Anti-isospectral Transformations, Orthogonal Polynomials and Quasi-Exactly Solvable Problems.

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Abstract

We consider the double sinh-Gordon potential which is a quasi-exactly solvable problem and show that in this case one has two sets of Bender-Dunne orthogonal polynomials . We study in some detail the various properties of these polynomials and the corresponding quotient polynomials. In particular, we show that the weight functions for these polynomials are *not* always positive. We also study the orthogonal polynomials of the double sine-Gordon potential which is related to the double sinh-Gordon case by an anti-isospectral transformation. Finally we discover a new quasi-exactly solvable problem by making use of the anti-isospectral transformation.

I. INTRODUCTION

Recently, in an interesting paper, Krajewska *et al.* [1] have introduced an anti-isospectral transformation (which they also called as duality transformation). In particular using this transformation they relate the spectra of quasi-exactly solvable (QES) potentials V_1 and V_2 given by

$$V_1(x) = x^2(ax^2 + b)^2 - \hbar a(2M + 3)x^2 \qquad (1.1)$$

$$V_2(x) = x^2(ax^2 - b)^2 - \hbar a(2M + 3)x^2 \qquad (1.2)$$

where $a, b > 0$ and M is nonnegative integer. It may be noted here that the duality property does not hold for other (exactly non-calculable) levels. It is clearly of great interest to understand this new transformation in some detail and explore it's various consequences. For example, one would like to know whether one can discover new QES problems by using this transformation. Secondly are the QES levels of the dual potentials related even if they are valid over different domain? Further, whether the number of QES levels in the two cases are identical or not.

Another recent development is the work of Bender-Dunne [2] where they have shown that the eigenfunctions of the Schrödinger equation for a quasi-exactly solvable (QES) problem is the generating function for a set of orthogonal polynomials $\{P_n(E)\}$ in the energy variable E. It was further shown in one specific example that these polynomials satisfy the three-term recursion relation

$$P_n(E) = EP_{n-1}(E) + C_n P_{n-2}(E) \qquad (1.3)$$

where C_n is E independent quantity. Using the well known theorem [3,4], " the *necessary and sufficient* condition for a family of polynomials $\{P_n\}$ (with degree $P_n = n$) to form an orthogonal polynomial system is that $\{P_n\}$ satisfy a three-term recursion relation of the form

$$P_n(E) = (A_n E + B_n) P_{n-1}(E) + C_n P_{n-2}(E), \qquad n \geq 1 \qquad (1.4)$$

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where the coefficients A_n, B_n and C_n are independent of E, $A_n \neq 0$, $C_1 = 0$, $C_n \neq 0$ for $n \geq 1$", it then followed that $\{P_n(E)\}$ for this problem forms an orthogonal set of polynomials with respect to some weight function, $w(E)$. Recently several authors have studied the Bender-Dunne polynomials in detail [1,4–6]. In fact it has been claimed that the Bender-Dunne construction is quite universal and valid for any quasi-exactly solvable model in both one as well as multi-dimensions.

However, in a recent note [7] we have discussed three QES problems for all of which the Schroödinger equation can be transformed to Heun's equation and further in all these cases the three-term recursion relation satisfied by the Bender-Dunne polynomials is not of the form as given by Eq. (1.4) and hence does not form an orthogonal set. It is worth pointing out that in all these cases the Hamiltonian can not be written in terms of quadratic generator of the $Sl(2)$ algebra. We suspect that this may be the reason why the polynomials in these examples do not form an orthogonal set.

The question which we would like to raise and study in this paper is regarding the properties of the Bender-Dunne polynomials in case they form an orthogonal set. For example in the example discussed by Bender-Dunne (as well as in most other QES examples discussed so far), for a given potential, the QES states have either even or odd number of nodes. The question is what happens in a QES problem if one can obtain states with even as well as odd number of nodes. In particular, do the polynomials corresponding to the even and odd number of nodes together form an orthogonal set or if the polynomials corresponding to even number of nodes form one orthogonal set and polynomials with odd number of nodes form a separate orthogonal set ? Secondly, how universal are the properties of Bender-Dunne orthogonal polynomials in such cases? For example, are the weight functions always positive? Do the moments of weight function have pure power growth? [4]

The purpose of this note is to explore in some detail the various issues raised above. In particular, in Sec. II, we discuss the double sinh-Gordon (DSHG) model which is an QES problem but where states with odd as well as even number of nodes are known for a given potential. We show that in this case there are two sets of orthogonal polynomials

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with polynomials corresponding to even number of nodes forming one orthogonal set while the polynomials corresponding to odd number of nodes forming another orthogonal set. We study the properties of these polynomials as well as the corresponding quotient polynomials in detail and show that in many respect they are similar to the Bender-Dunne polynomials. However, unlike the Bender-Dunne case, the weight functions are not always positive for either of the orthogonal polynomial sets. In Sec. III, we discuss several consequences of the duality symmetry in the context of the QES problems. In particular, we study two QES problems, in both of which the domain of validity of the potential and it's dual are different. As a result we show that in the case of the double sine-Gordon (DSG) potential, the number of QES levels are almost half of those in the double sinh-Gordon case (which is dual to it). However, even then many of the predictions about their corresponding Bender-Dunne polynomials still go through. For example, the QES levels of DSG are still related to the corresponding levels of DSGH case. Further, the weight functions here are also not always positive. Finally using duality, we discover a new QES problem which has so far not been discussed in the literature. Sec. IV is reserved for discussions.

