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A Number of Quasi-Exactly Solvable N-body Problems

Avinash Khare

Institute of Physics, Sachivalaya Marg,

Bhubaneswar-751005, India,

Email: khare@iopb.stpbh.soft.net

Bhabani Prasad Mandal

Theory Group, Saha Institute of Nuclear Physics,

1/AF, Bidhannagar Calcutta-700064, India,

Email: bpm@tnp.saha.ernet.in

Abstract

We present several examples of quasi-exactly solvable N-body problems in one, two and higher dimensions. We study various aspects of these problems in some detail. In particular, we show that in some of these examples the corresponding polynomials form an orthogonal set and many of their properties are similar to those of the Bender-Dunne polynomials. We also discuss QES problems where the polynomials do not form an orthogonal set.

I. INTRODUCTION

In last few years, the quasi-exactly solvable (QES) problems have attracted a lot of attention [1]. In particular, by now a detailed study has been made of several QES problems in (one particle) non-relativistic quantum mechanics in one dimension and several interesting features have been uncovered. However, very few QES problems in multi-dimensions or N-body problems in one dimension [2,3] have been discussed so far. Further, to the best of our knowledge, no QES N-body problem in two and higher dimensions has been discussed so far. The purpose of this note is to initiate a systematic study of the various aspects of the N-body QES problems in multi-dimensions. In particular, one would first like to discover several N-body QES problems in two and higher dimensions. Further, one would like to know if the corresponding polynomials form an orthogonal set or not. Besides, one would like to know if there is an underlying hidden algebra. In this context, it is worth recalling that in the case of the one body problem in one dimension, it is known that in most QES cases (though not all [4]), there is an underlying hidden algebra $Sl(2)$. Further, the polynomials form an orthogonal set provided the Hamiltonian can be written in terms of the quadratic generators of this $Sl(2)$ algebra [5] and that in this case one has one [6] or two sets [7] of Bender-Dunne polynomials.

In this paper we discuss in some detail three different N-body QES problems namely, Calogero-Marchioro type N-body problem in D-dimensions [8,9] in Sec.II ($D \geq 2$), N-body system exhibiting novel correlations in two dimensions [10] in Sec. IIIA, and Calogero-Sutherland type N-body problem in one dimension [11] in Sec. IIIB. In each case we discuss QES problem in case the potential is either of the sextic-type or is a sum of oscillator plus Coulomb type-potential. We show that whereas in the former case, the polynomials form an orthogonal set, in the later case they do not. We discuss the various properties of the orthogonal polynomials in these problems and show that they are similar to those of the Bender-Dunne polynomials in the one-body case. As a special case, we also discuss a self-dual system [12] in which case we can obtain several more QES levels analytically. Further, in this case we obtain a novel relation between the weight functions of the system. Finally, in Sec. IV we summarize our conclusions and point out some of the open problems.

II. CALOGERO-MARCHIORO TYPE PROBLEM IN D-DIMENSIONS

The N-body Hamiltonian corresponding to the Calogero- Marchioro Model in D-dimension for arbitrary potential V can be written as [9]

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 + g\sum_{i<j}^{N}\frac{1}{r_{ij}^2} + G\sum_{i<j,\ i,j\neq k}\frac{\mathbf{r}_{ki}^2\cdot\mathbf{r}_{kj}^2}{r_{ki}^2 r_{kj}^2} + V\left(\sum_{i<j}^{N} r_{ij}^2\right) \tag{2.1}$$

where $\mathbf{r}_{ij} = (\mathbf{r}_i - \mathbf{r}_j)$ and $\mathbf{r}_i$ is the D-dimensional position vector for the i-th particle.

