Black hole entropy and attractors

2006 Class. Quantum Grav. 23 S957

(http://iopscience.iop.org/0264-9381/23/21/S04)

INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) S957–S980 doi:10.1088/0264-9381/23/21/S04

Black hole entropy and attractors

Atish Dabholkar

Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Received 1 September 2006
Published 3 October 2006
Online at stacks.iop.org/CQG/23/S957

Abstract

The introductory lectures delivered at the *2006 RTN Winter School on Strings, Supergravity and Gauge Theories* at CERN and the *2005 Shanghai Summer School on Recent Trends in M/String Theory* were aimed at Ph.D students and recent postdocs. We describe recent progress in understanding quantum aspects of black holes after reviewing the relevant background material, and illustrate the basic concepts with a few examples.

1. Introduction

One of the important successes of string theory is that one can obtain a statistical understanding of the thermodynamic entropy [1, 2] of certain supersymmetric black holes in terms of microscopic counting [3]. The entropy of black holes supplies us with very useful quantitative information about the fundamental degrees of freedom of quantum gravity.

In this lecture we describe some recent progress in our understanding of the quantum structure of black holes. We begin with a brief review of black holes, their entropy, and relevant aspects of string theory and then discuss a few illustrative examples. For more details on black holes we refer the reader to [4, 5] and for the relevant aspects of string theory to [6–9].

2. Black holes

To understand the relevant parameters and the geometry of black holes, let us first consider the Einstein–Maxwell theory described by the action

$$\frac{1}{16\pi G}\int R\sqrt{g}\,\mathrm{d}^4x - \frac{1}{16\pi}\int F^2\sqrt{g}\,\mathrm{d}^4x, \tag{2.1}$$

where G is Newton's constant, $F_{\mu\nu}$ is the electromagnetic field strength, R is the Ricci scalar of the metric $g_{\mu\nu}$. In our conventions, the indices μ, ν take values 0, 1, 2, 3 and the metric has signature $(-, +, +, +)$.

2.1. Schwarzschild metric

Consider the Schwarzschild metric which is a spherically symmetric, static solution of the vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ that follow from (2.1) when no electromagnetic fields are excited. This metric is expected to describe the spacetime outside a gravitationally collapsed non-spinning star with zero charge. The solution for the line element is given by

$$ds^2 \equiv g_{\mu\nu}\,dx^\mu\,dx^\nu = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2,$$

where t is the time, r is the radial coordinate and Ω is the solid angle on a 2-sphere. This metric appears to be singular at $r = 2GM$ because some of its components vanish or diverge, $g_{00} \to \infty$ and $g_{rr} \to \infty$. As is well known, this is not a real singularity. This is because the gravitational tidal forces are finite or in other words, components of Riemann tensor are finite in orthonormal coordinates. To better understand the nature of this apparent singularity, let us examine the geometry more closely near $r = 2GM$. The surface $r = 2GM$ is called the 'event horizon' of the Schwarzschild solution. Much of the interesting physics relating to the quantum properties of black holes comes from the region near the event horizon.

To focus on the near-horizon geometry in the region $(r - 2GM) \ll 2GM$, let us define $(r - 2GM) = \xi$, so that when $r \to 2GM$ we have $\xi \to 0$. The metric then takes the form

$$ds^2 = -\frac{\xi}{2GM}\,dt^2 + \frac{2GM}{\xi}(d\xi)^2 + (2GM)^2\,d\Omega^2, \tag{2.2}$$

up to corrections that are of order $\left(\frac{1}{2GM}\right)$. Introducing a new coordinate ρ,

$$\rho^2 = (8GM)\xi \quad \text{so that} \quad d\xi^2\frac{2GM}{\xi} = d\rho^2,$$

the metric takes the form

$$ds^2 = -\frac{\rho^2}{16G^2M^2}\,dt^2 + d\rho^2 + (2GM)^2\,d\Omega^2. \tag{2.3}$$

From the form of the metric it is clear that ρ measures the geodesic radial distance. Note that the geometry factorizes. One factor is a 2-sphere of radius $2GM$ and the other is the (ρ, t) space,

$$ds_2^2 = -\frac{\rho^2}{16G^2M^2}\,dt^2 + d\rho^2. \tag{2.4}$$

We now show that this (1+1)-dimensional spacetime is just a flat Minkowski space written in funny coordinates called the Rindler coordinates.

2.2. Rindler coordinates

To understand the Rindler coordinates and their relation to the near-horizon geometry of the black hole, let us start with $1 + 1$ Minkowski space with the usual flat Minkowski metric,

$$ds^2 = -dT^2 + dx^2. \tag{2.5}$$

In light-cone coordinates,

$$U = (T + X) \qquad V = (T - X), \tag{2.6}$$

the line element takes the form

$$ds^2 = -dU\,dV. \tag{2.7}$$

Now we make a coordinate change

$$U = \frac{1}{\kappa}\,\mathrm{e}^{a\kappa u}, \qquad V - \frac{1}{\kappa}\,\mathrm{e}^{-\kappa u}, \tag{2.8}$$

to introduce the Rindler coordinates (u, v). In these coordinates the line element takes the form

$$\mathrm{d}s^2 = -\mathrm{d}U\,\mathrm{d}V = -\mathrm{e}^{\kappa(u-v)}\,\mathrm{d}U\,\mathrm{d}V. \tag{2.9}$$

Using further coordinate changes

$$u = (t + x), \qquad v = (t - x), \qquad \rho = \frac{1}{\kappa}\,\mathrm{e}^{\kappa x}, \tag{2.10}$$

we can write the line element as

$$\mathrm{d}s^2 = \mathrm{e}^{2\kappa x}(-\mathrm{d}t^2 + \mathrm{d}x^2) = -\rho^2\kappa^2\,\mathrm{d}t^2 + \mathrm{d}\rho^2. \tag{2.11}$$

Comparing (2.4) with this Rindler metric, we see that the (ρ, t) factor of the Schwarzschild solution near $r \sim 2GM$ looks precisely like the Rindler spacetime with metric

$$\mathrm{d}s^2 - \rho^2\kappa^2\,\mathrm{d}t^2 + \mathrm{d}\rho^2 \tag{2.12}$$

with the identification

$$\kappa = \frac{1}{4GM}.$$

This parameter κ is called the surface gravity of the black hole. For the Schwarzschild solution, one can think of it heuristically as the Newtonian acceleration $GM\big/r_H^2$ at the horizon radius $r_H = 2GM$. Both these parameters—the surface gravity κ and the horizon radius r_H—play an important role in the thermodynamics of black hole.

This analysis demonstrates that the Schwarzschild spacetime near $r = 2GM$ is not singular at all. After all it looks exactly like flat Minkowski space times a sphere of radius $2GM$. So the curvatures are inverse powers of the radius of curvature $2GM$ and hence are small for large $2GM$.

2.3. *Kruskal extension*

One important fact to note about the Rindler metric is that the coordinates u, v do not cover all of Minkowski space because even when they vary over the full range

$$-\infty \leqslant u \leqslant \infty, \qquad -\infty \leqslant v \leqslant \infty,$$

the Minkowski coordinate varies only over the quadrant

$$0 \leqslant U \leqslant \infty, \qquad -\infty < V \leqslant 0. \tag{2.13}$$

If we had written the flat metric in these 'bad', 'Rindler-like' coordinates, we would find a fake singularity at $\rho = 0$, where the metric appears to become singular. But we can discover the 'good', Minkowski-like coordinates U and V and extend them to run from $-\infty$ to ∞ to see the entire spacetime.

Since the Schwarzschild solution in the usual (r, t) Schwarzschild coordinates near $r = 2GM$ looks exactly like Minkowski space in Rindler coordinates, it suggests that we must extend it in properly chosen 'good' coordinates. As we have seen, the 'good' coordinates near $r = 2GM$ are related to the Schwarzschild coordinates in exactly the same way as the Minkowski coordinates are related to the Rindler coordinates.

In fact one can choose 'good' coordinates over the entire Schwarzschild spacetime. These 'good' coordinates are called the Kruskal coordinates. To obtain the Kruskal coordinates, we first introduce the 'tortoise coordinate'

$$r^* = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right). \tag{2.14}$$

In the (r^*, t) coordinates, the metric is conformally flat, i.e., flat up to rescaling

$$ds^2 = \left(1 - \frac{2GM}{r}\right)(-dt^2 + dr^{*2}). \tag{2.15}$$

Near the horizon the coordinate r^* is similar to the coordinate x in (2.11) and hence $u = t + r^*$ and $v = t - r^*$ are like the Rindler (u, v) coordinates. This suggests that we define U, V coordinates as in (2.8) with $\kappa = 1/4GM$. In these coordinates the metric takes the form

$$ds^2 = -e^{-(u-v)\kappa}\, dU\, dV = -\frac{2GM}{r}\, e^{-r/2GM}\, dU\, dV. \tag{2.16}$$

We now see that the Schwarzschild coordinates cover only a part of spacetime because they cover only a part of the range of the Kruskal coordinates. To see the entire spacetime, we must extend the Kruskal coordinates to run from $-\infty$ to ∞. This extension of the Schwarzschild solution is known as the Kruskal extension.

Note that now the metric is perfectly regular at $r = 2GM$ which is the surface $UV = 0$ and there is no singularity there. There is, however, a real singularity at $r = 0$ which cannot be removed by a coordinate change because physical tidal forces become infinite. Spacetime stops at $r = 0$, and at present we do not know how to describe physics near this region.