II. DSHG SYSTEM AND ORTHOGONAL POLYNOMIALS

The DSHG system is one of the few double well problems in quantum mechanics which is quasi-exactly solvable [8,9]. This double well system has found application in several different branches of physics starting from the theory of diffusion in bistable field [10] , quantum theory of instantons [11,12], quantum theory of molecules and also as a model for nonlinear coherent structure [13,14]. The Hamiltonian for this case is given by (we shall assume $\hbar = 2m = 1$ throughout this paper)

$$H = -\frac{d^2}{dx^2} + (\zeta \cosh 2x - M)^2 \tag{2.1}$$

where ζ is a positive parameter. The value of M is not restricted in principle but it has been shown [8] that the solutions for first M levels are exactly known in case M is a positive

integer. Note further that for $M > \zeta$, this potential has a double well structure with the two minima lying at $\cosh 2x_0 = \frac{M}{\zeta}$. Let us now derive the three-term recursion relation in this case. On substituting

$$\psi(x) = e^{-\frac{\zeta}{2}\cosh 2x}\phi(x) \tag{2.2}$$

in the Schroödinger equation $H\psi = E\psi$ with H as given by Eq. (2.1) we obtain

$$\phi''(x) - 2\zeta \sinh 2x\phi'(x) + \left[(E - M^2 - \zeta^2) + 2(M-1)\zeta \cosh 2x\right]\phi(x) = 0 \tag{2.3}$$

On further substituting

$$z = \cosh 2x - 1; \quad \phi = z^s \sum_{n=0}^{\infty} \frac{R_n(E)}{n!}\left(\frac{z+2}{2}\right)^{\frac{n}{2}} \tag{2.4}$$

we obtain the three-term recursion relation $(n \geq 0)$.

$$R_{n+2}(E) - \left[n^2 + 4(s+\zeta)n + 4s^2 + E - M^2 - \zeta^2 - 2(M-1)\zeta\right]R_n(E)$$
$$- \quad 4\zeta\left[M + 1 - 2s - n\right]n(n-1)R_{n-2}(E) = 0 \tag{2.5}$$

provided $2s^2 = s$ i.e. either $s = 0$ or $s = \frac{1}{2}$. Thus we have two sets of independent solutions : the even states (i.e. states with even number of nodes) for $s = 0$ and the odd states for $s = \frac{1}{2}$. Note that unlike the Bender-Dunne (or most other QES) cases, s is not contained in the potential and this is perhaps related to the fact that for any integer M (≥ 2) the QES solutions corresponding to both even and odd states are obtained. From Eq. (2.5) we observe that the even and odd polynomials $R_n(E)$ do not mix with each other and hence we have two separate three-term recursion relations depending on whether n is odd or even. In particular, it is easily shown that the three-term recursion relations corresponding to the even and odd n cases respectively are given by $(n \geq 1)$

$$P_n(E) - \left[4n^2 + 8n(s+\zeta-1) + 4s^2 - 8s + 4 - 6\zeta + E - (M+\zeta)^2\right]P_{n-1}(E)$$
$$- 8\zeta(n-1)(2n-3)(M+3-2s-2n)P_{n-2} = 0 \tag{2.6}$$
$$Q_n(E) - \left[4n^2 + 4n(2s+2\zeta-1) + 4s^2 - 4s + 1 - 2\zeta + E - (M+\zeta)^2\right]Q_{n-1}(E)$$
$$- 8\zeta(n-1)(2n-1)(M+2-2s-2n)Q_{n-2} = 0 \tag{2.7}$$

with $P_0(E) = 1, Q_0(E) = 1$. These recursion relations generate a set of monic polynomials, of which the first few are

$$P_0(E) = 1$$

$$P_1(E) = \mathcal{E} + 4s^2 + 2\zeta$$

$$P_2(E) = \mathcal{E}^2 + \mathcal{E}\left[8s^2 + 8s + 12\zeta + 4\right]$$

$$+ \left(4s^2 + 4s + 10\zeta + 4\right)\left(4s^2 + 2\zeta\right) + 8\zeta\left(M - 2s - 1\right) \tag{2.8}$$

and

$$Q_0(E) = 1$$

$$Q_1(E) = \mathcal{E} + 4s^2 + 4s + 1 + 6\zeta$$

$$Q_2(E) = \mathcal{E}^2 + \mathcal{E}\left[8s^2 + 8s + 20\zeta + 10\right]$$

$$+ \left(4s^2 + 4s + 6\zeta + 1\right)\left(4s^2 + 4s + 14\zeta + 9\right) + 24\zeta\left(M - 2s - 2\right) \tag{2.9}$$

