma j)w^{-1}z\cap N=z^{-1}w(j^{-1}\Gamma j\cap N^-)w^{-1}z$. Since $j^{-1}\Gamma j\cap N^-$ is a (uniform) lattice in N^-, it follows that $y\Gamma y^{-1}\cap N$ is a lattice in N. Since the latter is the isotropy subgroup of $y\Gamma/\Gamma$ for the N-action we conclude that $Ny\Gamma/\Gamma$ is compact. Thus for the case at hand the theorem is satisfied with $L=N$.

We now consider the general case. Let G_i and Γ_i for $i\in I$ and Γ' be the subgroups as obtained from Proposition 8.1. In view of Lemma 1.11 the theorem need be proved only for Γ'; equivalently, we may assume $\Gamma=\prod_{i\in I}\Gamma_i$.

Now for $i\in I$ let $\eta_i\colon G/\Gamma\to G_i/\Gamma_i$ be the natural projection maps. For any i the measure $\eta_i(\sigma)$ is invariant and ergodic under the maximal horospherical subgroup $N_i=N\cap G_i$. Therefore by the special case considered earlier, for any $i\in I$, there exist, a subgroup L_i and an element $y_i\in G_i$ such that $L_iy_i\Gamma_i/\Gamma_i$ is a closed orbit supporting the measure $\eta_i(\sigma)$ and $\eta_i(\sigma)$ is L_i-invariant. Let $L=\prod_{i\in I}L_i$ and $y\in G$ be such that $\eta_i(y)=y_i$. Then clearly $Ly\Gamma/\Gamma$ is a closed L-orbit admitting a finite L-invariant measure. Further σ is obviously concentrated on $Ly\Gamma/\Gamma$. It only remains to show that σ is L-

invariant. A Borel subset E of G/Γ is said to be rectangular if for any $x \in G/\Gamma$, $\eta_i(x) \in \eta_i(E)$ for all $i \in I$ implies that $x \in E$. Let us fix an index say $k \in I$. Let E be any rectangular subset of G/Γ. Let $\tilde{E}$ be the rectangular set such that $\eta_i(\tilde{E}) = \eta_i(E)$ for all $i \neq k$ and $\eta_k(\tilde{E}) = G_k/\Gamma_k$. Then $\tilde{E}$ is G_k-invariant. Further the measure $\tilde{\sigma}$, defined by $\tilde{\sigma}(B) = \sigma(\tilde{E} \cap B)$ for any Borel subset B of G/Γ, is N_k-invariant. Consider the measure $\eta_k(\tilde{\sigma})$ on G_k/Γ_k. It is N_k-invariant and also absolutely continuous with respect to $\eta_k(\sigma)$. Since the latter is ergodic as a N_k-invariant measure we deduce that $\eta_k(\tilde{\sigma})$ is a scalar multiple of $\eta_k(\sigma)$. Since E is arbitrary one can deduce from this that for any rectangular subset E and $g \in L_k$, $\sigma(E)$ and $\sigma(gE)$ are $\sigma(\tilde{E})$ times the $\eta_k(\tilde{\sigma})$-measures of $\eta_k(E)$ and $\eta_k(gE)$ respectively. Therefore $\sigma(gE) = \sigma(E)$ for all $g \in L_k$. Since E is an arbitrary rectangular set it follows that σ is L_k-invariant. Now varying k over I we deduce that σ is L invariant.

(8.3) *Remark.* Case ii) in the above proof also follows from Theorem 10.1 in [8]. However the author notes with regret that the proof of Theorem 10.1 in [8], though valid for lattices in $\mathbb{R}$-rank-1 groups, is not valid in the generality that it is stated. This is because Lemma 10.3 in [8], which purports to reduce the task of proving the theorem to a special case, is incorrect. Also in the special case the notation is rather clumsy and the proof less transparent.

(8.4) *Remark.* In Theorem 8.2 if we choose N containing a maximal Γ-rational horospherical subgroup then the subgroup L has the following property: There exists a parabolic subgroup Q containing N such that a) the unipotent radical of Q is Γ-rational and b) if V is the smallest normal subgroup of Q containing N then $L = (\overline{V\Gamma})^0$. To verify this it is enough to consider the case of irreducible lattices and then apply it to the irreducible components. Consider the three possibilities as in the proof of the theorem. For arithmetic lattices this is simply assertion iii) in Theorem 6.1. For the first two cases L is G and N respectively and if we choose Q to be G and P respectively, the assertions a) and b) are satisfied.