2.4. Event horizon

We have seen that $r = 2GM$ is not a real singularity but a mere coordinate singularity which can be removed by a proper choice of coordinates. Thus, locally there is nothing special about the surface $r = 2GM$. However, globally, in terms of the causal structure of spacetime, it is a special surface and is called the 'event horizon'. An event horizon is a boundary of region in spacetime behind which no causal signals can reach the observers sitting far away at infinity.

To see the causal structure of the event horizon, note that in metric (2.11) near the horizon, the constant radius surfaces are determined by

$$\rho^2 = \frac{1}{\kappa^2} e^{2\kappa x} = \frac{1}{\kappa^2} e^{\kappa u}\, e^{-\kappa v} = -UV = \text{constant}. \tag{2.17}$$

These surfaces are thus hyperbolas. The Schwarzschild metric is such that at $r \gg 2GM$ an observer who wants to remain at a fixed radial distance $r =$ constant is almost like an inertial, freely falling observer in flat space. Her trajectory is timelike and is a straight line going upwards in a spacetime diagram. Near $r = 2GM$, on the other hand, the constant r lines are hyperbolas which are the trajectories of observers in uniform acceleration.

To understand the trajectories of observers at radius $r > 2GM$, note that to stay at a fixed radial distance r from a black hole, the observer must boost the rockets to overcome gravity. Far away, the required acceleration is negligible and the observers are almost freely falling. But near $r = 2GM$ the acceleration is substantial and the observers are not freely falling. In fact at $r = 2GM$, these trajectories are light like. This means that a fiducial observer who wishes to stay at $r = 2GM$ has to move at the speed of light with respect to the freely falling observer. This can be achieved only with infinitely large acceleration. This unphysical acceleration is the origin of the coordinate singularity of the Schwarzschild coordinate system.

In summary, the surface defined by $r =$ contant is timelike for $r > 2GM$, spacelike for $r < 2GM$ and light-like or null at $r = 2GM$.

In Kruskal coordinates, at $r = 2GM$, we have $UV = 0$ which can be satisfied in two ways. Either $V = 0$, which defines the 'future event horizon', or $U = 0$, which defines the 'past event horizon'. The future event horizon is a one-way surface where signals can be sent into but cannot come out of. The region bounded by the event horizon is then a black hole. It is literally a hole in spacetime which is black because no light can come out of it. Heuristically, a black hole is black because even light cannot escape its strong gravitational pull. Our analysis of the metric makes this notion more precise. Once an observer falls inside the black hole, she can never come out because to do so she will have to travel faster than the speed of light.

As we have noted already $r = 0$ is a real singularity that is inside the event horizon. Since it is a spacelike surface, once an observer falls inside the event horizon, she is sure to meet the singularity at $r = 0$ sometime in future no matter how much she boosts the rockets.

The summarize, an event horizon is a *stationary, null* surface. For instance, in our example of the Schwarzschild black hole, it is stationary because it is defined as a hypersurface $r = 2GM$ which does not change with time. More precisely, the timelike Killing vector $\frac{\partial}{\partial t}$ leaves it invariant. It is at the same time null because g^{rr} vanishes at $r = 2GM$. This surface that is simultaneously stationary and null, causally separates the inside and the outside of a black hole.

2.5. Black hole parameters

From our discussion of the Schwarzschild black hole we are ready to abstract some important general concepts that are useful in describing the physics of more general black holes.

To begin with, a *black hole* is an asymptotically flat spacetime that contains a region which is not in the backward lightcone of future timelike infinity. The boundary of such a region is a stationary null surface called the *event horizon*. The fixed t slice of the event horizon is a 2-sphere.

There are a number of important parameters of the black hole. We have introduced these in the context of Schwarzschild black holes. For general black holes their actual values are different but for all black holes, these parameters govern the thermodynamics of black holes:

1. The radius of the event horizon r_H is the radius of the 2-sphere. For a Schwarzschild black hole, we have $r_H = 2GM$.
2. The area of the event horizon A_H is given by $4\pi r_H^2$. For a Schwarzschild black hole, we have $A_H = 16\pi G^2M^2$.
3. The surface gravity is the parameter κ that we encountered earlier. As we have seen, for a Schwarzschild black hole, $\kappa = 1/4GM$.

3. Black hole entropy

3.1. Laws of black hole mechanics

One of the remarkable properties of black holes is that one can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is quite surprising because *a priori* there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

(1) *Zeroth law*. In thermal physics, the zeroth law states that the temperature T of a body at thermal equilibrium is constant throughout the body. Otherwise heat will flow from hot spots to the cold spots. Correspondingly for black holes one can show that the surface gravity κ is constant on the event horizon. This is obvious for spherically symmetric

horizons but is also true more generally for non-spherical horizons of spinning black holes.

(2) *First law*. Energy is conserved, $\mathrm{d}E = T\,\mathrm{d}s + \mu\,\mathrm{d}Q + \Omega dJ$, where E is the energy, Q is the charge with chemical potential μ and J is the spin with chemical potential Ω. Correspondingly for black holes, one has $\mathrm{d}M = \frac{\kappa}{8\pi G}\,\mathrm{d}A + \mu\,\mathrm{d}Q + \Omega\,\mathrm{d}J$. For a Schwarzschild black hole we have $\mu = \Omega = 0$ because there is no charge or spin.

(3) *Second law*. In a physical process the total entropy S never decreases, $\Delta S \geqslant 0$. Correspondingly for black holes one can prove the area theorem that the net area never decreases, $\Delta A \geqslant 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem since the area is proportional to the square of the mass and $(M_1 + M_2)^2 \geqslant M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

This formal analogy is actually much more than an analogy. Bekenstein and Hawking discovered that there is a deep connection between black hole geometry, thermodynamics and quantum mechanics.

3.2. Hawking temperature

Bekenstein asked a simple-minded but incisive question. If nothing can come out of a black hole, then a black hole will violate the second law of thermodynamics. If we throw a bucket of hot water into a black hole then the net entropy of the world outside would seem to decrease. Do we have to give up the second law of thermodynamics in the presence of black holes?

Note that the energy of the bucket is also lost to the outside world but that does not violate the first law of thermodynamics because the black hole carries mass or equivalently energy. So when the bucket falls in, the mass of the black hole goes up accordingly to conserve energy. This suggests that one can save the second law of thermodynamics if somehow the black hole also has entropy. Following this reasoning and noting the formal analogy between the area of the black hole and entropy discussed in the previous section, Bekenstein proposed that a black hole must have entropy proportional to its area.

This way of saving the second law is however in contradiction with the classical properties of a black hole because if a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

For example, for a Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$. Therefore, one would expect an inverse temperature that scales as M,

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M. \tag{3.1}$$

Now, if the black hole has temperature then like any hot body, it must radiate. For a classical black hole, by its very nature, this is impossible. Hawking showed that after including quantum effects, however, it is possible for a black hole to radiate. In a quantum theory, particles and antiparticles are constantly being created and annihilated even in a vacuum. Near the horizon, an antiparticle can fall in once in a while and the particle can escape to infinity. In fact, Hawking's calculation showed that the spectrum emitted by the black hole is precisely thermal with temperature $T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}$. With this precise relation between the temperature and surface gravity the laws of black hole mechanics discussed in the earlier section become identical to the laws of thermodynamics. Using the formula for the Hawking temperature and the first law of thermodynamics

$$\mathrm{d}M = T\,\mathrm{d}S = \frac{\kappa\hbar}{8\pi G\hbar}\,\mathrm{d}A,$$

one can then deduce the precise relation between entropy and the area of the black hole:

$$S = \frac{Ac^3}{4G\hbar}.$$

3.3. Euclidean derivation of Hawking temperature

Before discussing the entropy of a black hole, let us derive the Hawking temperature in a somewhat heuristic way using an Euclidean continuation of the near-horizon geometry. In quantum mechanics, for a system with Hamiltonian H, the thermal partition function is

$$Z = \mathrm{Tr}\,\mathrm{e}^{-\beta\hat{H}}, \tag{3.2}$$

where β is the inverse temperature. This is related to the time evolution operator $\mathrm{e}^{-\mathrm{i}tH/\hbar}$ by a Euclidean analytic continuation $t = -\mathrm{i}\tau$ if we identify $\tau = \beta\hbar$. Let us consider a single scalar degree of freedom Φ, then one can write the trace as

$$\mathrm{Tr}\,\mathrm{e}^{-\tau\hat{H}/\hbar} = \int \mathrm{d}\phi\,\langle\phi|\mathrm{e}^{-\tau_E\hat{H}/\hbar}|\phi\rangle$$

and use the usual path integral representation for the propagator to find

$$\mathrm{Tr}\,\mathrm{e}^{-\tau\hat{H}/\hbar} = \int \mathrm{d}\phi \int D\Phi\,\mathrm{e}^{-S_E[\Phi]}.$$

Here $S_E[\Phi]$ is the Euclidean action over periodic field configurations that satisfy the boundary condition

$$\Phi(\beta\hbar) = \Phi(0) = \phi.$$

This gives the relation between the periodicity in Euclidean time and the inverse temperature,

$$\beta\hbar = \tau \quad \text{or} \quad T = \frac{\hbar}{\tau}. \tag{3.3}$$

Let us now look at the Euclidean–Schwarzschild metric by substituting $t = -\mathrm{i}t_E$. Near the horizon the line element (2.11) looks like

$$\mathrm{d}s^2 = \rho^2\kappa^2\,\mathrm{d}t_E^2 + \mathrm{d}\rho^2.$$

If we now write $\kappa t_E = \theta$, then this metric is just the flat two-dimensional Euclidean metric written in polar coordinates provided the angular variable θ has the correct periodicity $0 < \theta < 2\pi$. If the periodicity is different, then the geometry would have a conical singularity at $\rho = 0$. This implies that Euclidean time t_E has periodicity $\tau = \frac{2\pi}{\kappa}$. Note that far away from the black hole at asymptotic infinity the Euclidean metric is flat and goes as $\mathrm{d}s^2 = \mathrm{d}\tau_E^2 + \mathrm{d}r^2$. With periodically identified Euclidean time, $t_E \sim t_E + \tau$, it looks like a cylinder. Near the horizon at $\rho = 0$ it is nonsingular and looks like a flat space in polar coordinates for this correct periodicity. The full Euclidean geometry thus looks like a cigar. The tip of the cigar is at $\rho = 0$ and the geometry is asymptotically cylindrical far away from the tip.