On substituting the ansatz

$$\psi = \left(\prod_{i<j}\mathbf{r}_{ij}^2\right)^{\Lambda_D/2}\phi(\rho) \tag{2.2}$$

in the Schrödinger equation ($H\psi = \epsilon\psi$) corresponding to the above system (with $\hbar = m = 1$), it is easily shown that the equation satisfied by $\phi(\rho)$ is [9]

$$\phi''(\rho) + \frac{2\Gamma_D + 1}{\rho}\phi'(\rho) - 2V(\rho)\phi(\rho) + E\phi(\rho) = 0\,. \tag{2.3}$$

Here $E = 2\epsilon$ while

$$\begin{aligned}\rho^2 &= \frac{1}{N}\sum_{i<j}^{N} r_{ij}^2\\ \Gamma_D &= \frac{1}{2}\left[D(N-1) - 2 + \Lambda_D N(N-1)\right]\\ \Lambda_D &= \sqrt{G} = \frac{1}{2}\left[\sqrt{(D-2)^2 + 4g} - (D-2)\right]\,.\end{aligned} \tag{2.4}$$

We now consider two different forms of $V(\rho)$ and obtain QES solutions in each case.

A. Sextic Potential and QES problem

Let us first consider $V(\rho)$ as given by

$$V(\rho) = \frac{1}{2}\left[B\rho^2 + C\rho^4 + H\rho^6 + \frac{F}{\rho^2}\right]\,. \tag{2.5}$$

In the special case when $C = 0 = H$, an infinite class of exact solutions including the bosonic ground state have already been obtained [9,13].

On substituting

$$\phi(\rho) = \rho^a \exp\left(-\alpha\rho^2 - \beta\rho^4\right)\eta(\rho) \tag{2.6}$$

in Eq. (2.3) we obtain

$$\eta''(\rho) + \left(\frac{2a + 2\Gamma_D + 1}{\rho} - [4\alpha\rho + 8\beta\rho^3]\right)\eta'(\rho) + \\ \left(E - 4\alpha(a + \Gamma_D + 1) + \rho^2\left[4\alpha^2 - B - 8\beta(a + \Gamma_D + 2)\right]\right)\eta(\rho) = 0 \tag{2.7}$$

where we have chosen, $\alpha = \frac{C}{4\sqrt{H}}$; $\beta = \frac{\sqrt{H}}{4}$ and $F = a^2 + 2a\Gamma_D$. Thus a is zero or nonzero depending on if F is zero or nonzero. Finally, on substituting

$$\eta(\rho) = \sum_n \frac{P_n(E)\rho^{2n}}{4^n n!(n + a + \Gamma_D)!} \tag{2.8}$$

in Eq. (2.7) we obtain the recursion relation satisfied by $P_n(E)$ as

$$P_n(E) + \left[E - 4\alpha\left(2n - 1 + a + \Gamma_D\right)\right]P_{n-1}(E) + 4(n-1)(n - 1 + a + \Gamma_D) \\ \left[4\alpha^2 - B - 8\beta\left(2n + a + \Gamma_D - 2\right)\right]P_{n-2}(E) = 0 \tag{2.9}$$

with initial conditions $P_{-1} = 0$ and $P_0 = 1$. Using the well known theorem [14,15], " the *necessary and sufficient* condition for a family of polynomials $\{P_n\}$ (with degree $P_n = n$) to form an orthogonal polynomial system is that $\{P_n\}$ satisfy a three-term recursion relation of the form

$$P_n(E) = \left(A_n E + B_n\right)P_{n-1}(E) + C_n P_{n-2}(E)\,, \qquad n \geq 1 \tag{2.10}$$

where the coefficients A_n, B_n and C_n are independent of E, $A_n \neq 0, C_1 = 0, C_n \neq 0$ for $n \geq 1$" , it then follows that $\{P_n(E)\}$ for this problem forms an orthogonal set of polynomials with respect to some weight function, $\omega(E)$.

Let us write B in the form $B = 4\alpha^2 - 8\beta(2J + a + \Gamma_D)$ where J is any arbitrary number. We shall see that when J is a positive integer, then this represents a QES system. In terms of J, Eq. (2.9) then can be rewritten as

$$
\begin{aligned}
&P_n(E) + \left[E - 4\alpha\left(2n - 1 + a + \Gamma_D\right)\right] P_{n-1}(E) \\
&\qquad -64\beta(n-1)(n - 1 + a + \Gamma_D)\left(n - J - 1\right) P_{n-2}(E) = 0
\end{aligned} \tag{2.11}
$$

from where it is clear that so long as J is positive integer, this recursion relation will reduce to a two term recursion relation. Thus, when J is a positive integer, we have a QES system. These recursion relations generate a set of orthogonal polynomials of which the first few are

$$
\begin{aligned}
P_1 &= -E + 4\alpha(a + \Gamma_D + 1) \\
P_2 &= E^2 - E\left[8\alpha(a + \Gamma_D + 2)\right] + 16\alpha^2(a + \Gamma_D + 1)(a + \Gamma_D + 3) \\
&\qquad\qquad -64\beta(a + \Gamma_D + 1)(J - 1)
\end{aligned} \tag{2.12}
$$

