

# Chern-Simons Term and Charged Vortices in Abelian and Nonabelian Gauge Theories

Avinash Khare

Institute of Physics, Sachivalaya Marg,  
Bhubaneswar 751 005, India.

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## Abstract

In this article we review some of the recent advances regarding the charged vortex solutions in abelian and nonabelian gauge theories with Chern-Simons (CS) term in two space dimensions. Since these nontrivial results are essentially because of the CS term, hence, we first discuss in some detail the various properties of the CS term in two space dimensions. In particular, it is pointed out that this parity (P) and time reversal (T) violating but gauge invariant term when added to the Maxwell Lagrangian gives a massive gauge quanta and yet the theory is still gauge invariant. Further, the vacuum of such a theory shows the magneto-electric effect. Besides, we show that the CS term can also be generated by spontaneous symmetry breaking as well as by radiative corrections. A detailed discussion about Coleman-Hill theorem is also given which asserts that the parity-odd piece of the vacuum polarization tensor at zero momentum transfer is unaffected by two and multi-loop effects. Topological quantization of the coefficient of the CS term in nonabelian gauge theories is also elaborated in some detail.

One of the dramatic effect of the CS term is that the vortices of the abelian (as well as nonabelian) Higgs model now acquire finite quantized charge and angular momentum. The various properties of these vortices are discussed at length with special emphasis on some of the recent developments including the discovery of the self-dual charged vortex solutions.

# 1 INTRODUCTION

In 1957, Abrikosov [1] wrote down the vortex solutions in Ginzburg-Landau (GL) theory which is the mean-field theory of superconductivity. Subsequently, these vortices were experimentally observed in type-II superconductors. In 1973, Nielsen and Olesen [2] rediscovered these solutions in the context of the abelian Higgs model which is essentially a relativistic generalization of the GL theory. These people were looking for string like objects in field theory. It turns out that these vortex solutions have finite energy per unit length (i.e. finite energy in 2+1 dimensions as the vortex dynamics is essentially confined to the x-y plane), quantized flux but are electrically neutral and have zero angular momentum.

In 1975, Julia and Zee [3] showed that the  $SO(3)$  Georgi-Glashow model which admits 't Hooft-Polyakov monopole solution also admits its charged generalization i.e. the dyon solution with finite energy and finite, nonzero charge. It was then natural to enquire if the abelian Higgs model which admits neutral vortex solutions with finite energy (in 2+1 dimensions), also admits charged vortex solutions with finite charge and energy in 2+1 dimensions. Julia and Zee showed in the appendix of the same paper [3] that the answer to the question is no. More than ten years later Samir Paul (then my Ph.D. student) and myself showed [4] that the Julia-Zee result can be bypassed by adding the CS term [5, 6] to the abelian Higgs model in 2+1 dimensions. In particular, we showed that the abelian Higgs model with CS term in 2+1 dimensions admits charged vortex (soliton to be more precise) solutions of finite energy, nonzero finite charge and flux. As an extra bonus, one found that these vortices have nonzero angular momentum which is in general fractional. This strongly suggested that these objects could in fact be charged anyons [7] i.e. the objects which are neither bosons or fermions but which obey statistics which is interpolating between the two. Subsequently, Frohlich and Marchetti [8] have shown using axiomatic field theory that these objects are indeed charged anyons.

There is one question that has remained unanswered though i.e. can one overcome Julia-Zee objection [3] in 3+1 dimension itself? Recently we have answered the question in the affirmative. In particular we [9] have been able to construct self-dual topological as well as nontopological charged vortex solutions of finite energy per unit length in a generalized abelian Higgs model with a dielectric function and a neutral scalar field. The interesting

point is that in this case the Bogomol'nyi bound on energy per unit length is obtained as a linear combination of the magnetic flux and the electric charge per unit length.

By now our work on the CS vortices has been extended in several directions. Most of the developments till 1988 have been well summarized in my earlier review article on this subject [10]. However, several new advances have taken place since that time and one purpose of this article is to discuss some of those developments. Since the key role in this game is played by the CS term, it is only proper that at first I discuss the various properties of this term.

The plan of the paper is as follows: In Sec. II, the role of the CS term is discussed in the context of both abelian and nonabelian gauge theories. The key point to note is that whereas the Chern-Pontryagin term is topological but has no dynamics (being a total divergence), the CS term is topological and also contributes to equations of motion and hence has nontrivial dynamics. In particular, it is pointed out that in 2+1 dimensions, because of this term, one has at the same time a massive gauge field and a gauge invariant action. Quantum electrodynamics with CS term has some interesting and unusual properties which are discussed here. For example, the vacuum polarization tensor has an extra piece which is odd in both P and T. Various properties of this P and T-odd piece including Coleman-Hill (CH) theorem [14] and magneto-electric effect [15] are discussed at length. In particular, it is emphasised that contrary to the claim of CH, not only fermions but any P and T violating interaction including even scalar [16] or vector [17] particles can give nonzero contribution to the P and T-odd part of vacuum polarization tensor at zero momentum transfer. We also point out that the theorem is also valid in case the gauge symmetry is spontaneously broken and the theorem is stated in terms of effective action rather than vacuum polarization tensor [16]. In the nonabelian case, it turns out that the theory is not well defined unless the coefficient of the CS term is quantized [6]. Various issues in this regard are discussed at length. In particular, it is pointed out that the tree level quantization continues to remain valid at one [18] and higher loops [19] in the case of pure gauge theories. Further, it is also valid at one loop in the presence of matter fields. Finally, it is argued that the quantization of the coefficient of the CS term is also valid at one loop for an unbroken nonabelian gauge group in a theory where larger nonabelian gauge symmetry is spontaneously broken to that of the smaller gauge group [20].

