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A Calogero-Sutherland Type Model For Branched Polymers

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We show that a Calogero-Sutherland type model with anharmonic interactions of fourth and sixth orders leads to the matrix model corresponding to the branched polymers. We also show that by suitably modifying this model one can also obtain N-particle problems which are connected to matrix models corresponding to the pure gravity phase as well as corresponding to the transition point between the soap bubble and the branched polymer phase.

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In recent years, Calogero - Sutherland (CS) type N-body problems in one dimension have received considerable attention in the literature [1,2,3]. In particular, remarkable connections have been found between such models and seemingly totally different models [4]. Further, these models are shown to correspond to ideal gas in one dimension with fractional exclusion statistics. The purpose of this note is to point out one more such connection. In particular, we show that the N-body problem with equal mass in 1-dimension characterized by ($\hbar = 2m = 1,\ g > -1/2, B > 0$)

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N} \frac{g}{(x_i - x_j)^2} + B\sum_i x_i^2 + A(\sum_i x_i^2)^2 + C(\sum_i x_i^2)^3 \tag{1}$$

is related to the matrix model corresponding to the problem of branched polymers. We also show that by suitably modifying this model one can also obtain N-body problems which are connected to matrix models corresponding to pure gravity phase as well to the transition point between the pure gravity and the branched polymer phase.

Consider a many-body system with the hamiltonian given in eq.(1). The corresponding Schrödinger equation is given by

$$H\psi = E\psi \tag{2}$$

Following Sutherland[3] we will write the wave function as a product of two wave functions, one of which carries the Jastraw factor i.e. the antisymmetric part of the wave function and the remaining part which carries the exponential damping terms

$$\psi = \phi(x_i)\varphi(x_i) \tag{3}$$

where

$$\phi = \prod_{i<j} \mid x_i - x_j \mid^{\lambda} \tag{4}$$

and

$$\varphi = \exp\left(-\alpha \sum_{i=1}^{N} x_i^2 - \beta(\sum_{i=1}^{N} x_i^2)^2\right). \tag{5}$$

On substituting this wave function in eq.(2) we find the relation between the power of the Jastraw factor and the strength of the inverse square potential

$$\lambda^2 - \lambda = g/2. \tag{6}$$

Solving this for λ we get

$$\lambda = \frac{1}{2}[1 + (1 + 2g)^{\frac{1}{2}}]. \tag{7}$$

Further, we find that the coefficients in the wavefunction and the coupling constants appearing in the hamiltonian are related by

$$A = 16\alpha\beta, \qquad B = 4[\alpha^2 - \beta(N + 2 + \lambda N(N-1))], \qquad C = 16\beta^2. \tag{8}$$

In case these relations are satisfied then the wave function as given by eq. (3) is the ground state eigenfunction and the corresponding eigenvalue is given by

$$E_0 = \frac{A}{2(C)^{\frac{1}{2}}}(N + \lambda N(N-1)) \tag{9}$$

The ground state eigenfunction of this many body system can be interpreted in terms of a matrix model[3]. In particular, square of the modulus of this eigenfunction can be interpreted as the weight function of the corresponding matrix integral. For our case it reduces to the following matrix integral

$$Z_N = \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}M^2 - \frac{b'}{N}(\mathrm{Tr}M^2)^2\right) \tag{10}$$

involving hermitian matrices provided $\lambda = 1$. Of course, $\lambda = 1$ means, in the original model there in no inverse square interaction. This in turn makes the many body problem even simpler.

This matrix integral has been studied in detail [5]. The leading order solution is given by taking large N limit where the eigenvalues scale as

$$x_i = \sqrt{N}x(i/\sqrt{N}) = \sqrt{N}x(z) \tag{11}$$

with the continuous variable $z = i/\sqrt{N}$ taking values between 0 and 1. In the large N limit we can solve the integral in the saddle point approximation. The density of eigenvalues $u(x)$ is defined as

$$u(x) = \frac{dz}{dx}, \tag{12}$$

and its second moment is

$$c = \int_{-\mu}^{\mu} dx u(x) x^2. \tag{13}$$

Using these definitions the saddle point equation can be written as

$$\frac{1}{2}x + 2b'cx = P\int_{-\mu}^{\mu} dy \frac{u(y)}{x-y}. \tag{14}$$

Where P stands for principal value. The eigenvalue range $(-\mu, \mu)$ is determined by normalization of the eigenvalue density. The solution to the saddle point equation satisfying the proper asymptotics[6] is

$$u(x) = \frac{1}{\pi}(\frac{1}{2} + 2b'c)\sqrt{\mu^2 - x^2}. \tag{15}$$

From the normalization as well as the self-consistency conditions (13) it then follows that

$$b'\mu^4 = 4 - \mu^2. \tag{16}$$

The free energy of the model in the large N limit now takes the form

$$\begin{aligned} E(b') &= \lim_{N\to\infty} \ln Z_N \\ &= \int dx \frac{1}{2}u(x)x^2 + b'c^2 - \int dxdyu(x)u(y)\ln \mid x-y \mid \end{aligned} \tag{17}$$

from where we obtain

$$E_0(b') - E_0(0) = \frac{1}{16}(\mu^2 - 4) - \frac{1}{2}\log \mu^2/4 \tag{18}$$