It is easily seen that when J is a positive integer, exact energy eigenvalues for the first J levels are known. Further, when J is a positive integer, these polynomials exhibit the factorization property as given by

$$
P_{n+J}(E) = P_J(E) Q_n(E) \tag{2.13}
$$

where the polynomial set $Q_n(E)$ correspond to the non-exact spectrum for this problem with $Q_0(E) = 1$. For example, for $J = 1$, P_{n+1} will be factorized into P_1 and Q_n and the corresponding QES energy level (which in this case is the ground state) is obtained by putting $P_1 = 0$ i. e.

$$
E_1 = 4\alpha(a + \Gamma_D + 1) \tag{2.14}
$$

Similarly for $J = 2$, P_{n+2} will be factorized into P_2 and Q_n and the corresponding energy levels are,

$$
\begin{aligned}
E_1 &= 4\alpha(a + \Gamma_D + 2) - 4\sqrt{\alpha^2 + 4\beta(a + \Gamma_D + 1)} \\
E_2 &= 4\alpha(a + \Gamma_D + 2) + 4\sqrt{\alpha^2 + 4\beta(a + \Gamma_D + 1)}
\end{aligned} \tag{2.15}
$$

The quotient polynomials $Q_n(E)$ also form an orthogonal set as they satisfy the recursion relation

$$
\begin{aligned}
&Q_n(E) + \left[E - 4\alpha\left\{2(n+J) - 1 + a + \Gamma_D\right\}\right] Q_{n-1}(E) \\
&\qquad -64\beta(n+J-1)(n+J-1+a+\Gamma_D)\,(n-1)\,Q_{n-2}(E) = 0
\end{aligned}
\tag{2.16}
$$

with $Q_0(E) = 1$ and $Q_{-1}(E) = 0$.

The square norm of both P_n and Q_n polynomials can be calculated from the recursion relations in Eqs (2.11) and (2.16) respectively. We obtain

$$
\begin{aligned}
\gamma_n^P &= 64\beta \prod_{k=1}^{n} k\,(k+a+\Gamma_D)\,(J-n) \\
\gamma_n^Q &= 64\beta \prod_{k=1}^{n} (k+J)\,(k+J+a+\Gamma_D)
\end{aligned}
\tag{2.17}
$$

Note that a la the Bender-Dunne case [6] the norm for the P_n polynomials vanish for $n \geq J$ while it is positive definite for $n < J$. This is an alternative characterization of the QES system. On the other hand, the norm for the Q_n polynomials is always non-vanishing and positive definite.

One can also calculate the weight functions ω_k for the set $P_n(E)$ by using the relations

$$
\sum_{k=1}^{J} P_n(E_k)\omega_k = \delta_{n0}\,,\ \ n = 0, 1, ..., J-1\,. \tag{2.18}
$$

For example, for J =2 we obtain

$$
\begin{aligned}
\omega_1 &= \frac{1}{2} + \frac{\alpha}{2\sqrt{\alpha^2 + 4\beta(a+\Gamma_D+1)}} \\
\omega_2 &= \frac{1}{2} - \frac{\alpha}{2\sqrt{\alpha^2 + 4\beta(a+\Gamma_D+1)}}
\end{aligned}
\tag{2.19}
$$