We also point out that atleast in the nonabelian case, adding the CS term to the action is not really a luxury –in a sence one is forced to add it since even if one does not add it at the tree level, it is automatically induced by radiative corrections [21]. Finally, it is pointed out that a la gauge field mass term the abelian CS term can also be generated by spontaneous symmetry breaking [22]. This is possible because in 2+1 dimensions, even for scalar particles one can introduce a Pauli type nonminimal interaction term. Some other implications of this term are also discussed in Sec. II.

In Sec. III we discuss one of the most dramatic consequence of the CS term in 2+1 dimensions i.e. the existance of charged evortex solutions [4]. Only salient features of this solution are discussed here since the details are already contained in my previous review article [10]. In Sec. IV, we discuss some of the recent developments in this field. In particular, we discuss the self-dual charged vortex solutions in both relativistic [11] and nonrelativistic [12] theories as well as semi-local charged vortex solutions [13].

## 2 Some Important Properties of the Chern-Simons Term

It is well known that there is a gauge singlet (popularly known as axial) anomaly in any even dimension  $2n$  so that the divergence of the gauge singlet axial current is proportional to the corresponding Chern-Pontryagin density  $P_{2n}$  in that dimension which in turn can always be written as a total divergence i.e.

$$P_{2n} = \partial_\mu \Lambda^\mu; \quad \mu = 0, 1, 2, \dots, n-1 \quad (1)$$

The object  $\Lambda^\mu$ , for a particular value of  $\mu$  (say  $\mu_0$ ) naturally lives in  $(n-1)$  dimensions (excluding that particular dimension  $\mu_0$ ) and is known as the Chern-Simons density. For example the gauge singlet anomaly in 3+1 dimensional quantum electrodynamics is given by

$$\partial^\mu j_\mu^5 = \frac{e^2}{2\pi} \epsilon_{\mu\nu\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} \quad (2)$$

$$= \frac{e^2}{\pi} \partial^\mu (\epsilon_{\mu\nu\lambda\sigma} A^\nu F^{\lambda\sigma}) \quad (3)$$

so that the abelian CS term in 2+1 dimensions is given by

$$\Lambda^3 = \frac{e^2}{\pi} \epsilon_{3\nu\lambda\sigma} A^\nu F^{\lambda\sigma} \quad (4)$$

which clearly lives in 2+1 dimensions. In the nonabelian case, the CS term has an extra piece i.e.

$$\Lambda^3 \propto \epsilon^{\nu\lambda\sigma} \text{Tr}(F_{\nu\lambda} A_\sigma - \frac{2}{3} A_\nu A_\lambda A_\sigma) \quad (5)$$

where  $A_\mu$  and  $F_{\mu\nu}$  are matrices

$$A_\mu = g T^a A_\mu^a ; F_{\mu\nu} = g T^a F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (6)$$

Here we have used the representation matrices of the group (for SU(2):  $T^a = \tau^a/2i$ )

$$[T^a, T^b] = f^{abc} T^c. \quad (7)$$

Let us first discuss the properties of the abelian CS term as given by eq. (4).

**P and T Violation:** It is easily seen that the abelian (as well as the nonabelian) CS term is not invariant under P and T separately even though it is invariant under PT as well as charge conjugation C. It is worth noting here that in 2+1 dimensions, even the fermion mass term  $m\bar{\psi}\psi$  is odd under P as well as T and infact this is the underlying reason as to why CS term can be generated in perturbation theory by integrating over fermions in a massive fermionic theory [21].

**CS Term as Gauge Field Mass Term:** Let us consider electrodynamics in the presence of the CS term

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu}{4} \epsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda \quad (8)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Note that the corresponding equation of motion

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0 \quad (9)$$

as well as the action are invariant under  $U(1)$  gauge transformation

$$A_\mu \rightarrow A_\mu - \frac{1}{2} \partial_\mu \Lambda \quad (10)$$

The field eq. (9) can also be written as

$$(g^{\mu\nu} + \frac{1}{\mu}\epsilon^{\mu\nu\alpha}\partial_\alpha)^*F_\nu = 0 \quad (11)$$

where  $*F_\nu$  is the dual field strength which is a vector in three dimensions

$$*F_\nu = \frac{1}{2}\epsilon_{\nu\alpha\beta}F^{\alpha\beta}; \quad F_{\mu\nu} = \epsilon_{\mu\nu\alpha}*F^\alpha \quad (12)$$

On operating by  $(g_{\beta\mu} - \epsilon_{\beta\mu\delta}\partial^\delta/\mu)$  to eq. (11) we have

$$(\square + \mu^2)^*F_\beta = 0 \quad (13)$$

which clearly shows that the gauge field excitations are massive. This remarkable property of having a gauge invariant mass term for the gauge field in the action itself is very special to 2+1 dimensions. In all other dimensions one has to take recourse to the Higgs mechanism (or one could have dynamical symmetry breaking as in the 1+1 dimensional Schwinger model). It is worth noting here though that unlike in other dimensions, in this case, both massless and the CS-mass photon has only one degree of freedom. Further, whereas the massless photon in 2+1 dimensions has spin zero, the CS-mass photon has spin 1 (-1) if  $\mu > (<)0$ . Note that because of the CS term, one has necessarily a P and T violating theory. On the otherhand, the normal massive photon has two degrees of freedom and both spins  $\pm 1$  are present as they should be in a parity conserving theory.