Using the relation between Euclidean periodicity and temperature, we then conclude that Hawking temperature of the black hole is

$$T = \frac{\hbar\kappa}{2\pi}. \tag{3.4}$$

3.4. Bekenstein–Hawking entropy

Even though we have 'derived' the temperature and the entropy in the context of Schwarzschild black hole, this beautiful relation between area and entropy is true quite generally essentially because the near-horizon geometry is always Rindler-like. For *all* black holes with charge, spin and in number of dimensions, the Hawking temperature and the entropy are given in terms of the surface gravity and horizon area by the formulae

$$T_H = \frac{\hbar\kappa}{2\pi}, \qquad S = \frac{A}{4G\hbar}.$$

This is a remarkable relation between the thermodynamic properties of a black hole on one hand and its geometric properties on the other.

The fundamental significance of entropy stems from the fact that even though it is a quantity defined in terms of gross thermodynamic properties it contains nontrivial information about the *microscopic* structure of the theory through Boltzmann relation

$$S = k \log \Omega,$$

where Ω is the total number of microstates of the system for a given energy and k is Boltzmann constant. Entropy is not a kinematic quantity like energy or momentum but rather contains information about the total number of microscopic degrees of freedom of the system. Because of this relation, we can learn a great deal about the microscopic properties of a system from its thermodynamics properties.

Bekenstein–Hawking entropy behaves in every other respect like the ordinary thermodynamic entropy. It is therefore natural to ask what microstates might account for it. Since the entropy formula is given by this beautiful general form,

$$S = \frac{Ac^3}{4G\hbar},$$

that involves all the three fundamental dimensional constants of nature; it is a valuable piece of information about the degrees of freedom of a quantum theory of gravity.

String theory is a consistent quantum theory of gravity and should offer a statistical interpretation of black hole entropy. Indeed, this is a highly nontrivial consistency check of the formalism of string theory. At the moment, we still do not understand the entropy of a big Schwarzschild black hole in terms of its microstates. But for a large class of special supersymmetric black holes, it is possible to obtain a statistical account of the entropy with impressive numerical agreement.

4. Supersymmetric black holes

4.1. Reissner–Nordström metric

The most general static, spherically symmetric, charged solution of the Einstein–Maxwell theory (2.1) gives the Reissner–Nordström (RN) black hole. In what follows we choose units so that $G = \hbar = 1$. The line element is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \qquad (4.1)$$

and the electromagnetic field strength by

$$F_{tr} = Q/r^2.$$

The parameter Q is the charge of the black hole and M is the mass as for the Schwarzschild black hole.

Now, the event horizon for this solution is located at where $g^{rr} = 0$, or

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0.$$

Since this is a quadratic equation in r,

$$r^2 - 2QMr + Q^2 = 0,$$

it has two solutions

$$r_\pm = M \pm \sqrt{M^2 - Q^2}.$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$.

Following the steps similar to what we did for the Schwarzschild black hole, we can analyse the near-horizon geometry to find the surface gravity and hence the temperature,

$$T = \frac{\kappa\hbar}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{4\pi M(M + \sqrt{M^2 - Q^2}) - Q^2} \tag{4.2}$$

$$S = \pi r_+^2 = \pi\left(M + \sqrt{M^2 - Q^2}\right)^2. \tag{4.3}$$

These formulae reduce to those for the Schwarzschild black hole in the limit $Q = 0$.

4.2. Extremal black holes

For a physically sensible definition of temperature and entropy in (4.2) the mass must satisfy the bound $M^2 \geqslant Q^2$. Something special happens when this bound is saturated and $M = |Q|$. In this case $r_+ = r_- = |Q|$ and the two horizons coincide. We choose Q to be positive. Solution (4.1) then takes the form

$$\mathrm{d}s^2 = -(1 - Q/r)^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{(1 - Q/r)^2} + r^2\,\mathrm{d}\Omega^2, \tag{4.4}$$

with a horizon at $r = Q$. In this extremal limit (4.2), we see that the temperature of the black hole goes to zero and it stops radiating, but nevertheless its entropy has a finite limit given by $S \to \pi Q^2$. When the temperature goes to zero, thermodynamics does not really make sense but we can use this limiting entropy as the definition of the zero temperature entropy.

For extremal black holes it is more convenient to use isotropic coordinates in which the line element takes the form

$$\mathrm{d}s^2 = H^{-2}(\vec{x})\,\mathrm{d}t^2 + H^2(\vec{x})\,\mathrm{d}\vec{x}^2,$$

where $\mathrm{d}\vec{x}^2$ is the flat Euclidean line element $\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ and $H(\vec{x})$ is a harmonic function of the flat Laplacian

$$\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

The Reissner–Nordström solution is obtained by choosing

$$H(\vec{x}) = \left(1 + \frac{Q}{r}\right),$$

and the field strength is given by $F_{0i} = \partial_i H(\vec{x})$.

One can in fact write a multi-centred Reissner–Nordström solution by choosing a more general harmonic function

$$H = 1 + \sum_{i=1}^{N}\frac{Q_i}{|\vec{x} - \vec{x}i|}. \tag{4.5}$$

The total mass M equals the total charge Q and is given additively:

$$Q = \sum Q_i. \tag{4.6}$$

The solution is static because the electrostatic repulsion between different centres balances gravitational attraction among them.

Note that the coordinate r in the isotropic coordinates should not be confused with the coordinate r in the spherical coordinates. In the isotropic coordinates the line element is

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^2 dt^2 + \left(1 + \frac{Q}{r}\right)^{-2} (dr^2 + r^2\, d\Omega^2),$$

and the horizon occurs at $r = 0$. Contrast this with the metric in the spherical coordinates (4.4) that has the horizon at $r = M$. The near horizon geometry is quite different from that of the Schwarzschild black hole. The line element is

$$\begin{aligned} ds^2 &= -\frac{r^2}{Q^2}\, dt^2 + \frac{Q^2}{r^2}(dr^2 + r^2\, d\Omega^2) \\ &= \left(-\frac{r^2}{Q^2}\, dt^2 + \frac{Q^2}{r^2}\, dr^2\right) + (Q^2\, d\Omega^2). \end{aligned}$$

The geometry thus factorizes as for the Schwarzschild solution. One factor is the 2-sphere S^2 of radius Q but the other (r, t) factor is now not Rindler any more but is a two-dimensional anti-de Sitter or AdS_2. The geodesic radial distance in AdS is $\log r$. As a result the geometry looks like an infinite throat near $r = 0$, and the radius of the mouth of the throat has radius Q.

Extremal RN black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. We now try to see if the entropy can be explained by counting of microstates. In doing so, supersymmetry proves to be a very useful tool.

4.3. Supersymmetry algebra

Some of the special properties of external black holes can be understood better by embedding them in $N = 2$ supergravity.

The supersymmetry algebra contains in addition to the usual Poincaré generators the supercharges Q^i_α, where $\alpha = 1, 2$ is the usual Weyl spinor index of four-dimensional Lorentz symmetry. Because we have $\mathcal{N} = 2$ symmetry we have an internal index $i = 1, 2$ so the supercharges transform in a doublet of an $SU(2)$, the R-symmetry of the superalgebra. The relevant anti-commutators for our purpose are

$$\left\{Q^i_\alpha, \bar{Q}_{\dot{\beta} j}\right\} = 2P_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta^i_j \tag{4.7}$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = Z\epsilon_{\alpha\beta}\epsilon^{ij} \qquad \{\bar{Q}_{\dot{\alpha} j}, \bar{Q}_{\dot{\beta} j}\} = \bar{Z}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ij}, \tag{4.8}$$

where σ^μ are (2×2) matrices with $\sigma_0 = -\mathbf{1}$ and σ^i, $i = 1, 2, 3$ are the usual Pauli matrices. Here P_μ is the momentum operator and Q are the supersymmetry generators and the complex number Z is the central charge of the supersymmetry algebra. We have altogether eight real supercharges since we have $\mathcal{N} = 2$ supersymmetry. In general for $\mathcal{N}$ extended supersymmetry in four dimensions, there are $4\mathcal{N}$ real supercharges.

Let us first look at the representations of this algebra when the central charge is zero. In this case the massive and massless representation are qualitatively different.

