Supersymmetry-inspired WKB approximation in quantum mechanics

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The supersymmetry-inspired WKB approximation (SWKB) in quantum mechanics is discussed in detail. The SWKB method can be easily applied to any potential whose ground-state wave function is known. It yields eigenvalues that are exact for large quantum numbers n (as any WKB approximation should in the classical limit). Furthermore, for the important special case of "shape-invariant" potentials, the SWKB approach gives the exact analytic expressions for the entire bound-state spectra. A study of some nonshape-invariant, but solvable, potentials suggests that shape invariance is not only sufficient but perhaps even necessary for the SWKB approximation to be exact. A comparison of the WKB and SWKB predictions for the bound-state spectra of a number of potentials reveals that in many cases the SWKB approach does better than the usual WKB approximation.

I. INTRODUCTION

The WKB method[1] is one of the most useful approximations for computing the energy eigenvalues of the Schrödinger equation. The purpose of this article is to describe and give applications of the recently developed, supersymmetric WKB method (henceforth called SWKB[2]), which has been inspired by supersymmetric quantum mechanics.[3] For quantum mechanical problems, the main implication of supersymmetry (SUSY) is that it relates the energy eigenvalues, eigenfunctions, and phase shifts of two supersymmetric partner potentials. Combining the ideas of supersymmetry with the lowest-order WKB method, Comtet *et al.*[2] obtained the lowest-order SWKB quantization condition, which can be applied to any potential whose ground state ($n=0$) wave function ψ_0 (and hence eigenvalue E_0) is known. SWKB yields energy eigenvalues that are exact for large quantum numbers n (as any semiclassical approximation scheme should). Being exact in the two extreme limits of $n=0$ and $n\to\infty$, one might reasonably expect SWKB to be better than the WKB approximation for all values of n. For shape-invariant potentials,[4] this is indeed the case. As has been shown by us,[5] the lowest-order SWKB approximation reproduces the exact bound-state spectrum for any shape-invariant potential. (This is a big improvement over the ordinary WKB method which only gives the exact spectrum for the harmonic oscillator and Morse potentials.) Further, the corrections to $O(\hbar^6)$ have been explicitly shown to be zero[6] for all known shape-invariant potentials.[7] A study of some nonshape-invariant, but solvable, potentials suggests that shape invariance is not only sufficient but perhaps also necessary for the lowest-order SWKB approximation to be exact.[8] This study also demonstrates[8,9] that in many cases the SWKB method gives more accurate eigenvalues than the usual WKB approach.

The supersymmetric WKB method is not only very useful and elegant, but, at the same time, it is simple enough to be profitably included in graduate-level quantum mechanics courses and textbooks. Accordingly, we have kept this article at a pedagogical level and made it as self-contained as possible.

In Sec. II we give a quick review of the main ideas of supersymmetric quantum mechanics. Section III is the heart of this article. In it, we first derive the lowest-order SWKB quantization condition by starting from the WKB approximation. Then, we define the notion of "shape-invariant" potentials[4] and show that for these potentials the lowest-order SWKB approximation gives the exact bound-state spectrum. In Sec. IV, we obtain and compare the bound-state spectra of a few analytically solvable, nonshape-invariant potentials by using the WKB as well as the SWKB quantization conditions. A discussion of related problems and concluding remarks is contained in Sec. V.

II. SUPERSYMMETRIC QUANTUM MECHANICS

Let us consider a potential $V_-(x)$ whose ground-state wave function $\psi_0^{(-)}(x)\equiv\psi_0(x)$ is known and whose ground-state energy has been adjusted so that $E_0^{(-)}=0$. Then the Schrödinger equation for the ground state is

$$H_-\psi_0=\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_-(x)\right)\psi_0=0, \tag{1}$$

so that

$$V_-(x)=(\hbar^2/2m)[\psi_0''(x)/\psi_0(x)]. \tag{2}$$

One now defines the operators

$$A=\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x),\quad A^+=-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x), \tag{3}$$

where the superpotential $W(x)$ is related to ψ_0 by

$$W(x)=-\hbar/\sqrt{2m}\,[\psi_0'(x)/\psi_0(x)],$$
$$\psi_0(x)=\exp\left(-\frac{\sqrt{2m}}{\hbar}\int^x W(x')dx'\right). \tag{4}$$