**Coleman-Hill Theorem:** It turns out that because of the P and T violating but gauge invariant CS term, the most general form for the vacuum polarization tensor (consistent with Lorentz and gauge invariance) is more general than in other dimensions i.e.

$$\Pi_{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi_1(k^2) - i\epsilon_{\mu\nu\lambda}k^\lambda\Pi_2(k^2) \quad (14)$$

Notice that the second term on the r.h.s. is P and T odd. It is clear that any P and T violating interaction will contribute to  $\Pi_2(k^2)$ . For example the fermion mass term which in 2+1 dimensions break both P and T, does contribute to  $\Pi_2(k^2)$  at one loop level. Remarkably enough, it was discovered that at two loops, though, there is no contribution to  $\Pi_2(0)$  [23]. Inspired by this result, Coleman and Hill [14] have in fact proved under very general

conditions that  $\Pi_2(0)$  receives no contributions from two and higher loops in any gauge and Lorentz invariant theory including particles of spin one or less. In particular, they have emphasized that their result is valid even for nonrenormalizable interactions in the presence of gauge and Lorentz invariant regularizations. These authors also claimed that at one loop the only contribution to  $\Pi_2(0)$  can come from fermion loop. This is however not true. In particular, there is no reason why P and T violating interactions involving spin 0 or 1 particles should not contribute to  $\Pi_2(0)$  at one loop. Indeed, Hagen et al. [17] as well as we [16] have shown that nonrenormalizable spin one and spin zero interactions respectively do contribute to  $\Pi_2(0)$  at one loop.

It might be added here that apart from these situations which were overlooked by Coleman-Hill, there are other situations where the initial assumptions of the theorem are not satisfied and where  $\Pi_2(0)$  does get further radiative corrections. One such situation is if there are massless particles present in which case infrared divergences spoil the proof of the theorem [24]. Another case is if Lorentz or gauge invariance is not satisfied, a situation found in the nonabelian case. A third case is that of spontaneously broken scalar electrodynamics [25] where the term quadratic in the gauge field explicitly violates one of the assumption of Coleman and Hill. However, even in this case we have recently shown [16] that if the theorem is formulated in terms of effective action rather than vacuum polarization tensor then the coefficient of the CS term in the effective action does not receive a radiative correction at one loop.

**Magnetoelectric Effect:** There are many crystals in nature like chromium oxide which show this effect i.e. they get magnetized in an electric field and electrically polarized in a magnetic field [26]. It is well known that this effect depends upon having a T-assymmetric medium. In this case the usual relation between  $\vec{D}$  and  $\vec{E}$  as well as between  $\vec{H}$  and  $\vec{B}$  is modified to

$$D_i = \chi_{ij}^{(e)} E_j + \chi_{ij}^{(em)} B_j \quad (15)$$

$$H_i = \chi_{ij}^{(m)} B_j + \chi_{ij}^{(me)} E_j \quad (16)$$

Since CS term violates P and T, it is natural to ask if the vacuum of the 2+1 dimensional QED with CS term also shows the magnetoelectric effect or not. We have shown [15] that indeed the vacuum in such a theory does show this effect and both  $\chi_i^{(em)}$  and  $\chi_i^{(me)}$  are proportional to  $k_i \Pi_2(k^2)$ .

**CS Term by Spontaneous Symmetry Breaking:** We have seen above that the CS term provides mass to the gauge field. Now usually the gauge field mass is generated by spontaneous symmetry breaking hence it is worth enquiring if the CS term can also be generated by this mechanism. The answer to the question turns out to be yes. This is because as has been shown by us [22], unlike other dimensions, 2+1 dimensions offer more general possibility of a covariant derivative. For example, it is possible to introduce a Pauli type nonminimal coupling for even scalar particles. In particular, notice that

$$D_\mu \phi = (\partial_\mu + ieA_\mu + ig\epsilon_{\mu\nu\lambda}\partial^\nu A^\lambda)\phi \quad (17)$$

also behaves like a covariant derivative since the nonminimal term by itself is gauge covariant. As a result one now finds that the matter field kinetic energy term  $\frac{1}{2}(D^\mu\phi)^*(D_\mu\phi)$  has a piece  $eg|\phi|^2\epsilon_{\mu\nu\lambda}(\partial^\mu A^\nu)A^\lambda$ . Thus if  $\phi$  acquires a nonzero vacuum expectation value then the (abelian) CS term is generated. Clearly a similar mechanism should also work for the nonabelian case, but technically it is a tougher problem since one also has to generate the nonlinear triple gluon coupling term. So far as I know, till today it is an open problem.

The resulting nonminimal theory has been shown to have some very interesting properties in case the magnetic coupling constant  $g$  (called critical magnetic moment) acquires a special value [27, 29]. In particular, this theory gives rise to an effective action which is renormalizable at one loop [28] and has similar properties to the one describing ideal anyons upto an additive contact term. It is worth adding here that this nonminimal magnetic coupling can be induced by radiative corrections even if it is not present at the tree level [30].