1. Massive representation, $M > 0$, $P^\mu = (M, 0, 0, 0)$
   In this case, (4.7) becomes $\left\{Q^i_\alpha, \bar{Q}_{\dot{\beta} j}\right\} = 2M\delta_{\alpha\dot{\beta}}\delta^i_j$ and all other anti-commutators vanish. Up to overall scaling, these are the commutation relations for four fermionic

oscillators. Each oscillator has a two-state representation, filled or empty, and hence the total dimension of the representation is $2^4 = 16$ which is CPT self-conjugate.

2. Massless representation $M = 0$, $P^\mu = (E, 0, 0, E)$
In this case (4.7) becomes $\left\{Q^1_\alpha, \bar{Q}_{\dot\beta 1}\right\} = 2E\delta_{\alpha\dot\beta}$ and all other anti-commutators vanish. Up to overall scaling, these are now the anti-commutation relations of *two* fermionic oscillators and hence the total dimension of the representation is $2^2 = 4$ which is not CPT self-conjugate. In local field theory we also get the CPT conjugate of the representation and hence the massless representation is eight-dimensional.

The important point is that for a massive representation, with $= \epsilon > 0$, no matter how small ϵ, the supermultiplet is long and precisely at $M = 0$ it is short. Thus the size of the supermultiplet has to change discontinuously if the state has to acquire mass. Furthermore, the size of the supermultiplet is determined by the number of supersymmetries that are *broken* because those have non-vanishing anti-commutations and turn into fermionic oscillators.

There are two massless representations that will be of interest to us.

1. Supergravity multiplet.
It contains the metric $g_{\mu\nu}$, a vector A^0_μ and two gravitini $\psi^i_{\mu\alpha}$.
2. Vector multiplet.
It contains a vector A^I_μ, a complex scalar field X^I and the gaugini χ^i_α, where the index I goes from $1, \ldots, n_v$ if we have n_v vector multiplets.

Note that in the case $Z = 0$ that we discussed above, there is a bound on the mass $M \geqslant 0$ which simply follows from the fact that using (4.7) one can show that the mass operator on the right-hand side of the equation equals a positive operator, the absolute value square of the supercharge on the left-hand side. The massless representation saturates this bound and is 'small' whereas the massive representation is long. There is an analogue of this phenomenon also for nonzero Z. In this case, following a similar argument using the supersymmetry algebra one can prove the BPS bound $M - |Z| \geqslant 0$ by showing that this operator is equal to a positive operator. When this bound is saturated then we have a BPS state with $M = |Z|$. In this case, half of supersymmetries are unbroken as in the case for massless representation. As a result, one gets only two fermionic oscillators and an eight-dimensional CPT conjugate representation. Indeed, the massless representation is a special case of a BPS representation with $M = Z = 0$. On the other hand, when the bound is not saturated, $M > |Z|$, then all supersymmetries are broken and one obtains a 16-dimensional representation. The $M = |Z|$ representation is often called the 'short' representation and the $M \geqslant |Z|$ representation 'long' because of the size of the representation.

With $\mathcal{N} = 4$ supersymmetry, which we will use later, there are altogether 16 real supercharges. For the non-BPS states, all are broken and there are effectively eight fermionic oscillators resulting in a 'long' representation that is 256 dimensional. The 'short' representation that preserves half the supersymmetries is on the other hand 16 dimensional.

The significance of BPS states in string theory and in gauge theory stems from the classic argument of Witten and Olive which shows that under suitable conditions, the spectrum of BPS states is stable under smooth changes of moduli and coupling constants. The crux of the argument is that with sufficient supersymmetry, for example $\mathcal{N} = 4$, the coupling constant does not get renormalized. The central charge Z of the supersymmetry algebra depends on the quantized charges and the coupling constant which therefore also does not get renormalized. This shows that for BPS states, the mass also cannot get renormalized because if the quantum corrections increase the mass, the states will have to belong a long representation. Then, the number of states will have to jump discontinuously from, say 16 to 256 which cannot happen under smooth variations of couplings unless there is a phase transition.

As a result, one can compute the spectrum at weak coupling in the region of moduli space where perturbative or semiclassical counting methods are available. One can then analytically continue this spectrum to strong coupling. This allows us to obtain invaluable nonperturbative information about the theory from essentially perturbative commutations.

4.4. Supersymmetric states and supergravity solutions

This was global, rigid supersymmetry. In supergravity we have *local* supersymmetry. The black holes are solutions of the supergravity equations of motion that are asymptotically flat. In asymptotically flat spacetime, one can consider supergravity gauge transformations (which include coordinate transformations as well as local supersymmetry transformations) with gauge parameters that do not vanish at infinity. These are rigid transformations. In asymptotically flat space we obtain a representation of this asymptotic symmetry. For example, a soliton can be moved around

$$x^\mu \to x^\mu + \varepsilon^\mu.$$

This is a coordinate transformation in a theory of gravity for a constant gauge parameter $\varepsilon^\mu(x) = \varepsilon^\mu$. Such global coordinate transformations generate the group of translations with generators $\mathrm{i}\partial/\partial x^\mu = P_\mu$. Similarly, the boosts and rotations of the asymptotic Poincaré algebra can be obtained by coordinate transformations with constant gauge parameters. States in asymptotic space form representations of this Poincaré algebra. Given a black hole solution we can boost it to consider a black hole moving with some momentum. In supergravity, in the same way, we look for representations of rigid supersymmetry algebra generated by supersymmetry transformations with gauge parameters that do not vanish at infinity. A BPS representation of this algebra then would preserve half the supersymmetries and we are thus led to look for solutions that preserve half (or more generally quarter, or some other fraction) of the supersymmetries.

In supergravity, to find a half-BPS solution ($M = |Z|$) we therefore try to find a solution that *preserves* half the supersymmetries with constant gauge parameters at infinity. Acting on bosonic fields, the supersymmetry transformations give fermionic fields. Since classically all fermionic fields have vanishing expectation value, these variations of bosonic fields vanish automatically. On the other hand, variations of the fermionic fields give bosonic fields and are not zero automatically. Setting these variations to zero gives a set of first-order equations often called the 'Killing spinor equations' which we have to solve to find the unbroken supersymmetries. The solutions are called 'Killing spinors' by analogy with Killing vectors which are the vector fields on which the fields do not depend. For example static solutions have $\partial/\partial t$ time translations as a Killing vector. Killing spinors generate supertranslations under which the fields are invariant.

Solving the Killing spinor equations is often easier because these are first-order equations compared to the original second-order equations of motion. The solutions of Killing spinor equations in supergravity corresponding to supersymmetric black hole solutions typically take the Reissner–Nordström form that we encountered earlier in isotropic coordinates,

$$\mathrm{d}s^2 = -\left(1 + \frac{|Z|}{r}\right)^2 + \left(1 + \frac{|Z|}{r}\right)^2 (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2), \tag{4.9}$$

where Z is the supersymmetry central charge. The black hole has mass $M = |Z|$ and entropy $S = \pi|Z|^2$ analogous to the extremal RN black hole. The near-horizon geometry is then $\mathrm{AdS}_2 \times S^2$ as before and the radius of both factors equals $|Z|$.

4.5. Perturbative BPS states

An instructive example of BPS of states is provided by an infinite tower of BPS states that exists in perturbative string theory [10, 11].

Consider, heterotic string theory compactified on $T^4 \times S^1 \times \tilde{S}^1$ to four dimensions. We denote spacetime coordinates by x^M, $M = 0, \ldots, 9$ and take (0123) to label the noncompact spacetime coordinates, (4) and (5) to label $\tilde{S}^1$ and S^1 coordinates respectively, and (6789) to label the T^4 coordinates.

Consider a perturbative string state wrapping around S^1 with winding number w and quantized momentum n. Let the radius of the circle be R and $\alpha' = 1$, then one can define left-moving and right-moving momenta as usual,

$$P_{L,R} = \sqrt{\frac{1}{2}}\left(\frac{n}{R} \pm wR\right). \tag{4.10}$$

The Virasoro constraints are then given by

$$\tilde{H} = -\frac{M^2}{4} + \tilde{N} + \frac{P_R^2}{2} = 0 \tag{4.11}$$

$$H = -\frac{M^2}{4} + N + \frac{P_L^2}{2} = 0, \tag{4.12}$$

where N and $\tilde{N}$ are the left-moving and right-moving oscillation numbers, respectively.

Recall that the heterotic strings consist of a right-moving superstring and a left-moving bosonic string. In the NSR formalism in the light-cone gauge, the worldsheet fields are

- right-moving superstring

$$X^i(\sigma^-)\tilde{\psi}^i(\sigma^-) \qquad i = 1, \ldots, 8,$$

- left-moving bosonic string

$$X^i(\sigma^+), X^I(\sigma^+) \qquad I = 1, \ldots, 16,$$

where X^i are the bosonic transverse spatial coordinates, $\tilde{\psi}^i$ are the worldsheet fermions and X^I are the coordinates of an internal $E_8 \times E_8$ torus.

The left-moving oscillator number is then

$$N = \sum_{n=1}^{\infty}\left(\sum_{i=1}^{8} n\alpha_{-n}^i\alpha_n^i + \sum_{I=1}^{16} n\beta_{-n}^I\beta_{-n}^I\right) - 1, \tag{4.13}$$

where α^i are the left-moving Fourier modes of the fields X^i and β^I are the Fourier modes of the fields X^I.