where

$$\mathcal{E} \equiv E - (M + \zeta)^2 \tag{2.10}$$

Note that unlike the Bender-Dunne case, the polynomails $P_n(E)$ and $Q_n(E)$ are not eigenfunctions of parity. It is easily seen that when M is a positive integer, exact solutions for first M levels are obtained. In particular, if M is odd (even) integer, then solutions with even number of nodes $(s = 0)$ are obtained when the coefficient of P_{n-2} (Q_{n-2}) vanishes. Similarly if M is odd (even) integer, the solutions with odd number of nodes $(s = \frac{1}{2})$ are obtained when the coefficient of Q_{n-2} (P_{n-2}) vanishes. Further for M even (say $2k+2$, $k = 0, 1, 2 \cdots$), half the levels, i.e. $(k+1)$ levels are obtained each from the zeros of the orthogonal polynomials $P_{k+1}(E)$ and $Q_{k+1}(E)$. On the other hand when M is odd (say $2k+1$, $k = 0, 1 \cdots$) then $k+1$ and k levels are obtained from the zeros of the orthogonal polynomials $P_{k+1}(E)$ and $Q_k(E)$ respectively.

We have studied the properties of the polynomial sets $\{P_n(E)\}$ and $\{Q_n(E)\}$ and we find that many of their properties are almost the same and in turn they are similar to the

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Bender-Dunne polynomials . In particular, since in both the cases the recursion relations are similar to those given by Eq. (1.4) , hence for all values of the parameters M and s, they form an orthogonal set . Secondly, the wave function $\psi(x, E)$ is the generating function for the polynomials $P_n(E)$ as well as $Q_n(E)$. Thirdly when M is a positive integer, both of these polynomials exhibit factorization property whose precise form depends on whether M is even or odd. In particular, if $M = 2k + 1, \ k = 0, 1, 2 \cdots$ then one has $(n \geq 0)$

$$P_{k+1+n}(E) = P_{k+1}(E)\bar{P}_n(E)$$

$$Q_{k+n}(E) = Q_k(E)\bar{Q}_n(E) \tag{2.11}$$

On the other hand if $M = 2k + 2, \ k = 0, 1, 2 \cdots$, then

$$P_{k+1+n}(E) = P_{k+1}(E)\bar{R}_n(E)$$

$$Q_{k+1+n}(E) = Q_{k+1}(E)\bar{S}_n(E) \tag{2.12}$$

where $\bar{P}_0(E) = \bar{Q}_0(E) = \bar{R}_0(E) = \bar{S}_0(E) = 1$. Following Ref. [5] it is easily shown that the polynomial sets $\{\bar{P}_n(E)\}$, $\{\bar{Q}_n(E)\}$, $\{\bar{R}_n(E)\}$ and $\{\bar{S}_n(E)\}$ correspond to the non-exact spectrum for this problem.

To illustrate this factorization we list in factored form, the first few polynomials of both the types in case $M = 3$ and $M = 4$.

$\underline{M = 3}$

$P_0(E) = 1$

$P_1(E) = \mathcal{E} + 2\zeta$

$P_2(E) = \mathcal{E}^2 + \mathcal{E}\left[12\zeta + 4\right] + 20\zeta^2 + 24\zeta$

$P_3(E) = \left[\mathcal{E} + 18\zeta + 16\right] P_2(E)$

$P_4(E) = \left[\mathcal{E}^2 + \mathcal{E}(46\zeta + 52) + (36 + 28\zeta)(16 + 18\zeta) - 240\zeta\right] P_2(E). \tag{2.13}$

$Q_0(E) = 1$

$$Q_1(E) = \mathcal{E} + 6\zeta + 4$$

$$Q_2(E) = \left[\mathcal{E} + 14\zeta + \frac{35}{2}\right] Q_1(E)$$

$$Q_3(E) = \left[\mathcal{E}^2 + \mathcal{E}(36\zeta + 52) + (36 + 22\zeta)(16 + 14\zeta) - 160\zeta\right] Q_1(E). \tag{2.14}$$

where $\mathcal{E}$ is given by Eq. (2.6). Let us notice that in this case $P_2(E)$, is a common factor of $P_n(E)$ for $n \geq 2$. The zeros of $P_2(E)$ are at

$$E_0 = 7 + \zeta^2 - 2\sqrt{1 + 4\zeta^2}$$

$$E_2 = 7 + \zeta^2 + 2\sqrt{1 + 4\zeta^2} \tag{2.15}$$

which give the ground and second excited state eigenvalues for the potential when $M = 3$. On the other hand $Q_1(E)$ is a common factor of $Q_n(E)$ for $n \geq 1$. The zeros of $Q_1(E)$ are at

$$E_1 = \zeta^2 + 5 \tag{2.16}$$

which give the first excited state eigenvalue for the same potential (i.e. when $M = 3$). In this way for $M = 3$ one obtains exact energy eigenvalues for the first three levels.

