Exact Solution of a N-body Problem in One Dmension

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Abstract

Complete energy spectrum is obtained for the quantum mechanical problem of N one dimensional equal mass particles interacting via potential

$$V(x_1, x_2, ..., x_N) = g \sum_{i<j}^{N} \frac{1}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}}$$

Further, it is shown that scattering configuration, characterized by initial momenta $p_i (i = 1, 2, ..., N)$ goes over into a final configuration characterized uniquely by the final momenta p'_i with $p'_i = p_{N+1-i}$.

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In recent years, the Calogero - Sutherland (CS) type of N-body problems in one dimension have received considerable attention in the literature [1,2,3,4]. It is believed that the CS model with inverse square interaction provides an example of an ideal gas in one dimension with fractional statistics [5]. Besides, these models are related to (1+1) - dimensional conformal field theory, random matrices as well as host of other things [6]. Inspired by these successes, it is of considerable interest to discover new exactly solvable N-body problems.

The purpose of this note is to present one such example. In particular I show that the N-body problem with equal mass in 1-dimension characterized by ($\hbar = 2m = 1,\ g > -1/2,\ \alpha > 0$)

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N} \frac{g}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}} \tag{1}$$

is exactly solvable. The interesting point about this model is that unlike most other exactly solvable models, it has both bound state and scattering solutions. In particular I show that the complete bound state spectrum (in the center-of-mass frame) is given by the formula

$$E_{n+k} = -\frac{\alpha^2}{4N[n+k+b+\frac{1}{2}]^2}, \quad n, k = 0, 1, 2, \tag{2}$$

where

$$b = \frac{N(N-1)}{2}a + \frac{N(N+1)}{4} - \frac{3}{2}; \quad a = \frac{1}{2}\sqrt{1+2g} \tag{3}$$

For positive energy one has only scattering states. I show that a scattering configuration, characterized by initial momenta $p_i (i = 1, 2, ..., N)$ goes over into final configuration characterized uniquely by the final momenta p_i' with

$$p_i' = p_{N+1-i} \tag{4}$$

However, unlike the pure inverse square scattering case ($\alpha = 0$), in our case the phase shift is energy dependent. Thus, as in other integrable cases, in our case too the scattering problem reduces to a sequence of 2-body processes.

Finally, a la Sutherland [3], I also solve a slightly diferent variant of the Hamiltonian (1) with $-\alpha/\sqrt{\sum_{i<j}(x_i - x_j)^2}$ being replaced by an external potential $-\alpha/\sqrt{\sum_i x_i^2}$ and obtain exact expressions for the ground state energy eigenvalues and eigenfunctions.

Consider the Hamiltonian as given by eq. (1). We need to solve the eigenvalue equation

$$H\psi = E\psi \tag{5}$$

where ψ is a translation invariant eigenfunction. Note that our Hamiltonian is very similar to the classic Calogero Hamiltonian (see eq. (2.1) of his paper [2]) except that whereas he has a pairwise quadratic potntial, we have a "N-body" potential $\frac{-\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}}$. However, we shall see that many of the key steps are very similar in the two cases and hence we avoid giving most steps which are already contained there [2]. Without any loss of generality, a la [2] we also restrict our attention to the sector of the configuration space corresponding to a definite ordering of particles, say

$$x_i \geq x_{i+1}, \quad i = 1, 2, ..., N-1. \tag{6}$$

A la [2] it is clear that the normalizable solutions of eq. (5) (with H being given by eq. (1)) can be cast in the form

$$\psi(x) = Z^{a+1/2}\phi(r)P_k(x) \tag{7}$$

where a is defined in eq.(3) while Z and r are given by

$$Z = \Pi^N_{i<j}(x_i - x_j), \quad r^2 = \frac{1}{N}\sum^N_{i<j}(x_i - x_j)^2, \tag{8}$$

and $P_k(x)$ is a homogeneous polynomial of degree k in the particle coordinates and satisfies the generalized Laplace equation i.e.

$$\left[\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+2(a+\frac{1}{2})\sum_{i<j}^{N}\frac{1}{(x_i-x_j)}(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j})\right]P_k(x)=0 \tag{9}$$

As discussed in detail in [2], the polynomials $P_k(x)$ are completely symmetrical under the exchange of any two coordinates. On inserting the ansatz (7) into the Schrödinger eq. (5) (with H given by eq. (1)) and using eq. (9) and following the procedure of ref [2], we find that $\phi(r)$ satisfies the equation

$$-\left[\phi^{"}(r)+\{2k+2b+1\}\frac{1}{r}\phi'(r)\right]-(\frac{\alpha}{\sqrt{N}r}+E)\phi(r)=0 \tag{10}$$

where prime denotes differentiation with respect to the argument. The normalizable solutions of this equation are

$$\phi_{n,k}(r)=exp(-\sqrt{\mid E\mid}r)L_n^{2b+2k}(2\sqrt{\mid E\mid}r). \tag{11}$$

while the corresponding energies are as given by eq. (2). Here $L_n^{\alpha}(r)$ is a Laguerre polynomial. Notice that in the expression (2) for the energy, n and k always come in the combination n+k (unlike in the Calogero case [2] where it comes in the combination 2n+k).