Multiplying the operators A and A^+ gives

$$A^+A=H_-,$$
$$AA^+=H_+=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_+(x), \tag{5}$$

where

$$V_{\pm}(x) = W^2(x) \pm (\hbar/\sqrt{2m})W'(x). \tag{6}$$

From Eqs. (3) and (4), it immediately follows that $A\psi_0 = 0$, which checks that $E_0^{(-)}$ is indeed zero. The Hamiltonians H_+, H_- and operators A, A^+ satisfy the algebra of SUSY. H_+ and H_- are called SUSY partner Hamiltonians since their energy eigenvalues and eigenfunctions are related. Let $\psi_n^{(+)}(\psi_n^{(-)})$ denote the eigenfunctions of H_+ (H_-) with eigenvalues $E_n^{(+)}(E_n^{(-)})$, where $n = 0,1,2,...$ gives the number of nodes of the wave function. Then, if $\psi_n^{(-)}$ is an eigenfunction of H_- with eigenvalue $E_n^{(-)}$ then $A\psi_n^{(-)}$ is an eigenfunction of H_+ with the same eigenvalue except when $n = 0$.[7] Similarly, if $\psi_n^{(+)}$ is an eigenfunction of H_+ with eigenvalue $E_n^{(+)}$, then $A^+\psi_n^{(+)}$ is an eigenfunction of H_- with the same eigenvalue.

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad E_0^{(-)} = 0, \tag{7}$$

$$\psi_n^{(+)} = (E_{n+1}^{(-)})^{-1/2} A\psi_{n+1}^{(-)},$$

$$\psi_{n+1}^{(-)} = (E_n^{(+)})^{-1/2} A^+\psi_n^{(+)}. \tag{8}$$

The operator $A(A^+)$ not only converts an eigenfunction of H_- (H_+) into an eigenfunction of H_+ (H_-) with the same energy, but it also destroys (creates) a node.

III. LOWEST-ORDER SWKB APPROXIMATION

The lowest-order SWKB quantization condition can be derived in two equivalent ways. The first method is to repeat the standard textbook derivation[1] starting from first principles using Eq. (6) for the potential $V_-(x)$. This involves writing the wave function as

$$\psi(x) = \exp\left(i\hbar \int S(x)dx\right),$$

obtaining a Riccati equation for $S(x)$, and expanding in $\hbar$. The second method is to start directly from the WKB quantization condition, insert Eq. (6) for the potential $V_-(x)$, and retain the lowest-order terms in $\hbar$.[2] Here, we shall use the second method.

In lowest order, the WKB quantization condition for one-dimensional problems is[1]

$$\int_{x_1}^{x_2} \{2m[E_n - V(x)]\}^{1/2} dx = (n + \tfrac{1}{2})\hbar\pi \quad (n = 0,1,2,...), \tag{9}$$

where x_1 and x_2 are the turning points defined by $E_n = V(x_1) = V(x_2)$. Since the potential $V_-(x)$ given by Eq. (6) explicitly contains $\hbar$, the quantization condition (9) takes the form

$$I \equiv \int_{x_1(\hbar)}^{x_2(\hbar)} f(x,\hbar)dx = \left(n + \frac{1}{2}\right)\hbar\pi,$$

$$f(x,\hbar) \equiv \left\{2m\left[E_n^{(-)} - W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x)\right]\right\}^{1/2}. \tag{10}$$

We now wish to expand I in powers of $\hbar$. The dependence on $\hbar$ comes explicitly from the integrand $f(x,\hbar)$ as well as from the WKB classical turning points $x_1(\hbar)$ and $x_2(\hbar)$. Note that in the limit $\hbar \to 0$, the classical turning points a and b are given by $E_n = W^2(a) = W^2(b)$, and $\lim_{\hbar\to 0} x_{1,2} = a,b$. The Taylor expansion in $\hbar$ of I is given by

$$I = I_0 + \hbar I_1 + O(\hbar^2),$$

where,

$$I_0 = \lim_{\hbar\to 0} I = \int_a^b \{2m[E_n^{(-)} - W^2(x)]\}^{1/2} dx,$$

$$I_1 = \lim_{\hbar\to 0} \frac{dI}{d\hbar}$$

$$= \lim_{\hbar\to 0}\left(\int_{x_1}^{x_2} \frac{\partial f}{\partial \hbar} dx + f(x_2,\hbar)\frac{dx_2}{d\hbar} - f(x_1,\hbar)\frac{dx_1}{d\hbar}\right).$$