$\underline{M = 4}$

$$P_0(E) = 1$$

$$P_1(E) = \mathcal{E} + 2\zeta + 1$$

$$P_2(E) = \mathcal{E}^2 + \mathcal{E}\left[12\zeta + 10\right] + 20\zeta^2 + 44\zeta + 9$$

$$P_3(E) = \left[\mathcal{E} + 18\zeta + 25\right] P_2(E)$$

$$P_4(E) = \left[\mathcal{E}^2 + \mathcal{E}(46\zeta + 64) + (28\zeta + 49)(18\zeta + 25) - 240\zeta\right] P_2(E). \tag{2.17}$$

$$Q_0(E) = 1$$

$$Q_1(E) = \mathcal{E} + 6\zeta + 1$$

$$Q_2(E) = \mathcal{E}^2 + \mathcal{E}\left[20\zeta + 10\right] + 84\zeta^2 + 116\zeta + 9$$

$$Q_3(E) = [\mathcal{E} + 22\zeta + 25] \, Q_2(E)$$

$$Q_4(E) = \left[\mathcal{E}^2 + \mathcal{E}(52\zeta + 74) + (30\zeta + 49)(22\zeta + 25) - 336\zeta\right] Q_2(E). \tag{2.18}$$

In this case P_2 (Q_2) is a common factor of P_n (Q_n) for $n \geq 2$. The zeros of $P_2(E)$ are at

$$E_1 = \zeta^2 + 2\zeta + 11 - 4\sqrt{\zeta^2 + \zeta + 1}$$

$$E_3 = \zeta^2 + 2\zeta + 11 + 4\sqrt{\zeta^2 + \zeta + 1} \tag{2.19}$$

which give the first and the third excited state energies for the potential when $M = 4$. On the other hand the zeros of $Q_2(E)$ are at

$$E_0 = \zeta^2 - 2\zeta + 11 - 4\sqrt{\zeta^2 - \zeta + 1}$$

$$E_2 = \zeta^2 - 2\zeta + 11 + 4\sqrt{\zeta^2 - \zeta + 1} \tag{2.20}$$

which give the ground and the second excited state energies for the same potential ($M = 4$). The corresponding eigenfunctions can be easily obtained by evaluating $\psi(x)$ in Eq. (2.4) at these values of E and not surprisingly they are the same as given by Razavy [8] (with appropriate change of parameters).

The norms (squared) of the orthogonal set of polynomials $P_n(E)$ and $Q_n(E)$ can be easily determined by using the recursion relations and we find (assuming $\gamma_0^{(p)} = \gamma_0^{(q)} = 1$)

$$\gamma_n^{(p)} = (-8\zeta)^n n! \prod_{k=1}^{n} [(2k - 1)(M - 2s - 2k + 1)]$$

$$\gamma_n^{(q)} = (-8\zeta)^n n! \prod_{k=1}^{n} [(2k + 1)(M - 2s - 2k)] \tag{2.21}$$

We observe from here that $\gamma_n^{(p)}$ vanishes for $2n \geq M - 2s + 1$ while $\gamma_n^{(q)}$ vanishes for $2n \geq M - 2s$ provided M is a positive integer and as remarked by Bender-Dunne, this vanishing of norm is an alternative characterization of the QES problem. It may be noted that unlike the Bender-Dunne case, the norms are alternative in sign for $2n < M - 2s + 1$ or $2n < M - 2s$ depending on the set $\{P_n(E)\}$ or $\{Q_n(E)\}$ respectively.

Using the factorization property as given by Eqs. (2.11) and (2.12) and using the recursion relations (2.6) and (2.7) satisfied by P_n and Q_n, it is easily shown that $\bar{P}_n$ $\bar{Q}_n$ $\bar{R}_n$ and $\bar{S}_n$ satisfy the following recursion relations

$$\bar{P}_n(E) = \left[(M + 1 + 2n)^2 + 4(M + 1 + 2n)(s + \zeta - 1) + 4s^2 - 8s + 4 - 6\zeta + \mathcal{E} \right] \bar{P}_{n-1}(E)$$

$$+ 4\zeta(M - 1 + 2n)(M + 2n - 2) \left[2 - 2s - 2n \right] \bar{P}_{n-2}(E)$$

$$\bar{Q}_n(E) = \left[(M - 1 + 2n)^2 + 2(M - 1 + 2n)(2s + 2\zeta - 1) + 4s^2 - 4s + 1 - 2\zeta + \mathcal{E} \right] \bar{Q}_{n-1}(E)$$

$$+ 4\zeta(M - 3 + 2n)(M + 2n - 2) \left[3 - 2s - 2n \right] \bar{Q}_{n-2}(E)$$

$$\bar{R}_n(E) = \left[(M + 2n)^2 + 4(M + 2n)(s + \zeta - 1) + 4s^2 - 8s + 4 - 6\zeta + \mathcal{E} \right] \bar{R}_{n-1}(E)$$

$$+ 4\zeta(M - 1 + 2n)(M + 2n - 3) \left[3 - 2s - 2n \right] \bar{R}_{n-2}(E)$$

$$\bar{S}_n(E) = \left[(M + 2n)^2 + 2(M + 2n)(2s + 2\zeta - 1) + 4s^2 - 4s + 1 - 2\zeta + \mathcal{E} \right] \bar{S}_{n-1}(E)$$

$$+ 4\zeta(M - 2 + 2n)(M + 2n - 1) \left[2 - 2s - 2n \right] \bar{S}_{n-2}(E) \tag{2.22}$$