Yet another remarkable property of the CS term is that in this case the Lorentz invariance of the action automatically follows from gauge invariance. In particular, whereas for the Maxwell case the most general gauge invariant Lagrangian is

$$\mathcal{L} = \vec{E}^2 + a\vec{B}^2 \quad (18)$$

it is only the demand of Lorentz invariance which fixes  $a$  to be -1. On the other hand, in the CS case the demand of gauge invariance automatically fixes the form of the CS term.

Let us now discuss some properties which are unique to the nonabelian CS term.



**Quantization of the CS Mass:** In the nonabelian gauge theory with CS term in 2+1 dimensions one finds that the gauge field is again massive and that the theory is well defined only if this mass is infact quantized. The gauge field Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2} Tr(F^{\mu\nu} F_{\mu\nu}) - \frac{\mu}{2g^2} \epsilon^{\mu\nu\alpha} Tr(F_{\mu\nu} A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha) \quad (19)$$

where  $A_\mu$  and  $F_{\mu\nu}$  are matrices as defined by eq. (6) The field equation which follows from here

$$D_\mu F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0 \quad (20)$$

where

$$D_\mu = \partial_\mu + [A_\mu, ] \quad (21)$$

is gauge covariant. As in the abelian case it immediately follows that the gauge field has mass  $\mu$ . Now notice that the action  $I_{CS} = \int d^3x \mathcal{L}_{CS}$  even though invariant under small gauge transformations, is not invariant under homotopically-nontrivial gauge transformations [6]. In particular, if the gauge group  $G$  is such that

$$\Pi_3(G) = Z \quad (22)$$

where  $Z$  is the group of integers (note in particular that eq. (22) is true for any gauge group of which  $SU(2)$  is a subgroup), then under these so called large gauge transformations the action transforms as

$$I_{cs} \longrightarrow I_{cs} + \frac{8\pi^2\mu}{g^2} m \quad (23)$$

where  $m$  is an integer. Now in the path integral formulation, the action itself may or may not be gauge invariant but what is required is that the exponential of the action should atleast be gauge invariant. We thus conclude that the nonabelian gauge theory with CS term does not make sence in 2+1 dimensions unless the CS mass is quantized in units of  $g^2/4\pi$  i.e.

$$\frac{8\pi^2\mu}{g^2} = 2n\pi \text{ or } \mu = \frac{g^2}{4\pi} n, n = 0, \pm 1, \pm 2, \dots \quad (24)$$

Please note that in 2+1 dimensions the gauge coupling  $g$  is not dimensionless but rather has dimension of  $(mass)^{\frac{1}{2}}$ .

**Parity Anomaly:** Someone might wonder as to why is one considering models with CS term in the first place since afterall this term violates both P and T. The answer to that is (atleast in nonabelian gauge theories) even if one does not add CS term to the action at the tree level, it is still generated by the radiative corrections—the effect due to the so called parity anomaly [21]. In particular, even though the action

$$I[A_\mu, \psi] = \int d^3x \left[ \frac{1}{2g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \gamma_\mu D^\mu \psi \right] \quad (25)$$

is invariant under both gauge transformations and P and T, the effective action  $I_{eff}[A]$  obtained by integrating out the fermionic degrees of freedom must violate one of the two symmetries in the case of odd number of massless fermions. In other words, there is no regularization which can simultaneously maintain the invariance of  $I_{eff}$  under parity as well as under large gauge transformations. In particular, under large gauge transformations with winding number  $n$ ,  $I_{eff}$  transforms as

$$I_{eff}[A] \longrightarrow I_{eff}[A] \pm n\pi \quad n = 0, \pm 1, \pm 2, \dots \quad (26)$$

Since gauge invariance is usually regarded as more important than parity, one can maintain gauge invariance at the cost of parity by adding the P and T-odd CS term in the action (or by regulating the theory in such a way that the CS term is automatically produced—say by using Pauli-Villars regularization). In this way one finds that because of the parity anomaly the CS term is induced by radiative corrections even if it is absent at the tree level. This is very similar to the way the Chern-Pontryagin term is induced in even dimensions due to the gauge singlet (chiral) anomaly. Note however that whereas the anomaly in even dimensions is in the divergence of the current, in 2+1 case there is no anomaly in the divergence of the current. The anomaly is in the current itself in that there is a piece with wrong parity. Also notice that the parity anomaly is only in the nonabelian gauge theories and not in the abelian U(1) case since in this case there are no large gauge transformations in the first place!

Finally, since the CS action depends only on  $\epsilon_{\mu\nu\lambda}$  tensor and not on the metric  $g_{\mu\nu}$  hence the gauge field action with only nonabelian (or even abelian) CS term is an example of topological field theory [31].

### 3 Charged Vortex Solutions

Let us consider the abelian Higgs model with CS term as given by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu - ieA_\mu)\phi^*(\partial^\mu + ieA^\mu)\phi - C_4(|\phi|^2 - \frac{C_2}{2C_4})^2 + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}A^\lambda F^{\mu\nu} \quad (27)$$