Notice that to the leading order, the weight functions are inversely proportional to the particle number N for a fixed D, while they are inversely proportional to the square root of the space dimension D, when the particle number N remains unchanged. It may also be noted that both the weight functions ω_1 and ω_2 are positive definite. Actually one can prove on very general grounds that for any J, the weight functions in this case will always be positive. As has been shown by Finkel et al. [15], if the three-term recursion relation is of the form

$$P_{k+1} = (E - b_k)P_k - a_k P_{k-1}\,, \;\; k \geq 0 \tag{2.20}$$

with $a_0 = 0$ and a_{n+1} =0, then (i) the weight functions are all positive (ii) the norm of the polynomials (γ_n^p) is positive for $n < J$, provided if b_k is real for $0 \leq k \leq n$ and $a_k > 0$ for $1 \leq k \leq n$. Using Eq. (2.11) it is then easy to see that hence for any J, the weight functions and γ_n^p (for $n < J$) will be positive.

One can also calculate the moments $\mu_n = \int dE E^n \omega(E)$ of the weight functions. It is easily shown that both the odd and the even moments are non-zero. Further, to the leading order

$$\mu_n = [4\alpha(a + \Gamma_D + 1)]^n + \cdots \,. \tag{2.21}$$

Thus to the leading order, the growth rate is proportional to N^{2n} for fixed D and proportional to D^n for fixed N.

Finally, let us discuss the consequences of the anti-isospectral transformation (also termed as duality transformation) [12] in the context of our QES problem. It is easily seen that because of the duality transformation ($x \longrightarrow ix$), the QES levels of the potential in Eq. (2.5) are related to that of a similar potential (say $\hat{V}$) where C (and hence α) has been replaced by $-C$. In particular, if J levels of the potential in Eq. (2.5) are QES levels with energy eigenvalues and eigenfunctions E_k and ψ_k respectively ($k = 1, 2, ..., J$), then the corresponding QES eigenvalues and eigenfunctions of $\hat{V}$ are given by

$$\hat{E}_k = -E_{J+1-k}\,, \;\;\; \hat{\psi}_k(x) = \psi_{J+1-k}(ix) \tag{2.22}$$

For example, for J =2, the QES energy eigenvalues of $\hat{V}$ are same as those given by Eq. (2.15) with the precise relationship being $\hat{E}_{1,2} = -E_{2,1}$. The other properties of the dual potential are also similarly related and can be easily worked out.

B. Self-dual Potential

Let us now discuss the special case of $C = 0$ in Eq. (2.5) in which case we have a self-dual QES potential which has remarkably simple properties. For example, the recursion relation

(2.11) now takes the simpler form (note that now $\alpha = C/4\sqrt{H} = 0$)

$$P_n(E) + EP_{n-1}(E) - 64\beta(n-1)(n-1+a+\Gamma_D)\,(n-J-1)\,P_{n-2}(E) = 0 \tag{2.23}$$

and similarly the set Q_n satisfy Eq. (2.16) with $\alpha = 0$. As a result, both $P_n(E)$ and $Q_n(E)$ are eigenfunctions of parity. First few polynomials generated by this recursion relation are given by ($s \equiv 1 + a + \Gamma_D$)

$$\begin{aligned}
P_1 &= -E \\
P_2 &= E^2 - 64\beta s(J-1) \\
P_3 &= -E^3 + E\,[64\beta s(J-1) + 128\beta(s+1)(J-2)] \\
P_4 &= E^4 - E^2\,[64\beta s(J-1) + 128\beta(s+1)(J-2) + 192\beta(s+2)(J-3)] \\
&\quad +(64\beta)^2 3s(s+2)(J-1)(J-3)
\end{aligned} \tag{2.24}$$

From the duality relation (2.22) it follows that for this self-dual system, all the energy eigenvalues must be symmetrically distributed around $E = 0$. Further, in this case QES eigen values can be analytically calculated very simply for $J = 1, 2, ..., 5$. For example, the eigen values for $J = 3$ and $J = 4$ are given as

$$J = 3 \quad :$$

$$\begin{aligned}
E_1 &= -8\sqrt{2\beta(2s+1)} \\
E_2 &= 0 \\
E_3 &= 8\sqrt{2\beta(2s+1)}
\end{aligned} \tag{2.25}$$

and

$$J = 4 \quad :$$

$$\begin{aligned}
E_1 &= -\sqrt{320\beta(s+1) + 64\beta\sqrt{16s(s+2)+25}} \\
E_2 &= -\sqrt{320\beta(s+1) - 64\beta\sqrt{16s(s+2)+25}} \\
E_3 &= \sqrt{320\beta(s+1) - 64\beta\sqrt{16s(s+2)+25}} \\
E_4 &= -\sqrt{320\beta(s+1) + 64\beta\sqrt{16s(s+2)+25}}
\end{aligned} \tag{2.26}$$