Using these recursion relations, it is straightforward to obtain the norms of theses polynomials and show that they are all positive. In particular, we obtain

$$\gamma_n^{\bar{P}} = (4\zeta)^n \prod_{k=1}^{n} \left[(M + 2k + 1)(M + 2k)(2k + 2s) \right]$$

$$\gamma_n^{\bar{Q}} = (4\zeta)^n \prod_{k=1}^{n} \left[(M + 2k)(M + 2k - 1)(2k + 2s - 1) \right]$$

$$\gamma_n^{\bar{R}} = (4\zeta)^n \prod_{k=1}^{n} \left[(M + 2k + 1)(M + 2k - 1)(2k + 2s - 1) \right]$$

$$\gamma_n^{\bar{S}} = (4\zeta)^n \prod_{k=1}^{n} \left[(M + 2k + 1)(M + 2k)(2k + 2s) \right]$$

$$\tag{2.23}$$

Note that in Eqs. (2.22) and (2.23) M and s take only specific values. For example, in the case of $\bar{P}_n(E)$ M is odd and $s = 0$, while in the case of $\bar{Q}_n(E)$ M is odd and $s = \frac{1}{2}$. On the other hand, in the case of $\bar{R}_n(E)$ M is even and $s = \frac{1}{2}$ while in the case of $\bar{S}_n(E)$ M is even but $s = 0$.

Finally, we can also obtain the weight function $w(E)$ for our polynomials $\{P_n(E)\}$ and $\{Q_n(E)\}$ by using the expression as derived by Krajewska et al [5]. In particular, the weight function $w^{(p)}(E)$ can be written as

$$w^{(p)}(E) = \sum_{k=1}^{M} w_k^{(p)} \delta(E - E_k) \tag{2.24}$$

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where the numbers $w_k^{(p)}$ satisfy the algebraic equation

$$\sum_{k=1}^{M} P_n(E_k)w_k^{(p)} = \delta_{n0} \tag{2.25}$$

A similar relation also exists for the weight function of the Q polynomials.

As an illustration we have computed the weight functions $w_n^{(p)}$ and $w_n^{(q)}$ in case $M = 3$ and 4. For example, when $M = 3$, using Eqs. (2.24) to (2.25) and (2.13) we find that

$$w_0^{(p)} = \frac{1}{2} - \frac{2\zeta + 1}{2\sqrt{1 + 4\zeta^2}}$$
$$w_2^{(p)} = \frac{1}{2} + \frac{2\zeta + 1}{2\sqrt{1 + 4\zeta^2}}$$
$$w_1^{(q)} = 1 \tag{2.26}$$

On the other hand, when $M = 4$, using Eqs. (2.24), (2.25),(2.17) and (2.18) we find

$$w_0^{(q)} = \frac{1}{2} - \frac{\zeta + 1}{2\sqrt{\zeta^2 - \zeta + 1}}$$
$$w_1^{(p)} = \frac{1}{2} - \frac{\zeta + 1}{2\sqrt{\zeta^2 + \zeta + 1}}$$
$$w_2^{(q)} = \frac{1}{2} + \frac{\zeta + 1}{2\sqrt{\zeta^2 - \zeta + 1}}$$
$$w_3^{(p)} = \frac{1}{2} + \frac{\zeta + 1}{2\sqrt{\zeta^2 + \zeta + 1}} \tag{2.27}$$

As a cross check on our calculation of weight functions, we have calculated the norm of the polynomials by using the basic relations

$$\gamma_n^{(p)} = \int dE w^{(p)}(E)[P_n(E)]^2; \quad \gamma_n^{(q)} = \int dE w^{(q)}(E)[Q_n(E)]^2 \tag{2.28}$$

and for $M = 3$ as well as 4 we have verified that we get the same answer as given by Eq. (2.21). It is worth pointing out that unlike the Bender-Dunne example, in none of our cases the weight functions are always positive. Actually this is not all that surprising. As has been proved by Finkel et $al.$ [4], if the three-term recursion relation is of the form

$$\hat{P}_{k+1} = (E - b_k)\hat{P}_k - a_k\hat{P}_{k-1}, \quad k \geq 0 \tag{2.29}$$

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with $a_0 = 0$ and $a_{n+1} = 0$, then the weight functions are all positive if b_k is real for $0 \leq k \leq n$ and $a_k > 0$ for $1 \leq k \leq n$. On comparing Eq. (2.29) with our recursion relation (2.6) and (2.7) we find that in both of our cases, a_k is in fact < 0 for $1 \leq k \leq n$.