Following the neutral vortex case [2], let us consider the following n-vortex ansatz

$$\vec{A}(\vec{\varphi}, t) = -\hat{e}_0 C_0 \frac{g(r) - n}{r}, \quad \phi(\vec{\varphi}, t) = C_0 e^{in\theta} f(r), \quad A_0(\vec{\varphi}, t) = C_0 h(r) \quad (28)$$

where

$$\rho = \frac{r}{eC_0}, \quad C_0 = \sqrt{\frac{C_2}{2C_4}} \quad (29)$$

We have rescaled the lengths and the fields so that one can work in terms of the dimensionless variables. It turns out that the dynamics essentially depends on two dimensionless parameters  $\delta$  and  $\lambda$  defined by

$$\lambda = \sqrt{8C_4/e^2}; \quad \delta = \mu/eC_0 \quad (30)$$

The field equations which follow from here are [4]

$$g''(r) - \frac{1}{r}g'(r) - gf^2 = r\delta h'(r) \quad (31)$$

$$h''(r) + \frac{1}{r}h'(r) - hf^2 = \frac{\delta}{r}g'(r) \quad (32)$$

$$f''(r) + \frac{1}{r}f'(r) - \frac{g^2 f}{r^2} + \frac{\lambda^2}{2}f(1 - f^2) = -h^2 f \quad (33)$$

while the field energy can be shown to be [4]

$$E_n = \pi C_0^2 \int_0^\infty r dr \left[ \frac{1}{r^2} \left( \frac{dg}{dr} \right)^2 + \left( \frac{df}{dr} \right)^2 + \left( \frac{dh}{dr} \right)^2 + h^2 f^2 + \frac{g^2 f^2}{r^2} + \frac{\lambda^2}{4} (1 - f^2)^2 \right] \quad (34)$$

Several remarks are in order at this stage.

(i) As expected, in the limit  $A_0 = 0$  (i.e.  $h = 0$ ) and  $\mu = 0$  (i.e.  $\delta = 0$ ) the field equations reduce to those of the neutral vortex case [2]. From the

Gauss law eq. (32) it also follows that if  $\mu$  is nonzero then  $A_o$  must also be nonzero.

(ii) The boundary conditions for finite energy solutions are

$$\lim_{r \rightarrow \infty} : f(r) = 1, \quad h(r) = 0, \quad g(r) = 0 \quad (35)$$

$$\lim_{r \rightarrow 0} : f(r) = 0, \quad h(r) = \beta, \quad g(r) = n \quad (36)$$

with  $\beta$  being an arbitrary number.

(iii) A la neutral vortex case, in this case also one has flux quantization since the boundary conditions are again such that  $\Pi_1(U(1)) = \mathbb{Z}$

$$\Phi \equiv \int B d^2x = -\frac{2\pi}{e} \int_0^\infty r dr \frac{1}{r} \frac{dg}{dr} = \frac{2\pi}{e} n \quad (37)$$

From the Gauss law eq. (32) it then follows that these vortices also have nonzero and finite quantized charge [4]

$$Q \equiv \int e^2 h f^2 d^2x = \mu \Phi = \frac{2\pi\mu}{e} n \quad (38)$$

thereby bypassing the Julia-Zee objection. As far as I am aware off, this is probably the first time that the quantization of the Noether charge has followed from purely topological considerations. In a way the relation (38) can be looked upon as the 2+1 analog of the Witten effect [32]. One can also compute the magnetic moment of these vortices and show that whereas the neutral vortices have it equal to the flux  $\Phi (= 2\pi n/e)$ , the charged ones have an extra piece

$$K_z = \int (\vec{r} \times \vec{j})_z d^2x = \frac{2\pi n}{e} + \frac{\delta}{e} \int_0^\infty h(r) d^2r \quad (39)$$

(iv) As an extra bonus, one also finds that unlike the neutral vortices, the charged vortices have nonzero angular momentum which is ingeneral fractional and quantized in units of  $Q/2e$  i.e.

$$J \equiv \int d^2x \epsilon^{ij} x_i T_{oj} = -\frac{nQ}{2e} \quad (40)$$

This strongly suggests that the charged vortices could infact be charged anyons [7]. By explicit construction of a quantum one vortex operator,

Frohlich and Marchetti [8] have rigorously shown that this is indeed the case. Thus charged vortices provide us with a relativistic field theory model of extended charged anyons.

So far, no analytic solution has been obtained to the field eqs. (31) to (33). However, it is easily seen that for large  $r$ , the asymptotic values of the gauge and Higgs field are reached exponentially fast

$$g(r) = \alpha \sqrt{r} e^{-m_v r} \quad (41)$$

$$h(r) = \frac{\alpha}{\sqrt{r}} e^{-m_v r} \quad (42)$$

$$f(r) = 1 + \beta e^{-\lambda r} \quad (43)$$

where  $\alpha, \beta$  are dimensional constants while

$$m_v = \sqrt{\frac{\mu^2}{4} + e^2 C_0^2} - \frac{\mu}{2}. \quad (44)$$

Naively, another solution with

$$m_v = \sqrt{\frac{\mu^2}{4} + e^2 C_0^2} + \frac{\mu}{2}, \quad (45)$$

is also possible but as has been shown in [33], such a solution does not exist for all  $r$ . It is worth noting here that because of the P and T violating CS term, the gauge field, after the Higgs mechanism, propagates two modes each with one degree of freedom and with  $J = 1(-1)$  if  $\mu > (<)0$  [18, 34]. It is easily seen that the field equations are invariant under  $r \rightarrow -r$  so that the behaviour of the fields around  $r=0$  is given by

$$g(r) = n + \alpha r^2 + 0(r^4) \quad (46)$$

$$h(r) = \beta + \alpha \delta \frac{r^2}{2} + 0(r^4) \quad (47)$$

$$f(r) = \alpha_2 r^{|n|} + 0(r^{|n|+2}) \quad (48)$$

The qualitative behaviour of the charged vortex is as follows: the magnetic field  $B$  decreases monotonically from its nonzero value at the core of the vortex ( $r=0$ ) to zero at  $r = \infty$  with penetration length  $1/m_v$  while the

scalar field increases from zero at origin to its vacuum value at infinity with coherence length  $1/m_s$ . Finally, the electric field  $E_\rho$  vanishes at both  $r=0$  and  $r = \infty$  reaching the maximum in between at some finite  $r$ . It is worth pointing out that as in the Hall effect, for the charged vortex solutions too  $\vec{E}$  ( $= E_\rho$ ) is at right angles to  $\vec{j}$  ( $=j_\phi$ ) and both in turn are at right angles to  $B$ . It is also worth pointing out here that according to the presently accepted explanation, the quasi-particles responsible for the fractionally quantized Hall effect are the charged vortices.