The square norms of the polynomial sets P_n and Q_ns are again given by Eq. (2.17). Since the polynomials have definite parity the odd moments vanish. The even moments of the weight functions in the leading order are given by

$$\mu_{2n} \sim [(1 + a + \Gamma_D)]^n + \cdots \tag{2.27}$$

This means μ_{2n} is proportional to N^{2n} for fixed D and proportional to D^n for fixed N. It is worth noting that in the limit $C = 0$ (and hence $\alpha = 0$), the leading term in the expansion of μ_n in Eq. (2.21) vanishes, and that is why the leading behavior of μ_n as function of $(1 + a + \Gamma)$ when C (and hence α) is non-zero is same as that of μ_{2n} when C (and hence α) $= 0$.

The weight functions in this self dual QES model are related through a nice relation

$$\omega_k = \omega_{J+1-k} \tag{2.28}$$

This can be proved explicitly using the relation in Eq. (2.18) as follows

We consider n to be odd in Eq. (2.18) , i.e.

$$\sum_k P_{2m+1}(E_k)\omega_k = 0 \quad \text{for} \quad k = 1, 2..., J \tag{2.29}$$

Now since the $P_n(E)$ are of definite parity of E, we can write

$$P_{2m+1}(E_k) = \sum_{n=0}^{m} a_n E_k^{2n+1} \quad , \quad m = 0, 1, 2 \cdots \tag{2.30}$$

when a_n are (E- independent) constant and $a_m = -1$.

Putting (2.30) in (2.29) we obtain

$$\sum_{k=1}^{J} \sum_{n=0}^{m} a_n E_k^{2n+1} \omega_k = 0 \tag{2.31}$$

This implies that

$$\sum_{k=1}^{J} E_k^{2m+1} \omega_k = 0 \tag{2.32}$$

Now in Eq. (2.32) we use the duality relation between the eigen values of the self-dual system, i.e. $E_k = -E_{J+1-k}$ to obtain

$$\sum_{k=1}^{J} E_{J+1-k}^{2m+1}\omega_k = 0 \tag{2.33}$$

Next we make a change of variable $k = J + 1 - j$ in Eq. (2.33) to obtain

$$\sum_{j=1}^{J} E_j^{2m+1}\omega_{J+1-j} = 0 \tag{2.34}$$

Subtracting Eq. (2.34) from Eq. (2.32)

$$\sum_{k=1}^{J} E_k^{2m+1}[\omega_k - \omega_{J+1-k}] = 0 \tag{2.35}$$

This is true for arbitrary E_k and hence the result, $\omega_k = \omega_{J+1-k}$.

We now explicitly calculate the weight functions for the case of $J = 3$ and $J = 4$. We find,

$$\begin{aligned} J = 3: & \\ & \omega_1 = \omega_3 = \frac{s}{2(2s+1)} \\ & \omega_2 = \frac{s+1}{2s+2} \end{aligned} \tag{2.36}$$

and for

$$\begin{aligned} J = 4: & \\ & \omega_1 = \omega_4 = \frac{1}{4}\left[1 - \frac{2s+5}{\sqrt{16s(s+2)+25}}\right] \\ & \omega_2 = \omega_3 = \frac{1}{4}\left[1 + \frac{2s+5}{\sqrt{16s(s+2)+25}}\right] \end{aligned} \tag{2.37}$$

Not surprisingly, the weight function satisfy relation (2.28) and $\sum_{k=1}^{J}\omega_k = 1$.