§ 9. *Theorems and Problems*

(9.1) **Theorem.** *Let G be a reductive Lie group and Γ be a lattice in G. Let N be a maximal horospherical subgroup of G. Let σ be a finite N-invariant ergodic measure on G/Γ. Then there exist, a connected Lie subgroup L of G containing N and a $y \in G$ such that the following conditions are satisfied*

i) *$Ly\Gamma/\Gamma$ is a closed orbit of L, which admits a finite L-invariant measure and*
ii) *σ is L-invariant and $\sigma(G/\Gamma\text{-}Ly\Gamma/\Gamma) = 0$.*

Proof. Since Γ is a lattice in G, $G^0\Gamma$ has only finitely many orbits on G/Γ. Because of ergodicity only one of these orbits has positive σ-measure. Hence there is no loss of generality in assuming G to be connected. Now let H be the smallest closed normal subgroup of G such that $G' = G/H$ is a semisimple Lie group with trivial center and without compact factors. Let $\eta: G \to G/H$ be the natural projection homomorphism. The subgroup $\Gamma' = \eta(\Gamma)$ is closed and the map $\bar{\eta}: G/\Gamma \to G'/\Gamma'$ induced by η is proper (cf. [8], Lemma 9.1). Clearly $\eta(N)$ is a maximal horospherical subgroup and $\bar{\eta}(\sigma)$ is a finite $\eta(N)$-invariant ergodic measure on G'/Γ'. Hence by Theorem 8.2 there exists a connected Lie subgroup L' of G' such that $\bar{\eta}(\sigma)$ is a L'-invariant measure supported on a closed orbit $L'y'\Gamma'/\Gamma'$, where $y' \in G'$. Further there exists a subgroup N' of $\eta(N)$ such that N' is normal in L' and L'/N' is reductive. Under these conditions Proposition 9.3 in [8] asserts that there exists an analytic subgroup L of G contained in $\eta^{-1}(L')$ and $y \in \eta^{-1}(y')$ such that the assertions i) and ii) of Theorem 9.1 are satisfied.

(9.2) *Remark.* Let G, Γ and N be as in Theorem 9.1. Suppose that there exists no homomorphism φ of G^0 onto a simple Lie group H of $\mathbb{R}$-rank 1 such that $\varphi(\Gamma)$ is a non-uniform, non-arithmetic lattice in H. Then every N-invariant ergodic measure is finite. This can be deduced using Theorem 4.1 in [9] and the decomposition as in the proof of Theorem 8.2. Thus in this case the hypothesis of finiteness of σ in Theorem 9.1 is redundant. If the following question is answered in the affirmative then this would be the situation for all lattices.

(9.3) *Question.* Let H be a simple Lie group of $\mathbb{R}$-rank 1, and let Γ be a (non-uniform, non-arithmetic) lattice in H. Let N be a maximal horospherical subgroup of H. Then is it true that every N-invariant ergodic measure is finite? More generally we may ask the following: Is it true that for any element $u \in N$ every u-invariant ergodic measure is finite? This is related to the question whether given $x \in H/\Gamma$ there exists a compact subset C of H/Γ such that the sequence $\{j \in \mathbb{N} \mid u^j x \in C\}$ has positive upper density (cf. [9]).

Since the above was written, Gopal Prasad has sent to the author a proof of the analogue of Margulis's lemma for any (not necessarily arithmetic) lattice in a simple Lie group of $\mathbb{R}$-rank 1. The result implies that the set $\{j \in \mathbb{N} \mid u^j x \in C\}$, as above, is unbounded. (Apparently, the result was also known to M.S. Raghunathan.) However, whether the set has positive upper density or not is not known.

The classification of ergodic invariant measures of horospherical subgroups can be applied to determine their minimal (invariant closed non-empty) sets. Since any horospherical subgroup is solvable, any compact invariant set supports an invariant measure; this may be deduced from the Markov-Kakutani fixed point theorem. It may be deduced that every compact minimal set is the support of an ergodic invariant measure. Thus we immediately have the following.

(9.4) **Corollary.** *Let G, Γ and N be as in Theorem 9.1. Let C be a compact minimal N-invariant (non-empty) subset of G/Γ. Then there exists a closed connected subgroup L of G containing N such that C is an orbit of L.*

If G is a semisimple Lie group without compact factors then we can obtain precise information about the subgroup L. From Theorem 8.2 and Remark 8.4 we can deduce the following result.