Once the weight function is known, then one can also calculate the moments of $w(E)$, defined by

$$\mu_n = \int dE w(E) E^n \tag{2.30}$$

Since in our case, the polynomials $\{P_n(E)\}$ and $\{Q_n(E)\}$ are not eigenfunctions of parity, hence unlike the Bender-Dunne case, odd moments do not vanish in our case. For example, the first few moments in our case for $M = 3$ are (we choose $\mu_0^{(p)} = \mu_0^{(q)} = 1$ without any loss of generality)

$$\mu_1^{(p)} = (3 + \zeta)^2 - 2\zeta; \quad \mu_2^{(p)} = -16\zeta + [(3 + \zeta)^2 - 2\zeta]^2$$
$$\mu_1^{(q)} = (3 + \zeta)^2 - 6\zeta - 4; \quad \mu_2^{(q)} = [(3 + \zeta)^2 - 6\zeta - 4]^2 \tag{2.31}$$

Following Finkel *et al.* [4], we can also calculate the growth rate of the moments. In particular, since in both of our cases, $a_k \neq 0$ for $1 \leq k < n$ where a_k as given by Eqs. (2.29), (2.6) and 2.7, hence following their discussion it is easily shown that for large n, to leading order

$$\mu_n^{(p)}, \quad \mu_n^{(q)} \sim (M + \zeta)^{2n} \tag{2.32}$$

Thus in our case the moments have a pure power growth.

III. NEW QES POTENTIALS FROM ANTI-ISOSPECTRAL TRANSFORMATIONS

In an interesting paper, Krajewska *et al.* [1] have recently discussed the consequences of the anti-isospectral transformation (also termed as duality transformation) in the context of the QES problems. In particular, they have shown that under the transformations $x \rightarrow ix = y$, if a potential $V(x)$ goes to $\bar{V}(y)$, i.e.

$$V(x) \longrightarrow \bar{V}(y) \tag{3.1}$$

then the QES levels of the two are also related. In particular, they have shown that if M levels of the potential $V(x)$ are QES levels with energy eigenvalues and eigenfunctions $E_k(k = 0, 1, \cdots, M - 1)$ and $\psi_k(x)$ respectively then the energy eigenfunctions of $\bar{V}(y)$ are given by

$$\bar{E}_k = -E_{M-1-k}, \quad \bar{\psi}_k(y) = \psi_{M-1-k}(ix) \tag{3.2}$$

As an illustration, these authors have discussed the x^6- potentials as given by Eqs. (1.1) and (1.2) and explained in details how the eigen spectra of these two dual potentials are related. However in this particular case, the domain of x and y are the same.

The purpose of this section is to explore the consequences of this symmetry when the domain of validity of $V(x)$ and $\bar{V}(y)$ are different and to see how many of the results derived in Ref. [1] go through in this case. In particular, do $V(x)$ and $\bar{V}(y)$, hold the same number of bound states ? Are the bound states of the two potentials still related by Eq. (3.2)? To begin with, we would also like to comment that, the duality symmetry is only useful in case the potential $V(x)$ is symmetric in x as otherwise the dual potential $\bar{V}(y)$ will be a complex potential .

A. DSG Example

Let us consider the potential corresponding to the DSHG case, and try to explore the consequences of the duality symmetry in this case. On applying the duality transformation $x \to i\theta$ to the Schroödinger equation $H\psi = E\psi$ with H as given by Eq. (2.1) we obtain the following Schrödinger equation for the DSG equation.

$$\left[-\frac{d^2}{d\theta^2} - (\zeta \cos 2\theta - M)^2 \right] \psi(\theta) = \hat{E}\psi(\theta), \quad \hat{E} = -E. \tag{3.3}$$

Note that whereas the domain of x in DSHG equation is $-\infty \leq x \leq \infty$ in the DSG case, the domain is $0 \leq \theta \leq 2\pi$ provided we are solving the problem on a circle as we do here. As

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a result, unlike the DSHG case, in the DSG case due to the periodicity constraint, one has to demand that the wave function $\psi(\theta)$ must be invariant under $\theta \to \theta + \pi$ i.e.

$$\psi(\theta + \pi) = \psi(\theta) \tag{3.4}$$

Looking at the QES eigenfunctions of the DSHG case, it is easy to see that when M is even (*i.e.* $M = 2, 4 \cdots$) then the eigenfunctions of the DSG case do not remain invariant under $\theta \to \theta + \pi$ but they change sign. As a result, the QES levels of DSG when M is even must be rejected. Thus on physical ground, we find that in case the domain of $V(x)$ and $\bar{V}(y)$ are different then the number of QES levels of $V(x)$ and $\bar{V}(y)$ need not be identical. In particular, boundary conditions may forbid some wave functions to be eigenfunctions. For example, for $M = 2$, naively one would have two exact energy eigenstates of DSG equation as given by

$$\psi_0^{DSG}(\theta) = \sin\theta e^{-\frac{\zeta}{2}\cos 2\theta}$$
$$\psi_2^{DSG}(\theta) = \cos\theta e^{-\frac{\zeta}{2}\cos 2\theta} \tag{3.5}$$