C. An unusual QES Problem

Let us now discuss an unusual QES problem. To that end consider the following potential

$$V(\rho) = \frac{1}{2}[B^2\rho^2 - \frac{C}{\rho} + \frac{F}{\rho^2}] \tag{2.38}$$

It is worth noting that for $C = 0$ as well as for $B = 0$, a class of energy levels can be analytically obtained [9,13]. For the special case of $N = 2$ i.e. 2-anyons in a magnetic field and experiencing Coulomb potential . This problem has been discussed by Myrheim et al. [17]. We want to show that when both of them are non-zero, it is an example of an unusual QES system in the sense that for one parameter family of potentials (when there is a specific relation between B, C and F), one always obtains one QES eigenvalue.

On substituting

$$\phi(\rho) = \rho^a \exp\left(-B\rho^2/2\right)\eta(\rho) \tag{2.39}$$

in Eq. (2.3) we obtain

$$\eta''(\rho) + \left[\frac{2a + 2\Gamma_D + 1}{\rho} - (2B\rho)\right]\eta'(\rho) + \left[E - 2(a + \Gamma_D + 1)B + \frac{C}{\rho}\right]\eta(\rho) = 0 \tag{2.40}$$

where we have chosen, $F = a^2 + 2a\Gamma_D$. Next we substitute

$$\eta(\rho) = \sum_n P_n(E)\rho^n \tag{2.41}$$

in Eq. (2.40) and obtain the recursion relation satisfied by $P_n(E)$ as

$$n(2a + 2\Gamma_D + n)P_n(E) + CP_{n-1}(E) + [E - 2(n - 1 + a + \Gamma_D)]P_{n-2}(E) = 0 \tag{2.42}$$

with initial conditions $P_{-1} = 0$ and $P_0 = 1$.

From the recursion relation it follows that the QES levels are obtained in case the coefficient of $P_{n-2}(E)$ vanishes, i.e. the energy of the QES levels is given by $E = 2(a+\Gamma_D+n+1)$ where $n = 1, 2, \ldots.$ Further, in that case B, C and F are not independent. For example, for $n = 1$, the relation is $C^2 = 2(2a + 2\Gamma_D + 1)B$ while for $n = 2$ the relation is $C^2 = 4(4a + 4\Gamma_D + 3)B$. The QES level corresponds to an excited (ground) state depending on if $C > (<)0$. On comparing Eq. (2.42) with Eq. (2.10), we conclude that the polynomials in this case do not correspond to orthogonal polynomials.

III. OTHER N-BODY PROBLEMS

In this section we briefly discuss two other many body problems and we will see that in both of these cases we have two QES problems which are very similar to those discussed in the last section.

A. Novel Correlations for N- particles

Recently, a class of exact solutions including the bosonic ground state has been obtained for a many body Hamiltonian in two dimensions such that the wave-functions have a novel correlation (of the form $X_{ij} = x_i y_j - x_j y_i$) built into them [10]. Here we would like to show that QES states with novel correlations can also be obtained in this case. To that purpose we start from the Hamiltonian

$$\begin{aligned} H = & -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \frac{\hbar^2}{2m} g_1 \sum_{i,j} \frac{\mathbf{r}_{ij}^2}{X_{ij}^2} + \frac{\hbar^2}{2m} g_2 \sum \frac{\mathbf{r}_j \cdot \mathbf{r}_k}{X_{ij} \cdot X_{ik}} \\ & + \frac{\hbar^2}{2m}\left[2\left\{B\sum_i r_i^2 + C(\sum_i r_i^2)^2 + H(\sum_i r_i^2)^3 + \frac{F}{\sum_i r_i^2}\right\}\right] \end{aligned} \tag{3.1}$$

On substituting the ansatz

$$\psi = \prod_{i<j}^{N} |X_{ij}|^g \phi(\rho) \tag{3.2}$$

in the Schrödinger equation $H\psi = \epsilon\psi$, $(m = \hbar = 1)$ one can show that ϕ satisfies the equation

$$\phi'' + \frac{2\Delta + 1}{\rho}\phi' + \left[E - B\rho^2 - C\rho^4 - H\rho^6 - \frac{F}{\rho^2}\right]\phi = 0 \tag{3.3}$$

Where

$$\begin{gathered} 2\Delta + 1 = 2N - 1 + 2gN(N-1) \\ \rho^2 = \sum_i r_i^2; \quad E = 2\epsilon; \quad g_1 = g(g-1)\,, g_2 = g^2 \end{gathered} \tag{3.4}$$