The last two terms vanish, since $f(x_1,\hbar) = f(x_2,\hbar) = 0$ (definition of the WKB classical turning points). Therefore, substituting the expansion for I into Eq. (10), one gets

$$I \equiv \int_a^b \{2m[E_n^{(-)} - W^2(x)]\}^{1/2} dx + \frac{\hbar}{2}\int_a^b \frac{W'(x)dx}{[E_n^{(-)} - W^2(x)]^{1/2}} + O(\hbar^2)$$

$$= \left(n + \frac{1}{2}\right)\hbar\pi. \tag{11}$$

Note that the quantity I, which is $O(\hbar^0)$ in the normal WKB scheme now contains all powers of $\hbar$. This result is a consequence of the explicit $\hbar$ dependence in the potential $V_-(x)$, which comes from supersymmetric quantum mechanics. The second integral in Eq. (11) can be easily done and gives $\frac{1}{2}\hbar\pi$, so that to leading order in $\hbar$ the SWKB quantization condition is[2]

$$\int_a^b \{2m[E_n^{(-)} - W^2(x)]\}^{1/2} dx = n\hbar\pi. \tag{12}$$

The method used above to obtain the lowest-order SWKB result can be extended to obtain higher-order corrections. The result carried out explicitly through $O(\hbar^8)$ is given in Ref. 6. Note that the key features that distinguish the lowest-order SWKB quantization condition (12) [from the usual WKB result (9)] are the use of $W^2(x)$ with its associated turning points a,b and $n\hbar\pi$ on the right-hand side.

Proceeding in the same way, the SWKB quantization condition for the potential $V_+(x)$ [see Eq. (6)] turns out to be

$$\int_a^b \{2m[E_n^{(+)} - W^2(x)]\}^{1/2} dx = (n+1)\hbar\pi. \tag{13}$$

Some remarks are in order at this stage:

(i) For $n = 0$ the turning points a and b in Eq. (12) are coincident since $E_0^{(-)} = 0$ and hence SWKB is exact by construction for the ground-state energy of the Hamiltonian H_-.

(ii) On comparing Eqs. (12) and (13), it follows that the lowest-order SWKB quantization condition preserves the SUSY level degeneracy $E_{n+1}^{(-)} = E_n^{(+)}$.

(iii) Since the lowest-order SWKB approximation is not only exact for large n, but is also exact by construction for $n = 0$, hence, unlike the WKB approach, the SWKB eigenvalues are constrained at both ends. One can thus reasonably expect better results than the WKB scheme.

How good is the SWKB quantization condition (12) vis-a-vis the WKB condition (9)? One could study this question by obtaining the bound-state spectra of several analytically solvable potentials by using both Eqs. (9) and (12).

In this way, it was soon discovered[2] that the lowest-order SWKB quantization condition (12) reproduces the exact bound-state spectrum for several analytically solvable potentials like Coulomb, harmonic oscillator, Morse, Rosen–Morse, Eckart, etc.

Why is the lowest-order SWKB approximation exact for many analytically solvable potentials? We now show that this is because all these analytically solvable potentials satisfy the "shape-invariance condition,"[4] and for shape-invariant potentials the lowest-order SWKB is necessarily exact.[5]

Let us first explain precisely what one means by shape-invariant potentials. If the pair of SUSY partner potentials $V_{\pm}(x)$ defined by Eq. (6) are similar in shape and differ only in the parameters that appear in them, then they are said to be shape invariant. More precisely, if the partner potentials $V_{\pm}(x; a_0)$ satisfy the condition[4]

$$V_{+}(x; a_0) = V_{-}(x; a_1) + R(a_1), \tag{14}$$

where a_0 is a set of parameters, a_1 is a function of a_0 [$a_1 = f(a_0)$], and the remainder $R(a_1)$ is independent of x then $V_{\pm}(x; a_0)$ are said to be shape invariant.[4] Using condition (14) one can immediately obtain the energy eigenvalues of any shape-invariant potential.[4] To that purpose, let us first construct a series of Hamiltonians $H^{(s)}$, $s = 0,1,2,...$, where $H^{(0)} \equiv H_{-}$, $H^{(1)} \equiv H_{+}$, and

$$H^{(s)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{-}(x; a_s) + \sum_{k=1}^{s} R(a_k). \tag{15}$$