(9.5) **Corollary.** *Let G be a connected semisimple Lie group without compact factors and let Γ be a lattice in G. Let N be a maximal horospherical subgroup of G containing a maximal Γ-rational horospherical subgroup, say U. Let P be the normaliser of U. Suppose that $N \subset P$ and V^* be the smallest normal subgroup of P containing N. Let $L^* = (\overline{V^* \Gamma})^0 = (\overline{V^*(P \cap \Gamma)})^0$. Then every compact minimal N-invariant subset of G/Γ is an orbit of L.*

Proof. Firstly we note that if M denotes the center of G then $M\Gamma$ is closed and the quotient map $\eta: G/\Gamma \to G/M\Gamma$ is a covering of finite order; this is a consequence of Borel's density theorem (cf. [8], Lemma 9.1, for an idea of the proof). Therefore for the proof we may without loss of generality assume that G has trivial center. Further, in view of Proposition 8.1, as in the proof of Theorem 8.2, we may assume Γ to be an irreducible lattice in G. Now let C be a compact minimal N-invariant subset of G/Γ. In view of the discussion in the beginning of this section, Theorem 8.2 and Remark 8.4 imply that there exists a parabolic subgroup Q of G such that a) the unipotent radical of Q is Γ-rational and b) Q contains N and if V is the smallest normal subgroup of Q containing N, then C is an orbit of $L=(\overline{V\Gamma})^0$. Consider the three possible cases as enumerated in the proof of Theorem 8.2. If G/Γ is compact, then there exists no non-trivial Γ-rational horospherical subgroup and hence $P=Q=G$ and hence $L=L^*$. Suppose next that G has $\mathbb{R}$-rank 1 and that G/Γ is not compact. Then N is a maximal Γ-rational horospherical subgroup and since Q contains N, the rank condition implies that either $Q=P$ or $Q=G$. Consequently, $L=L^*$ or G. However, in the latter case C would have to be G/Γ which is certainly not minimal, as N itself admits closed orbits. Finally suppose that Γ is an arithmetic lattice in G (with respect to a suitable $\mathbb{Q}$-structure). In this case P and Q are parabolic $\mathbb{R}$-subgroups defined over $\mathbb{Q}$. It is possible to make our choices for the notions in §5 in such a way that N is the standard maximal horospherical subgroup and P is the group of $\mathbb{R}$-elements of the standard minimal parabolic $\mathbb{Q}$-subgroup. Assertion iii) in Theorem 6.1 therefore implies that $P \subset Q$. Hence V^* is contained in V and in turn, $L^*=(\overline{V^*(P\cap\Gamma)})^0=(\overline{V^*\Gamma})^0 \subset (\overline{V\Gamma})^0=L$. Now, in view of Theorem 6.1, the orbit of L supporting the ergodic invariant measure contains a point of the form zq where $z\in P$ and $q\in G_{\mathbb{Q}}$. In particular, $zq\Gamma/\Gamma \subset C$. We now show that $L^*zq\Gamma$ is a closed subset of G. By Proposition 2.3, L^* is a normal subgroup of P and hence z normalises L^*. Also, in view of Lemma 2.2 $L^*q\Gamma$ is closed. Hence $L^*zq\Gamma=zL^*q\Gamma$ is a closed set. Therefore $L^*zq\Gamma/\Gamma$ is a closed subset of G/Γ. Since $L^*zq/\Gamma \subset Lzq\Gamma/\Gamma \subset C$ and $N\subset L^*$ by minimality of C we get $C=L^*zq\Gamma/\Gamma$.

(9.6) *Remark.* We recall that if $G=\mathbf{G}_{\mathbb{R}}^0$ where $\mathbf{G}$ is an algebraic $\mathbb{Q}$-group and Γ is an arithmetic lattice in G, then every non-empty closed subset of G/Γ invariant under a horospherical flow supports an ergodic invariant measure (cf. Proposition 7.1, [9]). Thus in this case the arguments as in the proof of Corollary 9.5 yield that every closed (not necessarily compact) invariant subset of a maximal horospherical flow contains a compact minimal invariant set. We do not know whether a similar assertion holds for lattices in simple Lie groups of $\mathbb{R}$-rank 1.

It may be conjectured that the closure of any orbit of a maximal horospherical subgroup N is the support of an ergodic invariant measure. By Theorem 9.1 it would then be an orbit of a suitable subgroup containing N. We note that in view of the results in [10] §4, this is true for the actions of certain horospherical subgroups of $SL(n,\mathbb{R})$ on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

The study of dynamics of horospherical subgroups can be applied to study the orbits of certain discrete linear groups. Observe that if G is a Lie group and Γ and N be two closed subgroup of G then for any $g,h\in G$ the closure of the Γ-orbit of gN/N contains hN/N if and only if the closure of the N orbit of $g^{-1}\Gamma/\Gamma$ contains $h^{-1}\Gamma/\Gamma$; either of these holds if and only if $\overline{\Gamma gN}$ contains ΓhN. Therefore using Corollary 9.5 and Remark 9.6 it is easy to deduce the following result.