However both of these are unacceptable eigenfunctions of DSG, being odd under $\theta \to \theta + \pi$. Thus we have shown that even though DSG and DSHG are dual system, DSG equation has approximately only half the number of QES levels compared to the DSHG case. In particular, when M is an odd integer ($M = 1, 3, 5 \cdots$) only then the DSG is a QES system and in that case the first M levels are exactly known, and in fact they can be immediately obtained from the corresponding DSHG case by making use of the duality relation (3.2). For example, for $M = 1$ the ground state of the above DSG is given by

$$\hat{E}_0 = -(1 + \zeta^2) \qquad \psi_0 = e^{-\frac{\zeta}{2}\cos 2\theta} \tag{3.6}$$

On the other hand for $M = 3$, the first 3 levels of the DSG equation are are given by

$$\hat{E}_0 = -7 - \zeta^2 - 2\sqrt{1 + 4\zeta^2}; \qquad \psi_0 = \left[2\zeta - \left\{\sqrt{1 + 4\zeta^2} - 1\right\}\cos 2\theta\right]e^{-\frac{\zeta}{2}\cos 2\theta}$$
$$\hat{E}_1 = -\zeta^2 - 5; \qquad\qquad\quad \psi_1 = \sin 2\theta e^{-\frac{\zeta}{2}\cos 2\theta}$$
$$\hat{E}_2 = -7 - \zeta^2 + 2\sqrt{1 + 4\zeta^2}; \qquad \psi_2 = \left[2\zeta + \left\{\sqrt{1 + 4\zeta^2} + 1\right\}\cos 2\theta\right]e^{-\frac{\zeta}{2}\cos 2\theta} \tag{3.7}$$

The energy eigenstates of the DSG equation have recently been obtained by Habib *et al.* [9] and not surprisingly our results are the same as obtained by them.

One can now study the Bender-Dunne polynomials of the DSG equation and it is easy to see that most of the discussion of the last section (for the DSHG case) also goes through in this case except that now only odd values of M give us a QES system. For example, the recursion relations for P_n and Q_n as given in the last section are also valid in this case provided we replace E by $-E$. Thus the three-term recursion relation for P_n as given by Eq. (2.6) with E changed to $-E$ will give an exact solution for the DSG case only if $s = 0$ while the three-term recursion relation for Q_n as given by Eq. (2.7) (with E changed to $-E$) will give an exact solution only if $s = \frac{1}{2}$. Most of the other properties in the last section continue to be valid in the DSG case also (with the obvious replacement of E by $-E$) except now only $\bar{P}_n(E)$ and $\bar{Q}_n(E)$ exist while $\bar{R}_n(E)$ and $\bar{S}_n(E)$ as given by Eq. (2.22) do not exist (Note that P_n and Q_n do not give QES solution if M is even integer). In particular, we want to emphasize that the the norms of the polynomial sets $\{P_n(E)\}, \{Q_n(E)\}, \{\bar{P}_n(E)\}$ and $\{\bar{Q}_n(E)\}$ are unchanged from those of the DSHG case. However the weight functions get interchanged . For example for $M = 3$ the weight functions for the DSG case are

$$
\begin{aligned}
w_0^{(p)} &= \frac{1}{2} + \frac{2\zeta + 1}{2\sqrt{1 + 4\zeta^2}} \\
w_2^{(p)} &= \frac{1}{2} - \frac{2\zeta + 1}{2\sqrt{1 + 4\zeta^2}} \\
w_1^{(q)} &= 1
\end{aligned}
\tag{3.8}
$$

Thues even in this case, the weight function are *not* always positive. Finally it is easily shown that the moments for the DSG and DSHG cases are related by

$$
\mu_n^{(p,q)}|_{DSG} = (-1)^n \mu_n^{(p,q)}|_{DSHG}
\tag{3.9}
$$

so that for large n, to leading order, the DSG moments also have a pure power growth rate as given by

$$
\mu_n \sim (-1)^n (M + \zeta)^{2n}.
\tag{3.10}
$$

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B. New QES Potential

We shall now point out a non-trivial application of the duality symmetry. In particular, using this, we obtain an entirely new QES potential which has so far not been discussed in the literature [15].

Consider the following potential

$$V(x) = \frac{\mu^2 \left[8\sinh^4 \frac{\mu x}{2} - 4(\frac{5}{\epsilon^2} - 1)\sinh^2 \frac{\mu x}{2} + 2\left(\frac{1}{\epsilon^4} - \frac{1}{\epsilon^2} - 2\right) \right]}{8 \left[1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2} \right]^2} \tag{3.11}$$

which is obtained in the context of the stability analysis of the ϕ^6 -kink solution in 1+1 dimensions [12]. As has been shown before, in this case the Schroödinger equation can be converted to Heun's equation with four regular singular points. Further, for any ϵ, the ground stat of the Schroödinger equation corresponding to this potential is given by $(\hbar = 2m = 1)$

$$\psi_0 = N_0 \left[\frac{\epsilon^2 + 1}{\epsilon^2 + 1 + \epsilon^2 \sinh^2 \frac{\mu x}{2}} \right] \left[\frac{\epsilon^2 \sinh^2 \frac{\mu x}{2}}{\epsilon^2 + 1 + \epsilon^2 \sinh^2 \frac{\mu x}{2}} + \frac{1}{2} \right]^{\frac{1}{2}}, \quad E_0 = 0 \tag{3.12}$$