Here, $a_s = f^s(a_0)$ i.e., the function f applied s times. Clearly, then, the ground-state energy of $H^{(s)}$ is

$$E_0^{(s)} = \sum_{k=1}^{s} R(a_k). \tag{16}$$

On going back from $H^{(s)}$ to $H^{(s-1)}$, we eventually reach $H^{(1)}$ ($\equiv H_{+}$) and $H^{(0)}$ ($\equiv H_{-}$) whose ground-state energy is, of course, zero and its nth energy level is coincident with the ground state of Hamiltonian $H^{(n)}$ ($n = 0,1,2,...$) [see Eq. (7)]. Hence the complete spectrum of H_{-} is given by

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k), \quad E_0^{(-)} = 0. \tag{17}$$

Let us now show that the leading-order SWKB quantization condition (12) has the nice property of reproducing the exact bound-state spectrum as given by Eq. (17).[5] This follows because, as shown above, the SWKB condition [see Eqs. (12) and (13)] preserves the level degeneracy and the vanishing ground-state energy as given by Eq. (11). For the series of Hamiltonians, $H^{(s)}$ given by Eq. (15), the SWKB quantization condition takes the form

$$\int \left[2m\left(E_n^{(s)} - \sum_{k=1}^{s} R(a_k) - W^2(a_s; x)\right)\right]^{1/2} dx = n\pi\hbar. \tag{18}$$

Now, since by construction the SWKB quantization condition is exact for the ground-state energy when SUSY is unbroken, hence

$$E_0^{(s)} - \sum_{k=1}^{s} R(a_k) = 0 \tag{19}$$

as given by Eq. (18) must be exact for $H^{(s)}$. One can now go back in sequential manner from $H^{(s)}$ to $H^{(s-1)}$ to $H^{(1)}$ ($\equiv H_{+}$) and $H^{(0)}$ ($\equiv H_{-}$) and use the fact that the SWKB method preserves the level degeneracy $E_{n+1}^{(-)} = E_n^{(+)}$. On using this relation n times, we then find that for the shape-invariant potentials the lowest-order SWKB gives the exact energy eigenvalues.[5]

A complete list of all the known shape-invariant potentials with their exact bound-state energy eigenvalues has been given in Ref. 7. For all these potentials, we have explicitly checked that the SWKB formula (12) reproduces the exact bound-state spectrum. This is a very substantial improvement over the usual WKB formula (9), which is not exact for most shape-invariant potentials. The SWKB approach is also more natural than making *ad hoc* Langer corrections, which are different for different potentials.[10] Besides, even with such corrections, the higher-order WKB contributions are nonzero for most of these potentials.[10,11] What about the higher-order SWKB contributions? Since the lowest-order SWKB energies are exact for shape-invariant potentials, it would be nice to check that higher-order corrections vanish order by order in $\hbar$. By starting from the higher-order WKB formalism, we have recently developed[6] the higher-order SWKB formalism and explicitly checked that the $O(\hbar^2)$ to $O(\hbar^6)$ corrections indeed vanish for all known shape-invariant potentials. For the special case of the Rosen–Morse potential, Raghunathan *et al.* have further shown using a complex integration technique that all higher-order corrections are zero.[12]

IV. SWKB VERSUS WKB FOR NONSHAPE-INVARIANT POTENTIALS

We now have two semiclassical quantization schemes at hand. It is clearly necessary to study the two schemes in detail and explore their merits and demerits. For example, for potentials for which the ground-state wave function (and hence the superpotential W) is not known, clearly the WKB approach is preferable to SWKB, since one can directly use the WKB quantization condition (9) but cannot use (12). On the other hand, for shape-invariant potentials, clearly SWKB is superior to WKB. The obvious interesting question concerns potentials that are not shape invariant but for which the ground-state wave function is known. In this section, we wish to compare the predictions of the WKB and SWKB methods for some of these potentials.