On substituting

$$\phi(\rho) = \rho^a \exp\left(-\alpha\rho^2 - \beta\rho^4\right)\eta(\rho) \tag{3.5}$$

in Eq. (3.3) we obtain

$$\eta''(\rho) + \left[\frac{2a+2\Delta+1}{\rho} - (4\alpha\rho + 8\beta\rho^3)\right]\eta'(\rho) +$$
$$\left[E - 4\alpha(a+\Delta+1) + \rho^2\left(4\alpha^2 - B - 8\beta(a+\Delta+2)\right)\right]\eta(\rho) = 0 \tag{3.6}$$

where we have chosen, $\alpha = \frac{C}{4\sqrt{H}}$; $\beta = \frac{\sqrt{H}}{4}$ and $F = a^2 + 2a\Delta$. Now notice that this equation is similar to the corresponding equation in the last section and hence the rest of the analysis of that section will go through step by step in this case. Thus this is another example of QES N-body problem in two dimensions where the polynomials satisfy most of the properties of the Bender-Dunne polynomials. Similarly, one can also consider the admixture of the oscillator and Coulomb type potential and as in the last section, again obtain similar QES solutions. In particular, in this case the polynomials do not form an orthogonal set.

B. N-body QES problem in 1-dimension

Finally, we would like to briefly discuss the Calogero-Sutherland type N-body QES problems in one dimension [2,3].

The Hamiltonian corresponding to the N-body QES problem in 1-dimension is given by

$$H = -\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \sum_{i<j}\frac{g}{(x_i - x_j)^2} + B\sum_i x_i^2 + C[\sum x_i^2]^2$$
$$H[\sum x_i^2]^3 + \frac{F}{\sum x_i^2} \tag{3.7}$$

We substitute the standard ansatz for the wave function ψ,

$$\psi = Z^{\lambda+\frac{1}{2}}\phi(\rho) \tag{3.8}$$

where $Z = \prod_{i<j}(x_i - x_j)$, $\rho^2 = \sum x_i^2$, $\lambda = \frac{1}{2}\sqrt{1+4g}$ in $H\psi = \epsilon\psi$ ($\hbar = m = 1$) and obtain the ϕ-equation

$$\phi'' + \frac{2\Delta+1}{\rho}\phi' + \left[E - B\rho^2 - C\rho^4 - H\rho^6 - \frac{F}{\rho^2}\right]\phi = 0 \tag{3.9}$$

where $E = 2\epsilon$ and

$$2\Delta + 1 = N - 1 + N(N-1)\lambda \tag{3.10}$$

This equation is very similar to the Eq. (3.3) and hence the rest of the analysis goes through as in the last section, i.e. the polynomials of the QES problem indeed form an orthogonal set and share most of the properties of the Bender-Dunne polynomials. Similarly, we can show that even in this case the other QES solution corresponding to the admixture of the oscillator and Coulomb potential also exists, and in this case, the polynomials do not form an orthogonal set.

IV. SUMMARY AND OPEN PROBLEMS

In this paper, we have obtained QES solutions of a number of N-body problems in one,two and higher dimensions. We have shown that with sextic type potential, in all these cases one has a set of orthogonal polynomials which satisfy almost all the properties as satisfied by the Bender-Dunne polynomials. However, the hidden algebra is not clear in these cases. In this context, recall that for the case of one particle in one dimension experiencing the sextic potential, the underlying symmetry algebra is $Sl(2)$. In particular, in that case one can express the Hamiltonian in terms of the quadratic (and linear) generators of $Sl(2)$. Further, it will be nice to discover some other QES many-body problems. Finally, it would be most interesting if one can find some application of these QES N-body problems. In this context it is worth noting that one application has already been found by one of the present author (AK) and Jatkar [18]. In particular, they have considered the model discussed in Sec. IIIB (with F =0) and have shown that the square of the ground state wave function of this model is related to the matrix model corresponding to branched polymers. It will really be interesting, if one can find some other application, of the N-body QES problems in two and higher dimensions.

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