A BPS state is obtained by keeping the right movers in the ground state with no oscillations thus setting $\tilde{N} = 0$. From the Virasoro constraint (4.11) we see that the state saturates the BPS bound

$$M = \sqrt{2}p_R, \tag{4.14}$$

and thus $\sqrt{2}p_R$ can be identified with the central charge of the supersymmetry algebra. The right-moving ground state after the usual GSO projection is indeed 16 dimensional as expected for a BPS state in a theory with $\mathcal{N} = 4$ supersymmetry. To see this, note that the right-moving fermions satisfy anti-periodic boundary conditions in the NS sector and have half-integral moding, and satisfy periodic boundary conditions in the R sector and have integral moding. The oscillator number operator is then given by

$$\tilde{N} = \sum_{n=1}^{\infty}\sum_{i=1}^{8}\left(n\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r - \frac{1}{2}\right), \tag{4.15}$$

with $r \equiv -\left(n - \frac{1}{2}\right)$ in the NS sector and by

$$\tilde{N} = \sum_{n=1}^{\infty}\sum_{i=1}^{8}\left(n\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + r\tilde{\psi}^i_{-r}\tilde{\psi}^i_r\right), \tag{4.16}$$

with $r \equiv (n-1)$ in the R sector.

In the NS sector then one has $\tilde{N} = \frac{1}{2}$ and the states are given by

$$\tilde{\psi}^i_{-\frac{1}{2}}|0\rangle, \tag{4.17}$$

that transform as the vector representation $\mathbf{8_v}$ of $SO(8)$. In the R sector the ground state is furnished by the representation of fermionic zero mode algebra $\left\{\psi^i_0, \psi^j_0\right\} = \delta^{ij}$ which after GSO projection transforms as $\mathbf{8_s}$ of $SO(8)$. Altogether the right-moving ground state is thus 16 dimensional $\mathbf{8_v} \oplus \mathbf{8_s}$.

We thus have a perturbative BPS state which looks pointlike in four dimensions with two integral charges n and w that couple to two gauge fields $g_{5\mu}$ and $B_{5\mu}$, respectively. It saturates a BPS bound $M = \sqrt{2}p_R$ and belongs to a 16-dimensional short representation. This point-like state is our 'would be' black hole. Because it has a large mass, as we increase the string coupling it would begin to gravitate and eventually collapse to form a black hole.

Microscopically, there is a huge multiplicity of such states which arise from the fact that even though the right movers are in the ground state, the string can carry arbitrary left-moving oscillations subject to the Virasoro constraint. Using $M = \sqrt{2}p_R$ in the Virasoro constraint for the left movers gives us

$$N = \tfrac{1}{2}\left(p_R^2 - p_L^2\right) = nw. \tag{4.18}$$

We would like to know the degeneracy of states for a given value of charges n and w which is given by exciting arbitrary left-moving oscillations whose total worldsheet energy adds up to N. Let us denote the degeneracy by $\Omega(N)$ which we want to compute. As usual, it is more convenient to evaluate the canonical partition function and then read off the microcanonical degeneracy $\Omega(N)$ by inverse Laplace transform

$$Z(\beta) = \mathrm{Tr}\,\mathrm{e}^{-\beta N} \tag{4.19}$$

$$\equiv \sum \Omega(N) q^N, \tag{4.20}$$

where $q = \mathrm{e}^{-\beta}$. Using expression (4.13) for the oscillator number N and the fact that

$$\mathrm{Tr}(q^{-n\alpha_{-n}\alpha_n}) = 1 + q^n + q^{2n} + q^{3n} + \cdots = \frac{1}{(1-q^n)}, \tag{4.21}$$

the partition function can be readily evaluated to obtain

$$Z(\beta) = \frac{1}{q}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{24}}. \tag{4.22}$$

The degeneracy $\Omega(N)$ is given by the inverse Laplace transform

$$\Omega(N) = \frac{1}{2\pi \mathrm{i}}\int \mathrm{d}\beta\, \mathrm{e}^{\beta N} Z(\beta). \tag{4.23}$$

We would like to evaluate this integral for large N which corresponds to large worldsheet energy. We would therefore expect that the integral will receive most of its contributions from high temperature or small β region of the integrand. To compute the large N asymptotics, we then need to know the small β asymptotics of the partition function. Now, $\beta \to 0$ corresponds to $q \to 1$ and in this limit the asymptotics of $Z(\beta)$ are very difficult to read off from (4.22) because it is a product of many quantities that are becoming very large. It is more convenient to use the fact that $Z(\beta)$ is a modular form of weight 12 which means that

$$Z(\beta) = (\beta/2\pi)^{12} Z\left(\frac{4\pi^2}{\beta}\right). \tag{4.24}$$

This allows us to relate the $q \to 1$ or high temperature asymptotics to $q \to 0$ or low temperature asymptotics as follows. Now, $Z(\tilde{\beta}) = Z\left(\frac{4\pi^2}{\beta}\right)$ asymptotics are easy to read off because as $\beta \to 0$ we have $\tilde{\beta} \to \infty$ or $\mathrm{e}^{-\tilde{\beta}} = \tilde{q} \to 0$. As $\tilde{q} \to 0$

$$Z(\tilde{\beta}) = \frac{1}{\tilde{q}} \prod_{n=1}^{\infty} \frac{1}{(1-\tilde{q}^n)^{24}} \sim \frac{1}{\tilde{q}}. \tag{4.25}$$

This allows us to write

$$\Omega(N) \sim \frac{1}{2\pi\mathrm{i}} \int \left(\frac{\beta}{2\pi}\right)^{12} \mathrm{e}^{\beta N + \frac{4\pi^2}{\beta}}\, \mathrm{d}\beta. \tag{4.26}$$

This integral can be evaluated easily using saddle point approximation. The function in the exponent is $f(\beta) \equiv \beta N + \frac{4\pi^2}{\beta}$ which has a maximum at

$$f'(\beta) = 0 \quad \text{or} \quad N - \frac{4\pi^2}{\beta_c} = 0 \quad \text{or} \quad \beta_c = \frac{2\pi}{\sqrt{N}}. \tag{4.27}$$

The value of the integrand at the saddle point gives us the leading asymptotic expression for the number of states

$$\Omega(N) \sim \exp(4\pi\sqrt{N}). \tag{4.28}$$

This implies that the black holes corresponding to these states should have nonzero entropy that goes as

$$S \sim 4\pi\sqrt{nw}. \tag{4.29}$$

We would now like to identify the black hole solution corresponding to this state and test if this microscopic entropy agrees with the macroscopic entropy of the black hole.

4.6. Cardy's formula

Before turning to black hole geometry and the entropy, let us discuss one general point that has proved to be useful in the stringy explorations of black hole physics.

The formula that we derived for the degeneracy $\Omega(N)$ is valid more generally in any $1+1$ CFT. In general the partition function is a modular form of weight k

$$Z(\beta) \sim Z\left(\frac{4\pi^2}{\beta}\right)\beta^k,$$

which allows us to high temperature asymptotics to low temperature asymptotics for $Z(\tilde{\beta})$ because

$$\tilde{\beta} \equiv \frac{4\pi^2}{\beta} \to \infty \qquad \text{as} \quad \beta \to 0. \tag{4.30}$$

At low temperature only the ground state contributes

$$
\begin{aligned}
Z(\tilde{\beta}) &= \operatorname{Tr}\exp(-\tilde{\beta}(L_0 - c/24)) \\
&\sim \exp(-E_0\tilde{\beta}) \sim \exp\left(\frac{\tilde{\beta}c}{24}\right),
\end{aligned}
$$

where c is the central charge of the theory. Using the saddle point evaluation as above we then find

$$
\Omega(N) \sim \exp\left(2\pi\sqrt{\frac{cN}{6}}\right). \tag{4.31}
$$

In our case, because we had 24 left-moving bosons, $c = 24$, and then (4.31) reduces to (4.28).

This formula has been used extensively in black hole physics in string theory. Typically, one reduces the black hole counting problem to a counting problem in 1+1 CFT. For example, in the well-known Strominger–Vafa black hole in five dimensions, the microscopic configuration consists of Q_5 D5-branes wrapping $K3 \times S^1$, Q_1 D1-branes wrapping the S^1, with total momentum N along the circle. The bound states are described by an effective string wrapping the circle carrying left-moving momentum N. The central charge of the system can be computed at weak coupling and is given by $6Q_1Q_5$. Then applying Cardy's formula

$$
\Omega(N) = \exp\left(2\pi\sqrt{\frac{6Q_1Q_5N}{6}}\right), \tag{4.32}
$$

which implies a microscopic entropy $S = \log\Omega = 2\pi\sqrt{Q_1Q_5N}$. The corresponding BPS black hole solutions with three charges in five dimensions can be found in supergravity and the resulting entropy matches precisely with the macroscopic entropy.

We would now like to see if a similar comparison can be carried out for our two-charge states in four dimensions.

5. Black hole attractor geometry

Corresponding to our state with two charges, we have a point-like object that couples to two types of gauge fields with charges n, w. It also couples to the metric and two scalar moduli, the dilaton and the radius. An explicit solution in supergravity with these charges and coupling can be found easily. However, one finds that the resulting solution is not a black hole at all but is in fact mildly singular with vanishing area. The Bekenstein–Hawking entropy would thus be zero which seems to be in contradiction with the counting of states that we did earlier.

