

New Shape Invariant Potentials in Supersymmetric Quantum Mechanics

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Abstract: Quantum mechanical potentials satisfying the property of shape invariance are well known to be algebraically solvable. Using a scaling ansatz for the change of parameters, we obtain a large class of new shape invariant potentials which are reflectionless and possess an infinite number of bound states. They can be viewed as q -deformations of the single soliton solution corresponding to the Rosen-Morse potential. Explicit expressions for energy eigenvalues, eigenfunctions and transmission coefficients are given. Included in our potentials as a special case is the self-similar potential recently discussed by Shabat and Spiridonov.

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In recent years, supersymmetric quantum mechanics [1] has yielded many interesting results. Some time ago, Gendenshtein pointed out that supersymmetric partner potentials satisfying the property of shape invariance and unbroken supersymmetry are exactly solvable [2]. The shape invariance condition is

$$V_+(x, a_0) = V_-(x, a_1) + R(a_0), \quad (1)$$

where a_0 is a set of parameters and $a_1 = f(a_0)$ is an arbitrary function describing the change of parameters. The common x -dependence in V_- and V_+ allows full determination of energy eigenvalues [2], eigenfunctions [3] and scattering matrices [4] algebraically. One finds ($\hbar = 2m = 1$)

$$E_n^{(-)}(a_0) = \sum_{k=0}^{n-1} R(a_k), \quad E_0^{(-)}(a_0) = 0, \quad (2)$$

$$\psi_n^{(-)}(x, a_0) = \left[-\frac{d}{dx} + W(x, a_0) \right] \psi_{n-1}^{(-)}(x, a_1), \quad \psi_0^{(-)}(x, a_0) \propto e^{-\int^x W(y, a_0) dy} \quad (3)$$

where the superpotential $W(x, a_0)$ is related to $V_{\pm}(x, a_0)$ by

$$V_{\pm}(x, a_0) = W^2(x, a_0) \pm W'(x, a_0). \quad (4)$$

In terms of W , the shape invariance condition reads

$$W^2(x, a_0) + W'(x, a_0) = W^2(x, a_1) - W'(x, a_1) + R(a_0). \quad (5)$$

It is still a challenging open problem to identify and classify the solutions to eq.(5). Certain solutions to the shape invariance condition are known [5]. (They include the harmonic oscillator, Coulomb, Morse, Eckart and Poschl-Teller potentials). In all these cases, it turns out that a_1 and a_0 are related by a translation ($a_1 = a_0 + \alpha$). Careful searches with this ansatz have failed to yield any additional shape invariant potentials

[6]. Indeed it has been suggested [7] that there are no other shape invariant potentials. Although a rigorous proof has never been presented, no counter examples have so far been found either.

In this letter, we consider solutions of eq.(5) resulting from a new scaling ansatz

$$a_1 = qa_0, \quad (6)$$

where $0 < q < 1$. This choice was motivated by the recent interest in q -deformed Lie algebras. It enables us to find a large class of new shape invariant potentials all of which are reflectionless and possess an infinite number of bound states. As a special case, our approach includes the self-similar potential studied by Shabat [8] and Spiridonov [9].

Consider an expansion of the superpotential of the form

$$W(x, a_0) = \sum_{j=0}^{\infty} g_j(x) a_0^j. \quad (7)$$

Substituting into eq.(5), writing $R(a_0)$ in the form

$$R(a_0) = \sum_{j=0}^{\infty} R_j a_0^j, \quad (8)$$

and equating powers of a_0 yields

$$g'_0(x) = R_0, \quad g_0(x) = R_0 x + C_0, \quad (9)$$

$$g'_n(x) + 2d_n g_0(x) g_n(x) = r_n - d_n \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x), \quad (10)$$

where

$$R_n \equiv (1 - q^n) r_n, \quad d_n \equiv \frac{1 - q^n}{1 + q^n} \quad (n = 1, 2, 3 \dots). \quad (11)$$

This set of linear differential equations is easily solvable in succession yielding a general solution of eq.(5). Note that the limit $q \rightarrow 0$ is particularly simple yielding the one-soliton

Rosen-Morse potential of the form $W = \gamma \tanh \gamma x$. Thus our results can be regarded as multiparameter deformations of this potential corresponding to different choices of R_n . For simplicity, in this letter we shall confine our attention to the special case $g_0(x) = 0$ (i.e. $R_0 = C_0 = 0$) while the more general case will be discussed elsewhere [10].

For $g_0 = 0$, the solution is

$$g_n(x) = d_n \int dx \left[r_n - \sum_{j=1}^{n-1} g_j(x) g_{n-j}(x) \right]. \quad (12)$$

For the simplest case of $r_n = 0, n \geq 2$ we obtain the superpotential W as given by Shabat [8] and Spiridonov [9] provided we choose $d_1 r_1 a_0 = \gamma^2$ and replace q by q^2 . This shows that the self-similarity condition of these authors is in fact a special case of the shape invariance condition (5). This comment is also true in case any one r_n (say r_j) is taken to be nonzero and q^j is replaced by q^2 .