## 4 Recent Advances

After our discovery of the charged vortex solutions in 1986 [4], in last ten years this work has been extended in several directions. I shall only briefly discuss the developments till 1988 which are already contained in my previous review article on this subject [10] while will discuss in detail some of the latter developments.

**Charged Vortex-Vortex Interaction:** Perhaps the most interesting question is if we can directly observe the charged vortices in some condensed matter system. In this context recall that the neutral vortices with one unit of vorticity have been seen in type-II superconductors. However, none has been seen in type-I superconductors. This can be understood from the fact that in the neutral case, whereas the vortex-vortex interaction is attractive in type-I region ( $\lambda < 1$ ), it is repulsive in type-II region ( $\lambda > 1$ ) [35]. It is thus of great interest to study charged vortex-vortex interaction and see as to when is it repulsive. This has been done by us [36] using perturbation theory in CS mass as well as by a variational calculation. For example, when CS mass is small, one can expand the charged vortex fields in terms of the neutral vortex fields plus corrections in powers of CS mass  $\delta$ . In particular, it has been shown that to  $O(\delta^2)$ , the charged  $n$ -vortex fields are given by [36, 33]

$$g(\lambda, \delta) = g_0(\lambda) + \frac{r^2 \delta^2}{8} g_0(\lambda) + O(\delta^4) \quad (49)$$

$$h(\lambda, \delta) = \frac{\delta}{2} g_0(\lambda) + O(\delta^3) \quad (50)$$

$$f(\lambda, \delta) = f_0(\lambda) + O(\delta^4) \quad (51)$$

where  $g_o$  and  $f_o$  are the solutions to the corresponding neutral vortex solutions in the absence of the CS term. On substituting the solution as given by eqs. (49) to (51) in the expression for the field energy as given by eq. (34) one can show that [36]

$$E_n^{cha}(\lambda, \delta) = E_n^{neu}(\lambda) + \frac{n^2 \delta^2}{4} + O(\delta^4) \quad (52)$$

It is worth noting that the  $O(\delta^2)$  correction is positive, proportional to  $n^2$  and independent of  $\lambda$ . From this equation it immediately follows that

$$E_n^{cha}(\lambda, \delta) - n E_1^{cha}(\lambda, \delta) = E_n^{neu}(\lambda) - n E_1^{neu}(\lambda) + \frac{(n^2 - n)}{4} \delta^2 + O(\delta^4) \quad (53)$$

so that the charged vortex-vortex interaction is more repulsive than the corresponding neutral case with the extra repulsion coming from the electric field of the charged vortex. We have also performed a variational calculation (which is more reliable than the perturbative calculation for larger values of  $\delta$ ) and find that even in this case the same picture continues to hold good. For example, for  $\delta = 0.5$  the charged vortex-vortex interaction is repulsive even for  $\lambda > 0.45$  (note that in the neutral case the vortex-vortex interaction is repulsive only when  $\lambda > 1$ ).

A word of caution is in order here. Our analysis is only valid in the case of superimposed vortices. The problem of charged vortex-vortex interaction when the vortices are separated by distance  $d$ , is still an open unsolved problem.

**Pure CS vortices:** Can one obtain charged vortex solutions in abelian Higgs model with pure CS term (i.e. no Maxwell kinetic energy term)? This question is particularly sensible in the condensed matter context since in the long wave length limit the CS term dominates over the Maxwell term. In this context note that the Higgs mechanism is operative even in the absence of the Maxwell term and even in this case one obtains both massive gauge and scalar fields [37]. The question of the charged vortices in the abelian Higgs model with pure CS term was addressed by us [38] and we showed that the charged vortex solutions are indeed possible in this case. Their properties are almost same as those of the Maxwell-CS charged vortices except that the magnetic field is now zero at the core of the vortex and is concentrated in a ring surrounding the vortex core [38]. It is worth noting here that in the

absence of the Maxwell term, the gauge field eqs. (31) and (32) are already of first order

$$-gf^2 = r\delta h'(r) \quad (54)$$

$$hf^2 = \frac{\delta}{r}g'(r) \quad (55)$$

while eq. (33) remains unaltered and is still a coupled second order equation. The obvious interesting question is if one can also write it as a coupled first order equation so that a la Bogomol'nyi [39] one could obtain self-dual charged vortex solutions. This question was raised by us [38] but we were unable to obtain the first order equation. It was left to two other groups [11] to make this important breakthrough. They showed that in addition to dropping the Maxwell term one also has to replace the usual  $\phi^4$  potential with the following  $\phi^6$ -type potential

$$V(|\phi|) = \frac{e^4}{8\mu^2} |\phi|^2 (|\phi|^2 - v^2)^2 \quad (56)$$

so as to obtain the self-dual equations. In that case, the self-dual equations turn out to be

$$f' = \pm \frac{1}{r}fg \quad (57)$$

$$B \equiv -\frac{1}{r}\frac{dg}{dr} = \pm \frac{1}{2}f^2(1 - f^2) \quad (58)$$

while

$$h = \mp \frac{1}{2}(1 - f^2) \quad (59)$$

One can infact decouple the eqs. (57) and (58) and obtain the following uncoupled second order equation in  $f$

$$f''(r) + \frac{1}{r}f'(r) - \frac{f'^2}{f} + \frac{1}{2}f^3(1 - f^2) = 0 \quad (60)$$