Of course, to correctly carry out the comparison, one must remember that near the singularity, where the curvature becomes large, stringy corrections will become important. For the heterotic state that we have considered, the dilaton remains very small near the horizon so one expects that stringy loop corrections can be ignored but the alpha prime corrections to the geometry must be taken into account.

At first sight, it looks like a hopeless task to find a corrected geometry including the corrections. That would involve first finding the effective action including higher derivative corrections and then solving these equations of motion to find the corrected near-horizon geometry. It seems highly impossible to solve the nonlinear partial differential equations with higher order derivative.

It turns out that though using supersymmetry and the formalism of special geometry in supergravity one can solve the equations and evaluate the entropy explicitly. What makes this possible is a combination of supersymmetry and the so-called attractor mechanism in the presence of black holes [12–19]. To understand the attractor phenomenon, note that the black hole couples to a metric and a given set of vector fields and moduli. Hence in the black hole geometry all these fields vary as a function of radial distance

$$g_{\mu\nu}(r), \quad A^I_\mu(r), \quad X^I(r),$$

where A^I_μ are the vector fields and X^I are the moduli. The index I runs over $0, 1, 2, \ldots, n_v$ if there are n_v vector multiplets of $\mathcal{N} = 2$ supergravity. The relevant aspects of supergravity will be explained in the following section.

Now, since X^I are moduli fields that parametrize the size and shape of the internal manifold of string compactification, at asymptotic infinity as $r \to \infty$, they can take any value X^I_0. The solution for the black hole geometry and therefore the entropy would then seem to depend on these arbitrary continuous parameters. How can it possibly agree with the well-defined microscopic $\log \Omega(Q)$ which depends only on integral charges? The answer to this puzzle is provided by the attractor phenomenon. It turns out that the horizon of a black hole is a very special place. No matter what value X^I_0 the moduli have at asymptotic infinity, at the horizon they get àttracted' to the value $X^I_*(Q)$ that depends only on the charges of the black holes. Furthermore, in $\mathcal{N} = 2$ supergravity, the attractor values of the moduli are determined by solving purely *algebraic* equations.

For example, in our example, the radius modulus can take an arbitrary value at infinity R that corresponds to the radius of the internal circle. But as we will see, near the horizon of the black hole corresponding to our state with two charges n and w, the modulus gets attracted to the value $\sqrt{\frac{n}{w}}$. Similarly, the string coupling constant or the dilaton gets attracted to $\sqrt{\frac{1}{nw}}$.

What is more, the entire near-horizon geometry is determined in terms of the attractor values. This is because, the near-horizon geometry is $\mathrm{AdS}_2 \times S^2$ and the radius of the two factors is determined by the value of the central charge of the supersymmetry algebra which in turn is determined completely in a given theory by the attractor values of the moduli.

This is an enormous simplification. The problem of solving the higher order nonlinear partial differential equations is reduced to solving algebraic equations using supersymmetry and the attractor phenomenon. When one includes the higher derivative corrections to the two-derivative supergravity action, there is an additional subtlety that needs to be taken into account.

The higher derivative corrections are expected to modify not only the equations of motion and the solution but also the black hole entropy formula itself. Fortunately, there is an elegant generalization of the Bekenstein–Hawking formula due to Wald that allows us to take these corrections into account in a systematic way [20, 21].

While some of these ideas such as the Wald entropy have a broader applicability, the supergravity formalism of special geometry is what makes it possible to carry out explicit computations[1]. So we now turn to supergravity and special geometry

5.1. Special geometry

Consider $\mathcal{N} = 2$ supergravity with n_v vector multiplets. The bosonic fields from the gravity multiplet are $\left(g_{\mu\nu}, A^0_\mu\right)$ and those from the vector multiplets are $\left(A^A_\mu, X^A\right)$, $A = 1, 2, \ldots, n_V$.

In string theory, $\mathcal{N} = 2$ supergravity arises naturally in the context of compactification of Type-II string on a Calabi–Yau 3-fold to four dimensions. Special geometry formalism then describes the vector multiplet moduli space and the supergravity couplings of this sectors. In Type-IIA strings, vector fields arise from 3-form RR fields in ten dimensions that have two

[1] For a nice review of the attractor phenomenon including the higher derivative corrections and the Wald entropy in the resulting attractor geometries see [22].

indices coming from harmonic (1, 1)-forms on CY3 and hence the vector multiplet moduli space corresponds to the moduli space of Kähler deformations. In Type-II strings vector fields arise from 4-form RR fields in ten dimensional that have three indices, two indices coming from harmonic (1, 2)-forms of the CY3 and hence the vector multiplet moduli space corresponds to the moduli space of complex structure deformations. Our discussion of black holes will be mostly in the context of Type-IIA compactifications.

To discuss black holes, we need to know the low energy effective action coupling the metric $g_{\mu\nu}$ with the vector fields $\{A^0_\mu, A^A_\mu\}$ and the complex scalar fields $\{X^A\}$. This action is summarized elegantly using special geometry. The name 'special geometry' refers to the fact that the moduli space parametrized by the fields X^A is a special Kähler manifold. The geometry is completely specified by a single-holomorphic function $\mathcal{F}$ called the prepotential. We review a few relevant facts below.

Special geometry is most conveniently described using complex projective coordinates called special coordinates. To motivate this choice of coordinates note that we can collect the vector fields into $\{A^I_\mu\}$ with $I = 0, 1, 2, \ldots, n_v$. A generic state will therefore carry $(n_v + 1)$ electric charges $\{q_I\}$ and $(n_v + 1)$ magnetic charges $\{p^I\}$. If there are two states with charges (q_I, p^I) and (q'_I, p'^I) then they satisfy the Dirac quantization condition $q_I p'^J - p_I q^{J\prime} = 2\pi n$ for some integer n. The Dirac quantization condition is left invariant by an $Sp(n_v + 1; \mathbf{Z})$ transformation. To see this note that the product of charges that appears in the Dirac quantization condition can be written as

$$(\mathbf{q} \quad \mathbf{p}) \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{q}' \\ \mathbf{p}' \end{pmatrix}, \tag{5.1}$$

where $\mathbf{q}$ etc are $(n_v + 1)$-dimensional vectors $\mathbf{1}$ is a $(n_v + 1) \times (n_v + 1)$ block-diagonal matrix. The quantization condition is left invariant by transformation g with integer entries such that

$$g \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} g^t, \tag{5.2}$$

which is the definition of the symplectic group with integer entries. Now, since the symplectic group acts linearly on the charges it acts linearly on the vector fields but not on X^A. This is because while there are $(n_v + 1)$ vector fields, there are only n_V complex scalar fields. It is convenient to introduce an additional 'fake' coordinate X^0 and write

$$X^I = (X^0, X^A).$$

This additional unwanted degree of freedom can be removed by demanding that the vector (X^I) is defined only up to projective scaling

$$X^I \sim \lambda(X^I),$$

so that when X^0 is nonzero, one can simply choose $\lambda = (X^0)^{-1}$ for rescaling to get $X^I = (1, X^A/X^0)$. In this formalism then X^A/X^0 are the physical moduli fields which are the Kähler deformation of the Calabi–Yau manifold if we are discussing Type-IIA string theory. The advantage of using these projective coordinates is that the vector (X^I, F_I) transforms linearly under $Sp(n_v + 1; \mathbf{Z})$ exactly as (p^I, q_I), where F_I are given in terms of a holomorphic prepotential $\mathcal{F}$ as

$$F_I = \frac{\partial \mathcal{F}}{\partial X_I}.$$

The prepotential is a function of the projective coordinates that is homogenous of degree two, $\mathcal{F}(\lambda X^I) = \lambda^2 \mathcal{F}(X^I)$.

The formalism of special geometry is quite powerful because specifying the single-holomorphic function $\mathcal{F}$, the couplings of the vectors, scalars and metric are completely specified. What is more interesting is that this is not limited to only a two-derivative action but even higher derivative interactions of a certain type (those that can be written as chiral integrals on superspace) can be obtained by a slight generalization of the prepotential by letting it depend on an additional auxiliary field $\hat{A}$ of degree two under the rescaling. Thus, the couplings of all these massless fields with higher derivatives are completely specified by specifying the holomorphic prepotential $\mathcal{F}(X^I, \hat{A})$.

For our purposes, the relevant quantities are given in terms of the prepotential as follows. The Kähler potential which governs the kinetic terms of the scalars is given by

$$\mathrm{e}^{-K} = \mathrm{i}(\bar{X}^I F_I - \bar{F}_I X^I), \tag{5.3}$$

the central charge of the supersymmetry algebra is given by

$$Z = \mathrm{e}^{K/2}(p^I F_I - q_I X^I). \tag{5.4}$$

For a Type-IIA compactification on a given Calabi–Yau manifold $\mathcal{M}$, the prepotential is completely specified by topological data of the Calabi–Yau manifold. Let $\{\omega_A\}$ be a properly normalized basis of $(1, 1)$ harmonic forms and let $\{\Sigma_A\}$ be the basis of Poincaré dual 4-cycles. (Recall that the 4-cycle Σ_A Poincaré dual to ω_A is defined so that for any given 4-form β, $\int_{\Sigma_A} \beta = \int_{\mathcal{M}} \beta \wedge \omega_A$. Now, the prepotential is given by entirely in terms of the intersection numbers of the 4-cycles, $C_{ABC} \equiv \int_{\mathcal{M}} \omega_A \omega_B \omega_C$, and the second Chern numbers c_{2A} of the 4-cycles which are the integrals of the second Chern class on the 4-cycles. The prepotential then takes the form

$$F = \frac{1}{6}\frac{D_{ABC}X^A X^B X^C}{X^0} + d_A \frac{X^A}{X^0}\hat{A}, \tag{5.5}$$

where

$$D_{ABC} = -\frac{1}{6}C_{ABC}, \qquad d_A = -\frac{1}{24}\frac{1}{64}c_{2A}.$$

The first term in the prepotential is the classical supergravity piece that determines the two derivative terms in the action and the $\hat{A}$ term encapsulates higher curvature terms of the type $\int \frac{X^A}{X^0} R^2$, where R^2 is an appropriately contracted term quadratic in the Riemann tensor. There are further instanton corrections to the prepotential which will not be important for our purpose.