Another interesting and in a way related question is whether shape invariance is necessary for the lowest-order SWKB to be exact. This is a difficult question and, at present, we do know how to prove or disprove this conjecture. However, we shall study the spectra of some exactly solvable but nonshape-invariant potentials by using the lowest-order SWKB quantization condition (12). One potential that immediately comes to mind is the Ginocchio potential[13] through which Cooper *et al.*[14] have shown that though shape invariance is sufficient it is not necessary for the exact solvability of the Schrödinger equation, thereby disproving the conjecture of Gendenshtein.[4] Another choice is the class of strictly isospectral Hamiltonians (i.e., having same energy eigenvalues and phase shifts) corresponding to the various shape-invariant potentials.[15] The Ginocchio potential is given by[13]

$$V(x,\nu;\lambda) = (1-y^2)\{-\lambda^2\nu(\nu+1) + [(1-\lambda^2)/4] \times [2-(7-\lambda^2)y^2 + 5(1-\lambda^2)y^4]\}, \tag{20}$$

Table I. Comparison of the lowest-order WKB and SWKB predictions for the bound-state spectrum of the Ginocchio potential for different values of λ and ν. The exact answer is given by Eqs. (23) and (24) ($\hbar = 2m = 1$).

n	WKB $\lambda = 0.5, \nu = 5.5$	SWKB	Exact value	WKB $\lambda = 6.25, \nu = 5.5$	SWKB	Exact value
0	− 6.186 94	− 6.406 41	− 6.406 41	− 1372.281 12	− 1359.611 47	− 1359.611 47
1	− 3.079 65	− 3.169 71	− 3.129 98	− 1228.157 61	− 1212.7014	− 1213.8388
2	− 1.430 39	− 1.467 68	− 1.439 19	− 1012.523 12	− 999.6307	− 1003.703 01
3	− 0.584 66	− 0.600 78	− 0.584 74	− 733.588 61	− 727.7175	− 737.620 22
5	− 0.017 014	− 0.018 750	− 0.016 684	− 55.269 83	− 70.5885	− 109.5019
n	WKB $\lambda = 0.5, \nu = 10.5$	SWKB	Exact value	WKB $\lambda = 6.25, \nu = 10.5$	SWKB	Exact value
0	− 24.866 61	− 25.170 48	− 25.170 48	− 4659.878	− 4648.6161	− 4648.6161
1	− 17.091 83	− 17.272 13	− 17.233 52	− 4452.736	− 4438.321	− 4438.7989
2	− 11.570 71	− 11.677 90	− 11.633 52	− 4174.482	− 4158.478	− 4159.4341
3	− 7.701 99	− 7.766 73	− 7.728 00	− 3828.490	− 3812.566	− 3815.6502
5	3.151 00	− 3.176 11	− 3.153 52	− 2949.599	− 2978.741	− 2947.7017

where y is related to the independent variable x by

$$\frac{dy}{dx} = (1 - y^2)[1 - (1 - \lambda^2)y^2]. \tag{21}$$

Here, the parameters ν and λ measure the depth and the shape of the potential, respectively. The corresponding superpotential is[14]

$$W(x) = \tfrac{1}{2}(1 - \lambda^2)y(y^2 - 1) + \mu_0 \lambda^2 y, \tag{22}$$

where μ_n is given by[13]

$$\mu_n \lambda^2 = [\lambda^2(\nu + \tfrac{1}{2})^2 + (1 - \lambda^2)(n + \tfrac{1}{2})^2]^{1/2} - (n + \tfrac{1}{2}) \tag{23}$$

and the bound-state energies are

$$E_n = -\mu_n^2 \lambda^4, \quad n = 0,1,2,\ldots . \tag{24}$$

For the special case $\lambda = 1$, this potential reduces to the shape-invariant Rosen–Morse potential.

The one-parameter family of potentials that is strictly isospectral to the one-dimensional harmonic oscillator[15] is given by ($2m = 1$),

$$V_-(x;\lambda) = \frac{\omega^2 x^2}{4} - 2\frac{d^2}{dx^2}\ln[J(x) + \lambda], \qquad \lambda > 0 \text{ or } \lambda < -1, \tag{25}$$

where

$$J(x) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{\omega}{2}}\,x\right) \equiv \int_x^\infty \psi_0^2(y)\,dy \tag{26}$$

and $\psi_0(x)$ is the normalized ground-state wave function.