These self-dual equations are quite similar to those of the corresponding neutral case (at  $\lambda = 1$ ) which are given by

$$f' = \pm \frac{1}{r}gf \quad (61)$$



$$B \equiv -\frac{1}{r} \frac{dg}{dr} = \pm \frac{1}{2}(1 - f^2) \quad (62)$$

and hence the uncoupled second order equation in that case is

$$f''(r) + \frac{1}{r}f'(r) - \frac{f'^2}{f} + \frac{1}{2}f(1 - f^2) = 0 \quad (63)$$

Further, whereas the Lagrangian for the self-dual neutral vortex case is the bosonic part of a  $N = 1$  supersymmetric theory [40], the Lagrangian for the charged self-dual vortex case is the bosonic part of a  $N = 2$  supersymmetric theory [41].

Before we discuss the solutions to the self-dual eqs. (57) and (58) it may be worthwhile to point out that the usual  $\phi^4$  potential and the  $\phi^6$  potential as given by eq. (56) represent very different physical situations [42]. Whereas the  $\phi^4$  potential in eq. (27) corresponds to the case of second order phase transition with  $T < T_c^{II}$ , the  $\phi^6$  potential given above corresponds to the case of first order transition with  $T = T_c^I$ . One way to understand as to why  $\phi^6$ -type potential is required for the CS vortices while  $\phi^4$  potential is required for the neutral vortex case is that whereas in four space-time dimensions the coefficient of the  $\phi^4$ -term is dimensionless, it is the coefficient of the  $\phi^6$ -term which is dimensionless in 2+1 dimensions.

It turns out that the self-dual eqs. (57) and (58) admit both topological and nontopological self-dual charged vortex solutions. Let us first discuss the topological solutions.

**Topological Self-dual Solutions:** The topological solutions satisfy the boundary conditions as given by eqs. (35) and (36) with  $\beta = \mp 1/2$  (for  $n > (<) 0$  respectively). The flux, the charge and the angular momentum of these vortices are as given by eqs. (37) to (40) respectively and the energy is  $\pi v^2 |n|$ . Infinite number of sum rules have been derived for these vortices [43] (as well as the neutral self-dual vortices [44]) and using these we have shown that the magnetic moment of the self-dual pure CS vortex is  $2\pi n(n+1)\mu^2/e^3v^2$ . It is worth emphasizing that the self-dual solutions have been obtained not only with the cylindrical ansatz but also with an arbitrary ansatz. Further, following the work of Taubes for the neutral self-dual vortices [45], Wang [46] has given rigorous argument for the existence of self-dual charged vortex solutions even when the vortices are not superimposed on each other but lie at arbitrary positions in the plane.

**Nontopological Self-dual Solutions:** Since the potential as given by eq. (56) has degenerate minima at  $\phi = 0$  as well as at  $|\phi| = v$ , one finds that apart from the topological, one also has nontopological self-dual charged vortex solutions provided one chooses the following boundary conditions [43, 47]

$$\lim_{r \rightarrow \infty} : f(r) = 0, \quad g(r) = \mp \alpha, \quad \alpha > 0 \quad (64)$$

$$\lim_{r \rightarrow 0} : f(r) = 0, \quad g(r) = n, \quad \text{if } n \neq 0 \quad (65)$$

$$\lim_{r \rightarrow 0} : f(r) = \eta, \quad g(r) = 0 \quad \text{if } n = 0 \quad (66)$$

where  $+(-)\alpha$  is for  $n > (<)0$ . The flux, the charge and the angular momentum of these vortices can be shown to be ( $n > 0$ )

$$\Phi = \frac{2\pi}{e}(n + \alpha); \quad Q = \mu\Phi; \quad J = \frac{\pi\mu}{e^2}(\alpha^2 - n^2); \quad E = \pi v^2(n + \alpha) \quad (67)$$

It is worth pointing out that the finiteness of energy requires that  $\alpha > 1$  but otherwise  $\alpha$  is arbitrary. However, again in this case infinite number of sum rules have been derived by us [44] using which we have been able to show that  $\alpha$  must infact satisfy the lower bound  $\alpha \geq (n + 2)$  [48]. Further, from these sum rules it also follows that the magnetic moment of these nontopological vortices is  $-2\pi(\alpha + n)(\alpha - n - 1)\mu^2/e^3v^2$ . As far as I am aware off, this is the first instance when both topological and nontopological self-dual solutions simultaneously exist in a given model. So far as the decay to charged scalar mesons is concerned, these nontopological solutions are at the edge of their stability [49]. Finally, Spruck and Yang have rigorously shown the existance of self-dual nontopological CS vortices even when they are not superimposed on each other but lie at arbitrary positions in the plane [50].