In the special case of N = 3, we can check our expressions for E_n and $\psi_{n,k}$ with the exact expressions obtained by entirely different method (see eqs. (40) to (43) of [7]). On comparing the two we find (note coupling constant in [7] is $\sqrt{3}\alpha$ rather than α) that the two expressions agree provided k = 3l and $P_k(x)\propto r^{3l}C_l^{a+1/2}(cos3\phi)$ where C_i^a is a Gegenbauer polynomial.

Let us now consider the positive energy spectrum of the Hamiltonian (1). It is of course purely continuous spectrum. Following the treatment given above and as in [2]

(Sec. 4) it is clear that the complete set of stationary eigenfunctions of the problem (in the center-of-mass frame) is

$$\psi_{pk} = Z^{a+1/2}\phi_p(r)P_k(x), \; k = 0, 1, 2, ...; \quad p \geq 0 \tag{12}$$

where p is connected to the energy eigenvalue by $E = p^2 \geq 0$ (note that we have chosen $\hbar = 2m = 1$) while $\phi_p(r)$ satisfies eq. (10). It is easily shown that for $E \geq 0$, the solution of eq. (10) is given by

$$\phi_p(r) = e^{ipr}F(k + b + \frac{1}{2} - \frac{i\alpha}{2p\sqrt{N}}, 2k + 2b + 1; -2ipr) \tag{13}$$

One can now run through the arguments of [2] (Sec. 4) and show that if the stationary eigenfunction describing, in the center-of-mass frame, the scattering situation is characterized by the form

$$\psi_{in} \sim C(exp[i\sum_{i=1}^{N} p_i x_i] \tag{14}$$

with (note $x_i \geq x_{i+1}, \quad i = 1, 2, ..., N-1$)

$$p_i \leq p_{i+1}, \quad p^2 = \sum_{i=1}^{N} p_i^2, \quad \sum_{i=1}^{N} p_i = 0 \tag{15}$$

then ψ_{out} is given by

$$\psi_{out} \sim Ce^{2i\eta_p - ib\pi} exp[i\sum_{i=1}^{N} p_{N+1-i}x_i] \tag{16}$$

where

$$e^{2i\eta_p} = \frac{\Gamma(k + b + \frac{1}{2} - \frac{i\alpha}{2p\sqrt{N}})}{\Gamma(k + b + \frac{1}{2} + \frac{i\alpha}{2p\sqrt{N}})} \tag{17}$$

Thus we have the remarkable result that even in the presence of the potential $-\frac{\alpha}{\sqrt{\sum_{i<j}(x_i - x_j)^2}}$, the N-particle scattering problem reduces to a sequence of 2-body processes as characterized by eq. (4) but now one has an energy dependent phase shift.

Note that all the results about scattering are also valid in case α is negative but now the spectrum is purely continuous and there are no bound states.

Finally, let us discuss the "Sutherland variant" [3] of the Hamiltonian (1). Consider

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{g}{(x_i - x_j)^2} - \frac{\alpha}{\sqrt{\sum_i x_i^2}} \tag{18}$$

i.e. the N-body potential is now an external potential. Note that for the Calogero case, Sutherland [3] was able to obtain an exact expression for the ground state energy and eigenfunction ψ and find a remarkable connection of ψ^2 with the joint probability density function for the eigenvalues of matrices from a Gaussian ensembles in case $\beta = 2\lambda = 1$, 2 or 4. Using these connections he was the able to compute [3] the one particle density and the pair correlation function.

Following Sutherland, let us consider the Schrödinger equation $H\psi = E\psi$ with H as given by eq. (18). Further, let us write the wavefunction ψ as $\psi = \phi\Phi$ with

$$\phi = \Pi_{i<j} \mid x_i - x_j \mid^{\lambda}, \quad \lambda = \frac{1}{2} + a \tag{19}$$

On using this ansatz in the Schrödinger equation with H as given by eq.(18) we find that Φ must satisfy

$$-\sum_{i=1}^{N} \frac{\partial^2 \Phi}{\partial x_i^2} - 2\lambda \sum_{i<j} \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\Phi - \frac{e^2}{\sqrt{\sum_i x_i^2}}\Phi = E\Phi \tag{20}$$

It is easily verified that

$$\Phi = exp(-\sqrt{\mid E \mid}\sqrt{\sum_i x_i^2}) \tag{21}$$

is a solution to eq. (20) with the energy

$$E = -\frac{e^4}{[(N-1)(1+\lambda N)]^2} \tag{22}$$

Clearly, for each ordering of particles, ψ is nodeless and hence it is the solution for the ground state. If we rewrite ψ in terms of the variables

$$y_i = \frac{\sqrt{\mid E \mid}}{\sqrt{\lambda}} x_i \tag{23}$$

then one finds that

$$\psi^2 = C(exp(-\beta \sqrt{\sum_i y_i^2}) \Pi_{i<j} \mid y_i - y_j \mid^{\beta} \tag{24}$$

where C is the normalization constant. A la original Sutherland case [3], where ψ^2 was identical to the joint probability density function, it would indeed be remarkable if our ψ^2 as given by eq. (24) for atleast $\beta = 1, 2, 4$ can be mapped on to some known solvable problem and using these results if one could obtain the one particle density and the pair correlation function for our case.

This work raises several issues which need to be looked into. I hope to address some of these issues in the near future.

References

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