The energy eigenvalues of the potential (25) are

$$E_n^{(-)}(\lambda) = n\omega; \quad n = 0,1,2,..., \tag{27}$$

while the corresponding normalized ground-state wave function is given by

$$\psi_0^{(-)}(x) = \sqrt{\lambda(1 + \lambda)}\{\psi_0(x)/[J(x) + \lambda]\}. \tag{28}$$

Using Eqs. (20)–(28), the spectra of the two potentials have been computed[8] by using both the WKB and SWKB quantization conditions (9) and (12), respectively. The results are displayed in Tables I and II. From these tables we can draw the following conclusions:

(i) In general, the lowest-order SWKB and WKB methods do not give the exact spectrum for the potentials (20) and (25). The only exceptions are the shape-invariant limits [$\lambda = 1$ for the potential (20) while $\lambda = \pm\infty$ for the potential (25)], in which case the SWKB results are exact.

(ii) For potential (25), the SWKB results are better than the WKB ones for all values of λ and n, while for the Ginocchio potential there is no clear-cut pattern except that for $n = 0,1$, the SWKB values are consistently better.

(iii) For the potential (25) one finds that as one goes away from the shape-invariant limit ($\lambda = \pm\infty$), the SWKB (as well as the WKB) answer starts differing more and more from the exact answer.

(iv) These calculations reveal a certain nonuniqueness regarding the SWKB (as well as the WKB) approximation. The point is that the SUSY partner potential to the entire class of potentials (25) is

$$V_+(x) \equiv W^2(x) + \hbar W'(x) = \omega^2 x^2/4 + (\hbar/2)\omega, \tag{29}$$

Table II. Comparison of the lowest-order WKB and SWKB predictions for the bound-state spectrum of the potential given in Eq. (25) with different values of λ. The exact answer is $E_n^{(-)}(\lambda) = n(\hbar = 2m = 1)$.

n	WKB $\lambda = 100$	SWKB	WKB $\lambda = 10$	SWKB	WKB $\lambda = 1.0$	SWKB
1	1.0005	1.0004	1.003 72	1.002 66	1.069 43	1.048 36
2	2.000 03	2.000 02	2.002 46	2.001 63	2.047 12	2.030 39
3	3.000 03	3.000 02	3.001 96	3.001 27	3.037 88	3.023 88
5	5.000 02	5.000 01	5.001 50	5.000 15	5.029 00	4.9802
10	10.000 01	10.000 01	10.001 05	10.000 66	10.020 32	10.0125

whose exact spectrum is ($\hbar = 2m = 1$)

$$E_n^{(+)} = (n+1)\omega. \tag{30}$$

However, if we wish to obtain $E_n^{(+)}$ from $E_n^{(-)}$ by using the level degeneracy $E_n^{(+)} = E_{n+1}^{(-)}$ which is preserved by the SWKB approximation, then we would find from Table II that the answer would depend on which $V_-(x;\lambda)$ is used since $E_n^{(-)}(\lambda)$ is λ dependent. In this respect, the WKB method is much worse than SWKB, since it does not preserve either the level degeneracy or the $E_n^{(-)} = 0$ relation.

(v) These examples as well as various potentials studied in Ref. 4 support the conjecture that shape invariance is perhaps a necessary condition for the lowest-order SWKB approximation to reproduce the exact spectrum.

V. CONCLUSIONS

The ideas of SUSY quantum mechanics have many applications. Among these have been the study of atomic systems,[16] evaluation of the eigenvalues of bistable potentials,[17] improvement of large N expansions,[18] and analytically solvable potentials.[7,14] In this paper, we have focused on some of the properties of the supersymmetry-inspired WKB quantization scheme.[2] In order to use the SWKB scheme, one needs to know the exact ground-state wave function. We have shown that unlike the WKB method, the SWKB scheme correctly gives the exact bound-state spectrum for all shape-invariant potentials. We have also investigated a class of nonshape-invariant potentials, and compared the exact (numerical) energy eigenvalues with those obtained by using both the SWKB method and the usual WKB method.

In this paper we have concentrated on the energy eigenvalues. For completeness, let us note that several authors have also obtained the wave functions in the SWKB approximation.[2,19,20] As in the WKB method, the SWKB wave functions diverge at the classical turning points a, b defined below Eq. (11). These divergences can be regularized either by the uniform approximation[19,21] or by appropriate retention of higher orders in $\hbar$.[20] It would be interesting to investigate if the SWKB method proves to be as useful for scattering problems as it has for bound states.

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