Further for the special case of, $\epsilon^2 = \frac{1}{2}$, the second excited state is also analytically known and given by

$$\psi_2 = N_2 \left[\frac{3}{3 + \sinh^2 \frac{\mu x}{2}} \right] \left[\frac{\sinh^2 \frac{\mu x}{2}}{3 + \sinh^2 \frac{\mu x}{2}} - \frac{1}{4} \right], \quad E_2 = \frac{3}{4}\mu^2 \tag{3.13}$$

Thus for $\epsilon^2 = \frac{1}{2}$ this is a QES system. It may be noted that even though it is a QES system in one dimension, the Hamiltonian can not be written in this case in terms of the quadratic generators of $Sl(2)$ [16]. Further, as shown by us recently [7], in this case the Bender-Dunne polynomials do not form an orthogonal set . We shall now apply the duality transformation and obtain a new QES system which has not been discussed before.

On considering the duality transformation $x \to i\theta$ in the Schroödinger equation $H\psi = E\psi$ corresponding to the potential (3.11), we find that we have a new periodic potential

$$\bar{V}(\theta) = -\frac{\mu^2 \left[8\sin^4 \frac{\mu\theta}{2} + 4(\frac{5}{\epsilon^2} - 1)\sin^2 \frac{\mu\theta}{2} + 2\left(\frac{1}{\epsilon^4} - \frac{1}{\epsilon^2} - 2\right) \right]}{8 \left[1 + \frac{1}{\epsilon^2} - \sin^2 \frac{\mu\theta}{2} \right]^2} \tag{3.14}$$

Notice that the domain of $V(x)$ and $\bar{V}(\theta)$ are very different i.e. whereas $-\infty \leq \mu x \leq \infty,\ \ 0 \leq \mu\theta \leq 2\pi$. However, the predictions of duality symmetry as given by Eq. (3.2) are still valid. In particular, using Eq. (3.2) we predict that the potential $V(\theta)$ as given above is a QES system in case $\epsilon^2 = \frac{1}{2}$ and it's ground and second excited states energy eigenvalues are given by

$$E_0 = -\frac{3}{4}\mu^2, \quad \psi_0 = N_3 \left[\frac{3}{3 - \sin^2 \frac{\mu\theta}{2}}\right] \left[\frac{1}{4} + \frac{\sin^2 \frac{\mu\theta}{2}}{3 - \sin^2 \frac{\mu\theta}{2}}\right]$$

$$E_2 = 0, \quad \psi_2 = N_4 \left[\frac{\epsilon^2 + 1}{\epsilon^2 + 1 - \epsilon^2 \sin^2 \frac{\mu\theta}{2}}\right] \left[\frac{1}{2} - \frac{\epsilon^2 \sin^2 \frac{\mu\theta}{2}}{\epsilon^2 + 1 - \epsilon^2 \sin^2 \frac{\mu\theta}{2}}\right] \tag{3.15}$$

In particular, we would like to emphasize that for the QES potential (3.14), the second excited state ψ_2 is known at all real values of ϵ, while ψ_0 is only known at $\epsilon^2 = \frac{1}{2}$. One can solve the Schroödinger equation explicitly for the potential (3.14) and check that indeed these are the energy eigenstates of the system. In particular, note that both ψ_0 and ψ_2 are invariant under $\mu\theta \to \mu\theta + 2\pi$ i.e. $\psi(\mu\theta + 2\pi) = \psi(\mu\theta)$. It is really remarkable that using duality we are able to obtain a new QES system. In this case one can also obtain the three-term recursion relation for the Bender-Dunne polynomials [2] and show that they do not form an orthogonal set. This is related to the fact that the corresponding Schroödinger equation can be converted to Heun's equation with four regular singular points.

It may be worthwhile to look at all known QES systems and see if the corresponding dual systems have already been discussed in the literature or not. In this way one may discover some more QES systems.

IV. DISCUSSIONS

In this paper we have analyzed in some detail a QES system for which the energy eigenstates for levels with odd as well as even number of nodes are known for a given potential. We have seen that in this case one obtains two independent sets of orthogonal polynomials. Further, we have also seen that the weight functions in this case are not necessarily positive. It will be interesting to study few other examples of the same type

and enquire if in those cases too one obtains two sets of orthogonal polynomials or not. Further, whether the weight functions in such cases are always positive or not. We have also compared the spectrum of two dual potentials in case their domain of validity are quite different. We have seen in one case that because of different boundary conditions, the QES spectrum of the dual potentials is not the same. It will be interesting to study several such QES dual potentials and see if some general conclusions can be obtained in these cases.

Finally, it would be really interesting if one can discover some new QES potentials by making use of the duality symmetry.

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