(9.7) **Corollary.** *Let the notations be as in Corollary 9.5. Suppose also that G admits a $\mathbb{Q}$-structure such that Γ is an arithmetic lattice in G. Let $\varrho: G \to GL(V)$ be a finite dimensional representation of G and let $v \in V$ be a vector fixed by $\varrho(N)$. Then $\overline{\varrho(\Gamma)v}$ contains an L^*-orbit.*

Recall that for uniform lattices, L^* coincides with G and hence the $\varrho(\Gamma)$ orbit of every vector fixed by $\varrho(N)$ is dense in the $\varrho(G)$-orbit. For the non-uniform lattice $SL(n, \mathbb{Z})$ in $SL(n, \mathbb{R})$ we have the following result.

(9.8) **Corollary.** *Let V be a vector space of dimension n and let $SL(n, \mathbb{R})$ act on V via a basis $\{e_1, e_2, \ldots, e_n\}$. Let $f_1, f_2, \ldots, f_{n-1}$ be linearly independent vectors in V. Then there exist rational vectors $a_1, a_2, \ldots, a_{n-1}$ in V (with respect to the basis $\{e_1, \ldots, e_n\}$), non-zero scalars $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ in $\mathbb{R}$ and a sequence $\{\gamma_j\}$ in $SL(n, \mathbb{Z})$ such that*

$$\gamma_j(f_1 \wedge f_2 \wedge \ldots \wedge f_k) \to \lambda_k(a_1 \wedge a_2 \wedge \ldots \wedge a_k)$$

for all $k = 1, 2 \ldots, n-1$.

Proof. Let $\wedge^j V$, $j = 1, 2, \ldots, n-1$ be the jth exterior power of V and consider $W = \bigoplus_{j=1}^{n-1} \wedge^j V$. Let ϱ be the representation of $G = SL(n, \mathbb{R})$ on W obtained as the direct sum of exterior power representations of the natural representation of $SL(n, \mathbb{R})$ on V via the basis $\{e_1, e_2, \ldots, e_n\}$. Let $e = (e_1, e_1 \wedge e_2, \ldots, e_1 \wedge e_2 \wedge \ldots \wedge e_{n-1}) \in W$. It is easy to verify that the isotropy subgroup of e under the G-action via ϱ is the subgroup N consisting of all upper triangular unipotent matrices in $SL(n, \mathbb{R})$. The orbit $\mathfrak{O}$ of e in W consists of the vectors a of the form $(a_1, a_1 \wedge a_2, \ldots, a_1 \wedge a_2 \wedge \ldots \wedge a_{n-1})$, where $a_1, a_2, \ldots, a_{n-1}$ are any linearly independent vectors in V. The map $\eta: G/N \to \mathfrak{O}$ defined by $\eta(gN/N) = \varrho(g)e$ is a continuous bijection onto $\mathfrak{O}$.

We now recall that $\Gamma = SL(n, \mathbb{Z})$ is an arithmetic lattice in G. Let $f \in \mathfrak{O}$ and let $g \in G$ be such that $\eta(gN) = f$. Set $C = \overline{Ng^{-1}\Gamma}$. Then C/Γ is a closed N-invariant subset of G/Γ. By Remark 9.6 C/Γ supports a N-invariant ergodic measure. Therefore by Theorem 6.1, C/Γ must contain an element of the form $zq\Gamma/\Gamma$ where $q \in SL(n, \mathbb{Q})$ and z is a diagonal matrix with respect to $\{e_1, \ldots, e_n\}$ (i.e. in the present case Z may be chosen to be the subgroup consisting of all diagonal matrices). As in the earlier results, this implies that $\overline{\Gamma g N/N}$ contains $q^{-1}z^{-1}N/N$. Hence $\overline{\Gamma f}$ contains $\varrho(q^{-1}z^{-1})e$. It is easy to verify that there exist rational vectors $a_1, a_2, \ldots, a_{n-1}$ and non-zero scalars $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ such that $(q^{-1}z^{-1})e = (\lambda_1 a_1, \lambda_2 a_1 \wedge a_2, \ldots, \lambda_{n-1} a_1 \wedge \ldots \wedge a_{n-1})$, which proves the corollary.

It is possible to obtain similar results for other arithmetic lattices. Also, solving some of the questions raised earlier would enable their improvement. It would be possible to give sharper applications to orbits of arithmetic groups. The author hopes to return to the subject.

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