**Various Other Self-dual Solutions:** Once the self-dual CS vortices were discovered in 1990, there has been a flurry of activity and several people have obtained various other self-dual solutions. While it is clearly impossible to mention all these developments, we shall try to note atleast some of them. For example, since  $\phi^4$  and  $\phi^6$ -type models correspond to very different physical situations, hence it is of interest to enquire if self-dual charged vortex solutions can also be obtained in the original  $\phi^4$  model [4] itself. This has been done and Lee et al. [51] have shown that such solutions can also be constructed in the  $\phi^4$  model [4] provided one adds a neutral scalar field in the model. Further, self-dual solutions have also been constructed with unusual

properties [52] by essentially multiplying the Maxwell and/or the CS term by a dielectric function (which in almost all cases is assumed to be a function of the scalar field  $\phi$  alone) and also by including the nonminimal interaction term. For example, a class of self-dual solutions have been obtained which are degenerate in energy but have different flux which is not quantized. Further, nonabelian self-dual CS vortices have also been obtained [53].

**Nonrelativistic Self-dual Vortices:** In an interesting paper, Jackiw and Pi [12] started with the abelian Higgs model with pure CS term and the  $\phi^6$  potential as given by eq. (56) (which has both topological and nontopological self-dual charged vortex solutions) and considered its nonrelativistic limit. In particular, they showed that to leading order in the velocity of light  $c$ , this model reduces to

$$\mathcal{L}_{NR} = \frac{\mu}{4} \epsilon_{\mu\nu\lambda} F^{\mu\nu} A^\lambda + i\psi^*(\partial_t + iA_0)\psi - \frac{1}{2m}(D_i\psi)^*(D_i\psi) + \frac{1}{2mc|\mu|}(\psi^*\psi)^2 \quad (68)$$

where  $\psi$  represents the particle part of the mode expansion of the scalar field  $\phi$  (the anti-particle part having been put equal to zero) and  $m$  is the mass of the field  $\phi$ . This nonrelativistic model can be looked upon either as a classical field theory or as a second quantized N-body problem with 2-body attractive  $\delta$ -function interaction. These authors were able to obtain self-dual nontopological charged vortex solutions with zero energy in the above model and showed that these vortex solutions are precisely the nonrelativistic limit of the corresponding nontopological charged vortex solutions. In fact, the flux, the charge and the angular momentum of these objects turn out to be the same as given by eq. (67) but where  $\alpha = (n+2)$  i.e. the lower bound on  $\alpha$  [44] is saturated in the nonrelativistic case. By now, this work has been extended in several directions. Mention may be made of the time dependent solutions by Ezawa et al. [54] and the nonrelativistic Maxwell-CS vortices [55]. Further, nonrelativistic limit of the nonabelian self-dual CS vortices has also been considered and interesting connections with integrable models have been discovered [56].

**Interaction Between Self-Dual CS Vortices:** Following the work of Manton [57] for the case of monopole and neutral vortices, recently Kim and Min [58] have considered the slow motion of two well separated CS vortices and have shown that the effective Lagrangian (with finite degrees of freedom) has a statistical interaction term and that this term reflects the

anyonic nature of the CS vortices. This analysis has recently been extended to the nonrelativistic case [59].

**Semi-Local Self-Dual CS Vortices:** Recently, semi-local neutral vortex solutions have been obtained in an abelian Higgs model with  $SU(N)_{global} \otimes U(1)_{local}$  symmetry. The key point of the argument is that even though topologically trivial, these solutions are stable under small perturbations due to the gradient energy term [60]. These semi-local solutions gave some initial hope of finding energetically stable solutions in the Weinberg-Salam model. However, subsequent analysis has shown that these semi-local vortex solutions are always unstable for the realistic values of the Weinberg angle. Inspired by this work, we have constructed [13] semi-local self-dual CS vortex solutions in an abelian Higgs model with pure CS term and with  $SU(N)_{global} \otimes U(1)_{local}$  symmetry. Most of the results for the CS vortices can be easily extended to this case.

**Charged Vortices of Finite Energy Per Unit length in 3+1 Dimensions:** So far we have shown that the Julia-Zee objection [3] can be bypassed in 2+1 dimensions by adding CS term to the action. However, the question remains if one can also overcome their objection in 3+1 dimensions itself (remember that their original argument is infact in 3+1 dimensions)? Recently, we [9] have been able to do that. In particular, we showed that if one generalizes the abelian Higgs model by adding a dielectric function and a neutral scalar field then one can obtain self-dual topological as well as nontopological charged vortex solutions with finite energy per unit length. In particular, we considered the following generalized abelian Higgs model

$$\mathcal{L} = -\frac{1}{4}G(|\phi|)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|\partial_\mu - ieA_\mu\phi|^2 + \frac{1}{2}G(|\phi|)\partial_\mu N\partial^\mu N - \frac{e^2}{8G(|\phi|)}(|\phi|^2 - v^2)^2 - \frac{e^2}{2}N^2|\phi|^2 \quad (69)$$

where  $N$  is the neutral scalar field. We showed that if the dielectric function  $G(|\phi|)$  is chosen to have the form

$$G(|\phi|) = \frac{\alpha}{|\phi|^2} \quad (70)$$

then the self-dual equations of the model can essentially be mapped to those of the pure CS vortices thereby explaining as to why one has both topological

as well as nontopological charged vortex solutions. However, unlike in that case, the Bogomol'nyi bound on energy is expressed not only in terms of the flux but also the vortex charge per unit length.

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