Let us now consider a somewhat more general case when $r_n = 0, n \geq 3$. Using eq.(12) we can readily calculate all $g_n(x)$. The first three are

$$\begin{aligned} g_1(x) &= d_1 r_1 x, \quad g_2(x) = d_2 r_2 x - \frac{1}{3} d_1^2 r_1^2 d_2 x^3, \\ g_3(x) &= -\frac{2}{3} d_1 r_1 d_2 r_2 d_3 x^3 + \frac{2}{15} d_1^3 r_1^3 d_2 d_3 x^5. \end{aligned} \quad (13)$$

Note that $W(x)$ contains only odd powers of x . This makes the potential $V_-(x)$ symmetric in x and also guarantees the situation of unbroken supersymmetry. The energy eigenvalues follow immediately from eqs.(2) and (8) ($n=0,1,2,\dots,\infty; 0 < q < 1$)

$$E_n^{(-)}(a_0) = d_1 r_1 a_0 \frac{(1 - q^n)}{(1 - q)} + d_2 r_2 a_0^2 \frac{(1 - q^{2n})}{(1 - q^2)}. \quad (14)$$

Note that the energy eigenvalues, the superpotential W and hence the eigenfunctions only depend on the two combinations of parameters $\gamma_1^2 \equiv d_1 r_1 a_0$ and $\gamma_2^2 \equiv d_2 r_2 a_0^2$. The unnormalized ground state wave-function is

$$\psi_0^{(-)}(x, a_0) = \exp \left[-\frac{x^2}{2} (\gamma_1^2 + \gamma_2^2) + \frac{x^4}{12} (d_2 \gamma_1^4 + 2d_3 \gamma_1^2 \gamma_2^2 + d_4 \gamma_2^4) + 0(x^6) \right]. \quad (15)$$

The excited state wave-functions can be recursively calculated by using eq.(3) with $a_1 = qa_0$.

The transmission coefficient of two symmetric partner potentials are related by [11]

$$T_-(k, a_0) = \left[\frac{ik - W(\infty, a_0)}{ik + W(\infty, a_0)} \right] T_+(k, a_0), \quad (16)$$

where $k = [E - W^2(\infty, a_0)]^{1/2}$. For a shape invariant potential

$$T_+(k, a_0) = T_-(k, a_1). \quad (17)$$

Repeated application of eqs.(16) and (17) gives

$$T_-(k, a_0) = \left[\frac{ik - W(\infty, a_0)}{ik + W(\infty, a_0)} \right] \left[\frac{ik - W(\infty, a_1)}{ik + W(\infty, a_1)} \right] \cdots \left[\frac{ik - W(\infty, a_{n-1})}{ik + W(\infty, a_{n-1})} \right] T_-(k, a_n), \quad (18)$$

where

$$W(\infty, a_j) = \sqrt{E_\infty^{(-)} - E_j^{(-)}}. \quad (19)$$

As $n \rightarrow \infty$, since $a_n = q^n a_0$ and we have taken $g_0(x) = 0$, one gets $W(x, a_n) \rightarrow 0$. This corresponds to a free particle for which the transmission coefficient is unity. Thus, for the potential $V_-(x, a_0)$, the reflection coefficient vanishes and the transmission coefficient is

$$T_-(k, a_0) = \prod_{j=0}^{\infty} \left[\frac{ik - W(\infty, a_j)}{ik + W(\infty, a_j)} \right]. \quad (20)$$

Clearly, $|T| = 1$ and the poles of T_- correspond to the energy eigenvalues of eq.(14). Note that one does not get reflectionless potentials for the case $g_0(x) \neq 0$. This will be further discussed in ref.[10].

The above discussion, keeping only $r_1, r_2 \neq 0$, can readily be generalized to an arbitrary number of nonzero r_j . The energy eigenvalues for this case are given by ($\gamma_j^2 = d_j r_j a_0^j$)

$$E_n^{(-)}(a_0) = \sum_j \gamma_j^2 \left(\frac{1 - q^{jn}}{1 - q^j} \right), \quad n = 0, 1, 2, \dots \quad (21)$$

All of these potentials are also reflectionless with T_- as given by eqs.(19) to (21). One expects that these symmetric reflectionless potentials can also be derived using previously developed methods [12] and the spectrum given in eq.(21).

In eqs.(7) and (8) we have only kept positive powers of a_0 . If instead we had only kept negative powers of a_0 , then the spectrum would be similar except that one has to choose the deformation parameter $q > 1$. However, a mixture of positive and negative powers of a_0 is not allowed in general since neither $q < 1$ nor $q > 1$ will give an acceptable spectrum. For the enlarged class of shape invariant potentials discussed in this letter, it is clear [3] that the lowest order supersymmetric WKB approximation [13] will yield the exact spectrum.

We conclude with two brief remarks on extensions of the work described in this letter. We have been able to construct new shape invariant potentials which are q -deformations of the potentials corresponding to multi-soliton systems [10]. Also, it is possible to show [10] that with the choice $g_0(x) \neq 0$, one gets q -deformations of the one dimensional harmonic oscillator potential.

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