5.2. Attractor equations

Having specified the prepotential, we have specified the supergravity action including higher derivative corrections. One would now like to find the BPS black hole solutions of this action. What is of more direct interest to us is the near-horizon geometry of this black hole so that we can read off the entropy of the black hole.

Fortunately, as we mentioned earlier, the geometry near the horizon is determined completely in terms of the values of the moduli fields at the horizon. These in turn get attracted to some special values that depend only on the charge configuration (p^I, q_I) of the black hole that we have chosen independent of their asymptotic values far away from the black hole. Furthermore, attractor values of the moduli are determined completely by solving the following *algebraic* equations. To write down these equation it is convenient to define the rescaled variables $Y^I = \mathrm{e}^{\frac{K}{2}}\bar{Z}X^I$ and $\Upsilon = \mathrm{e}^K \bar{Z}^2$. The scaling is chosen so that under the projective scalings $X^I \to \lambda X^I$, $\hat{A} \to \lambda \hat{A}$ the new variables are invariant. These new variables thus are no longer projective. The attractor equations then take the form

$$Y^I - \bar{Y}^I = \mathrm{i}p^I \tag{5.6}$$

$$F_I - \bar{F}_I = \mathrm{i}q_I \qquad \Upsilon = -64. \tag{5.7}$$

Furthermore, the metric is determined completely because the near-horizon geometry is $\mathrm{AdS}_2 \times S^2$ with the radius of curvature of both factors given by $|Z|$ as in (4.9). On the attractor solution, $|Z|$ is determined in terms of the charges by

$$Z\bar{Z} = p^I F_I(Y, \Upsilon) - q_I Y^I. \tag{5.8}$$

Given the geometry, one can readily evaluate the Bekenstein–Hawking entropy which would be given by $S = \pi|Z|^2$. This however is not the complete story. Our action now has higher derivative interactions in addition to the Einstein–Hilbert term. The Bekenstein–Hawking entropy on the other hand was derived from the first law of thermodynamics that followed from the Einstein–Hilbert action. To do everything self-consistently, we must also include the corrections to the black hole entropy formula when there are additional higher derivative interactions present in the action. This is provided by Wald's formula for the black hole entropy.

5.3. Bekenstein–Hawking–Wald entropy

In our discussion of Bekenstein–Hawking entropy of a black hole, the Hawking temperature could be deduced from surface gravity or alternatively the periodicity of the Euclidean time in the black hole solution. These are geometric asymptotic properties of the black hole solution. However, to find the entropy we needed to use the first law of black hole mechanics which was derived in the context of Einstein–Hilbert action,

$$\frac{1}{16\pi}\int R\sqrt{g}\,\mathrm{d}^4x.$$

Generically, in string theory, we expect corrections (both in α' and g_s) to the effective action that has higher derivative terms involving Riemann tensor and other fields:

$$I = \frac{1}{16\pi}\int (R + R^2 + R^4F^4 + \cdots).$$

How do the laws of black hole thermodynamics get modified?

Wald derived the first law of thermodynamics in the presence of higher derivative terms in the action. This generalization implies an elegant formal expression for the entropy S given a general action I including higher derivatives

$$S = 2\pi\int_{\rho^2} \frac{\delta I}{\delta R_{\mu\gamma\alpha\beta}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\sqrt{h}\,\mathrm{d}^2\Omega,$$

where $\epsilon^{\mu\nu}$ is the binormal to the horizon, h is the induced metric on the horizon and the variation of the action with respect to $R_{\mu\nu\alpha\beta}$ is to be carried out regarding the Riemann tensor as formally independent of the metric $g_{\mu\nu}$.

As an example, let us consider the Schwarzschild solution of the Einstein–Hilbert action. In this case, the event horizon is S^2 which has two normal directions along r and t. We can construct an antisymmetric 2-tensor $\epsilon_{\mu\nu}$ along these directions so that $\epsilon_{rt} = \epsilon tr = -1$:

$$\mathcal{L} = \frac{1}{16\pi}R_{\mu\gamma\alpha\beta}g^{\nu\alpha}g^{\mu\beta}, \qquad \frac{\partial\mathcal{L}}{\partial R_{\mu\gamma\alpha\beta}} = \frac{1}{16\pi}\frac{1}{2}(g^{\mu\alpha}g^{\gamma\beta} - g^{\nu\alpha}g^{\mu\beta}).$$

Then the Wald entropy is given by

$$\begin{aligned} S &= \frac{1}{8}\int \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})(\epsilon_{\mu\nu}\epsilon_{\alpha\beta})\sqrt{h}\,\mathrm{d}^2\sigma \\ &= \frac{1}{8}\int g^{tt}g^{rr}\cdot 2 = \frac{1}{4}\int_{S^2}\sqrt{h}\,\mathrm{d}^2\sigma = \frac{A_H}{4}, \end{aligned}$$

giving us the Bekenstein–Hawking formula as expected.

In our case, our action has many higher derivative terms in the effective action that follows from our generalized prepotential $\mathcal{F}(X^I, \hat{A})$. Now that we have the generalized entropy formula, we can apply it to black hole solutions of this effective action. The Wald entropy in this case takes a simple form in terms of the prepotential,

$$S = \pi[|Z|^2 - 256\,\mathrm{Im}[\mathcal{F}_{\hat{A}}(X^I, \hat{A})]],$$

where $\mathcal{F}_{\hat{A}} = \partial_{\hat{A}}\mathcal{F}$. Equipped with formula for entropy we can now apply it to our two-charge black hole by solving the attractor equations for the specific compactification that we have chosen.

5.4. Solution of attractor equations

For a general charge configuration, we would like to solve the attractor equations and then evaluate the entropy. We restrict our attention to states with $p^0 = 0$.

5.5. Large black holes

Let us first solve the attractor equations for 'large' black holes that have large area of the event horizon in the supergravity approximation. Later, to make contact with our two-charge states, we consider 'small' black holes that have vanishing area in the supergravity approximation but which develop nonzero area after including the higher derivative corrections.

Given a prepotential of the form (5.5) and a state with charges (p^I, q_I), we can define the following quantities:

$$\begin{aligned} &D = D_{ABC}p^Ap^Bp^C, \qquad D_{AB} = D_{ABC}p^C, \\ &\hat{q}_0 = q_0 + \tfrac{1}{12}D^{AB}q_Aq_B D^{AB}D_{BC}, \qquad \text{where} \quad D^{AB}D_{BC} = \delta^A_C. \end{aligned}$$

Then the first set of attractor equations (5.7) is solved by

$$Y^0 - \bar{Y}^0 = 0 \quad \Rightarrow \quad Y^0 = \bar{Y}^0 \tag{5.9}$$

$$Y^A - \bar{Y}^A = p^A \quad \Rightarrow \quad Y^A = w^A + \frac{\mathrm{i}p^A}{2}, \tag{5.10}$$

for some real w^A. Now the second set of attractor equations (5.7) result in $(n_v + 1)$ real equations for $(n_v + 1)$ real unknowns w^A and Y^0 which in our case can be explicitly solved as follows.

Using the equations

$$F_A - \bar{F}_{\bar{A}} = \frac{6\mathrm{i}D_{AB}w^B}{Y^0} = \mathrm{i}q_A, \tag{5.11}$$

we can express all w^A in terms of Y^0 and the charges. Then using the equation

$$F_0 - \bar{F}_0 = \frac{D_{ABC}p^Ap^Bp^C}{4(Y^0)^2} - \frac{d_Ap^A\gamma}{(Y^0)^2} - 3D_{ABC}W^AW^BY^0p^C = q_0, \tag{5.12}$$

we obtain

$$(Y^0)^2 = \frac{D - 4d_A p^A \Upsilon}{4\hat{q}_0}. \tag{5.13}$$

The final solution then takes the form

$$Y^0 = \bar{Y}^0, \qquad (Y^0)^2 = \frac{D - 4d_A p^A \gamma}{4\hat{q}_0} \tag{5.14}$$

$$Y^A = \frac{1}{6} Y^0 D^{AB} q_B + \frac{\mathrm{i}}{2} p^A. \tag{5.15}$$

The area of the horizon and the Wald entropy are given by

$$A_H = 4\pi |Z|^2 = 16\pi Y^0 |\hat{q}_0| - \frac{8\pi d_A p^A \Upsilon}{Y^o} \tag{5.16}$$

$$S = 4\pi Y^0 |\hat{q}_0|. \tag{5.17}$$

5.6. Small black holes

We would now like to compare the microscopic entropy that we derived in by counting perturbative BPS states in heterotic string theory compactified on $T^4 \times S^1 \times \tilde{S}^1$. As we have already noted, the solution corresponding to this state is singular in leading two derivative supergravity action. We can now take the higher derivative corrections into account using the special geometry formalism described above by keeping the $\hat{A}$ term in the prepotential. Our state counting was done in the heterotic string but the formalism of special geometry for finding the macroscopic entropy is phrased more naturally in the Type-II language for Calabi–Yau compactifications. To make contact between the two, we use the Heterotic-Type-IIA duality which relates heterotic on $T^4 \times S^1 \times \tilde{S}^1$ to Type-II on $K3 \times \times S^1 \times \tilde{S}^1$.