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It is well known that the matrix models have interpretation in terms of summing over random surfaces. The interpretation in our case is that the quartic coupling β corresponds to the touching random surfaces. This phase, where there are several random surfaces touching each other is interpreted geometrically as the branched polymer phase. It is well known that in the branched polymer phase of the random surfaces the susceptibility exponent is positive and is given by

$$\gamma_s = 1/2. \tag{19}$$

We will not reproduce all the details here but it suffices to say that the analysis of [5] can be carried through in a straightforward manner. Following their analysis [5] it is easy to see that in our case the susceptibility exponent is indeed $1/2$ at $b' = -1/16$ thereby

establishing the connection of our N-body problem with the matrix model for the branched polymer phase.

If we modify the many body potential by adding the term

$$V_{new} = A' \sum_{i=1}^{N} x_i^4 + B' \sum_{i=1}^{N} x_i^6 + C' \sum_{i=1}^{N} x_i^4 \sum_{i=1}^{N} x_i^2 + D' \sum_{i<j} (x_i - x_j)^2, \tag{20}$$

and suitably adjust the couplings A', B', C' and D' we can get the susceptibility exponent $\gamma_s = 1/3$ as well as $\gamma_s = -1/2$. These cases correspond to the crossover from soap bubble phase to the branched polymer phase and the pure gravity phase respectively. For the crossover phase, the ground state eigenfunction is again as given by eq. (3) where ϕ is as given by eq. (4) while φ is given by

$$\varphi = \exp(-\alpha \sum_{i=1}^{N} x_i^2 - \beta (\sum_{i=1}^{N} x_i^2)^2 - \gamma \sum_{i=1}^{N} x_i^4) \tag{21}$$

where α, β and γ are related to the coefficients in the many body potential as follows

$$\begin{aligned} B &= 4[\alpha^2 - 3\gamma - 6\lambda\gamma(N-1) - \beta(2 + N + \lambda N(N-1))] \\ A &= 16\alpha\beta, \quad C = 16\beta^2, \quad A' = 16\alpha\gamma \\ B' &= 16\gamma^2, \quad C' = 32\beta\gamma, \quad D' = 4\lambda\gamma. \end{aligned} \tag{22}$$

In this case the ground state energy is as given by eq. (9).

In the pure gravity case, the ground state eigenfunction is again given by eq. (3) with ϕ being given as before by eq. (4) while φ is now of the form

$$\varphi = \exp(-\alpha \sum_{i=1}^{N} x_i^2 - \gamma \sum_{i=1}^{N} x_i^4) \tag{23}$$

where the relation between α, γ and the many body potential is

$$\begin{aligned} A = C = C' = 0, \quad B = 4\alpha^2 - 12\gamma(1 - 2\lambda(N-1), \\ A' = 16\alpha\gamma, \quad B' = 16\gamma, \quad D' = 4\lambda\gamma. \end{aligned} \tag{24}$$

The ground state energy is now given by

$$E_0 = \frac{A'}{2(C)^{\frac{1}{2}}} (N + N(N-1)\lambda) \tag{25}$$

We thus see that the N-body quantum mechanical problem on a line with the hamiltonian (1) corresponds to the problem of branched polymers. By adding suitable extra potential (20), we find that the new many body system corresponds to the pure gravity phase as well as the crossover of the soap bubble phase to the branched polymer phase. It would be interesting to see if this relation could be further explored to get better understanding of the branched polymer phase.

References

[1] F. Calogero, Jour. Math. Phys. **10** (1969), 2191, 2197.

[2] F. Calogero, Jour. Math. Phys. **12** (1971) 419.

[3] B. Sutherland, Jour. Math. Phys. **12** (1971) 246.

[4] B.D. Simons, P.A. Lee and B.L. Altshuler, Phys. Rev. Lett. **72** (1994) 64.

[5] S. Das, A. Dhar, A. Sengupta and S. Wadia, Mod. Phys. Lett. **A5** (1990) 1041.

[6] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, Commun. Math. Phys. **59** (1980) 35.