For IIA compactified on $K3 \times T^2$ results in various electric and magnetic charges which we denote as $(q_0, q_1, \ldots, q_{n_v})$ and $(p^0, p^1, \ldots, p^{n_v})$. These arise from wrapped D-branes as follows:

$$\begin{array}{ll} q_0 \equiv \text{D0-brane} & p^0 = \text{D6 brane}/K3 \times T^2 \\ q_1 \equiv \text{D2 on } T^2 & p^1 = \text{D4 on K3} \\ q_a \equiv \text{D2 on } \Sigma_a & p^a = \text{D4 on } \tilde{\Sigma}_a \times T^2, \end{array}$$

where $a = 2, 2, \ldots, 23$ labels the 22 2-cycles of K3. There are additional states from winding and momenta along the T^2 and their magnetic duals but those will not play an important role here. In the notation of the previous subsection we then have 23 vector multiplets and the index A runs over $(1, a)$ from $1, 2, \ldots, 23$.

Under duality map between heterotic and Type-IIA, a fundamental string of heterotic wrapping on S^1 is dual to the NS5 brane of IIA wrapping on $K3 \times S^1$. Starting with our winding momentum, F1-P configuration, we can then follow a chain of dualities to turn it into a collection of D-branes in Type-II description as follows:

$$\text{F1-P} \quad \to \quad \text{NS5-P} \quad \xrightarrow{T_5} \quad \text{NS5-F1} \quad \xrightarrow{S} \quad \text{D5-D1} \quad \xrightarrow{T_5} \quad \text{D4-D0},$$

where T_5 is T-duality along the S^1 coordinate that the string is wrapping on and S is the S-duality of Type-IIB.

We therefore have a charge configuration with $q_0 = n$ and $p^1 = w$ and all other charges zero. Since p^1 comes from a D4-brane wrapping, $K3$, the relevant second Chern number that appears in the prepotential is the second Chern number of $K3$ which equals 24. We are now ready to apply the entropy formula (5.17) which gives

$$S = 4\pi Y^0|\hat{q}_0| = 4\pi\sqrt{\frac{c_{2A}p^A|\hat{q}_0|}{24}} = 4\pi\sqrt{p^1 q_0} = 4\pi\sqrt{nw}, \tag{5.18}$$

in remarkable agreement with the microscopic entropy (4.29) computed by completely different means including the precise numerical coefficient [23–26].

6. Concluding remarks

We have seen that there is striking agreement between the macroscopic entropy of the black hole and the microscopic counting of states in string theory. This provides a nontrivial test of the consistency of string theory as a theory of quantum gravity.

Note that for our two-charge black holes, the stringy quantum corrections to the effective action were essential for this agreement. The classical limit of the attractor equations can be obtained by simply setting $\hat{A}$ to zero in the prepotential (5.5), because then the action reduces to the two-derivative supergravity action. In this limit we find that the attractor equations are singular for the two-charge states because for these states since $D \equiv D_{ABC}p^A p^B p^C = 0$. Then from (5.13) we see

$$(Y^0)^2_{\text{classical}} = \frac{D}{4\hat{q}_0} = 0.$$

Inclusion of quantum corrections to the prepotential is thus essential to get sensible answers from the attractor equations. After including the quantum corrections one can then consistently solve the attractor equations to find an entropy which we found to be in precise agreement with the microscopic counting.

In fact, one can do better. Under reasonable assumptions about an appropriate statistical ensemble [27], one can compute even the subleading corrections to the entropy. These corrections are then in precise agreement with the microscopic counting to all orders in inverse powers of $\sqrt{nw}$ [24, 28–30].

The entropy of black holes thus continues to be a rich source of important tests of string theory and offers us invaluable glimpses of the quantum structure of gravity.

References

[1] Bekenstein J D 1973 Black holes and entropy *Phys. Rev.* D **7** 2333–46

[2] Hawking S W 1975 Particle creation by black holes *Commun. Math. Phys.* **43** 199–220

[3] Strominger A and Vafa C 1996 Microscopic origin of the Bekenstein–Hawking entropy *Phys. Lett.* B **379** 99–104 (*Preprint* hep-th/9601029)

[4] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation (San Francisco)* p 1279

[5] Wald R M 1984 *General Relativity* (Chicago: Chicago University Press) p 491

[6] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory vol 1: Introduction* (*Cambridge Monographs On Mathematical Physics*) (Cambridge: Cambridge University Press) p 469

[7] Green M B, Schwarz J H and Witten E 1987 *Superstring Theory. vol 2: Loop Amplitudes, Anomalies and Phenomenology* (*Cambridge Monographs On Mathematical Physics*) (Cambridge: Cambridge University Press) p 596

[8] Polchinski J 1998 *String Theory* vol 1 (Cambridge: Cambridge University Press)

[9] Polchinski J 1998 *String Theory. vol 2: Superstring Theory and Beyond* (Cambridge: Cambridge University Press) p 531

[10] Dabholkar A and Harvey J A 1989 Nonrenormalization of the superstring tension *Phys. Rev. Lett.* **63** 478

[11] Dabholkar A, Gibbons G W, Harvey J A and Ruiz Ruiz F 1990 Superstrings and solitons *Nucl. Phys.* B **340** 33–55

[12] Ferrara S, Kallosh R and Strominger A 1995 $N = 2$ extremal black holes *Phys. Rev.* D **52** 5412–16 (*Preprint* hep-th/9508072)

[13] Ferrara S and Kallosh R 1996 Supersymmetry and attractors *Phys. Rev.* D **54** 1514–24 (*Preprint* hep-th/9602136)

[14] Strominger A 1996 Macroscopic entropy of $n = 2$ extremal black holes *Phys. Lett.* B **383** 39–43 (*Preprint* hep-th/9602111)

[15] Lopes Cardoso G, de Wit B and Mohaupt T 1999 Corrections to macroscopic supersymmetric black-hole entropy *Phys. Lett.* B **451** 309–16 (*Preprint* hep-th/9812082)

[16] Lopes Cardoso G, de Wit B and Mohaupt T 2000 Deviations from the area law for supersymmetric black holes *Fortsch. Phys.* **48** 49–64 (*Preprint* hep-th/9904005)

[17] Lopes Cardoso G, de Wit B and Mohaupt T 2000 Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes *Nucl. Phys.* B **567** 87–110 (*Preprint* hep-th/9906094)

[18] Lopes Cardoso G, de Wit B and Mohaupt T 2000 Area law corrections from state counting and supergravity *Class. Quantum Grav.* **17** 1007–15 (*Preprint* hep-th/9910179)

[19] Lopes Cardoso G, de Wit B, Kappeli J and Mohaupt T 2000 Stationary bps solutions in $n = 2$ supergravity with r**2 interactions *J. High Energy Phys.* JHEP12(2000)019 (*Preprint* hep-th/0009234)

[20] Wald R M 1993 Black hole entropy in the noether charge *Phys. Rev.* D **48** 3427–31 (*Preprint* gr-qc/9307038)

[21] Iyer V and Wald R M 1994 Some properties of noether charge and a proposal for dynamical black hole entropy *Phys. Rev.* D **50** 846–64 (*Preprint* gr-qc/9403028)

[22] Mohaupt T 2001 Black hole entropy, special geometry and strings *Fortsch. Phys.* **49** 3–161 (*Preprint* hep-th/0007195)

[23] Dabholkar A 2005 Exact counting of black hole microstates *Phys. Rev. Lett.* **94** 241301 (*Preprint* hep-th/0409148)

[24] Dabholkar A, Kallosh R and Maloney A 2004 A stringy cloak for a classical singularity *J. High Energy Phys.* JHEP12(2004)059 (*Preprint* hep-th/0410076)

[25] Sen A 1995 Extremal black holes and elementary string states *Mod. Phys. Lett.* A **10** 2081–94 (*Preprint* hep-th/9504147)

[26] Sen A 2005 How does a fundamental string stretch its horizon? *J. High Energy Phys.* JHEP05(2005)059 (*Preprint* hep-th/0411255)

[27] Ooguri H, Strominger A and Vafa C 2004 Black hole attractors and the topological string *Preprint* hep-th/0405146

[28] Dabholkar A, Denef F, Moore G W and Pioline B 2005 Exact and asymptotic degeneracies of small black holes *J. High Energy Phys.* JHEP08(2005)021 (*Preprint* hep-th/0502157)

[29] Dabholkar A, Denef F, Moore G W and Pioline B 2005 Precision counting of small black holes *Preprint* hep-th/0507014

[30] Sen A 2005 Black holes and the spectrum of half-bps states in $n = 4$ supersymmetric string theory *Preprint* hep-th/0504005