

NONRENORMALIZATION OF MASS
OF SOME NONSUPERSYMMETRIC STRING STATES

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ABSTRACT

It is argued that the quantum correction to the mass of some very massive, nonsupersymmetric states vanishes in inverse proportion to their tree-level mass to all orders in string loops. This approximate nonrenormalization can explain the agreement between the perturbative degeneracy of these states and the Sen entropy of the associated black holes.

1. Introduction

Supersymmetric states, or ‘super states’ for short, are the states that belong to short representations of the supersymmetry algebra [1]. The main significance of super states stems from the fact that the semiclassical spectrum of these states is often not renormalized as a consequence of supersymmetry. A super state preserves some of the supersymmetries, and it follows from the supersymmetry algebra that it is extremal, *i.e.* its mass M equals the absolute value of some charge Z . The renormalization of its mass is therefore related to the renormalization of the charge. With enough supersymmetries, the charge is sometimes not renormalized, which then implies that the mass is also not renormalized. Therefore, the states that are tagged by a particular mass and charges at weak coupling continue to have the same mass and charges even at strong coupling. Moreover, charge conservation together with energetic considerations can usually ensure the stability of these states. One can thus learn much about the strongly coupled regime of a theory from the spectrum of super states computed at weak coupling.

It is interesting to know if there are any nonsupersymmetric states for which such a nonrenormalization is possible. Such states, if they exist, can provide important additional information about the strongly coupled dynamics that does not rely entirely on supersymmetry. In this paper we investigate this question for a set of electrically charged, non supersymmetric states in toroidally compactified heterotic string theory. In this theory, which has $N = 4$ supersymmetry in four dimensions, the coupling constant, and consequently the charges of these states are not renormalized. However, the nonsuper states belong to long representations of the superalgebra, so there is no exact relation any more between their mass and their charges. Therefore, *a priori*, one expects nonzero and typically large mass renormalization. These states are not expected to be stable either. In fact, for a very massive state, there is a large phase space because the state can decay into a large number of lower mass states, and the decay rate is expected to be large. For the special states considered in this paper, we find, however, that for a large enough tree-level mass, the mass-shift and the decay rate are both vanishingly small to all orders in string loops.

To describe these states, let us consider, for definiteness, heterotic string theory compactified on a six torus. The states are specified by the mass M , the charge vector (q, \bar{q}) that lives on an even, self-dual Lorentzian lattice $\Gamma_{6,22}$, and the right-moving and the left-moving oscillator numbers N and \bar{N} respectively. For a general state, in the NSR formalism, the Virasoro constraints can be written as

$$M^2 = \bar{q}^2 + 2(\bar{N} - 1) = q^2 + 2(N - \frac{1}{2}). \quad (1.1)$$

Here and in what follows, the quantities with the bar, like \bar{q} , are left-moving, and the corresponding unbarred quantities are right-moving; we have also chosen $\alpha' = 2$. There are two special cases that are of interest:

- $N = \frac{1}{2}$, \bar{N} arbitrary. These states are supersymmetric. Their mass and charge are not renormalized [2], so the tree-level extremality relation, $M = |q|$, is exact.
- $\bar{N} = 1$, N arbitrary. These states are nonsupersymmetric, but classically extremal, *i.e.*, $M = |\bar{q}|$ at tree level. Now, there is no supersymmetric nonrenormalization theorem for the mass of these states. So, in general, the renormalized mass is not expected to satisfy the tree-level extremality.

In this paper we shall be interested in the second set of states. One of our main motivation for considering these states is their connection with black hole entropy. Let us briefly explain why we choose these particular nonsuper states and why we expect that their mass is not renormalized. Consider a set of states of given mass, charges, and degeneracy at weak coupling. As one increases the strength of the coupling, the state eventually undergoes a gravitational collapse to form a black hole. Now, if for some reason the semiclassical spectrum of these states is not renormalized, then the degeneracy of states is not expected to change as we vary the coupling. Therefore, the logarithm of the number of perturbative states must agree with the Bekenstein-Hawking entropy which counts the corresponding black-hole states. For the supersymmetric states this is certainly true. As shown by Sen [3], the entropy given by the area of the stretched horizon of four dimensional, supersymmetric, electrically charged, black holes does agree, up to a proportionality constant, with the statistical entropy given by the logarithm of the perturbative degeneracy of the super states. This works in other dimensions as well [4]. For

the super states, this agreement is a consequence of supersymmetric nonrenormalization of the spectrum.

What is more surprising is that a similar agreement holds [5,6] also for the set of nonsupersymmetric but extremal states considered above. Now, supersymmetry alone does not protect the spectrum from getting renormalized. On the other hand, the agreement between the perturbative degeneracy of states and the black hole entropy would not be possible if the quantum corrections to the tree-level mass were large. We are thus led to the expectation that perhaps the spectrum of these nonsuper states does not get renormalized for reasons other than supersymmetry. Partial support for this expectation can be found by looking at the classical self-energy of these states. It was argued in [7,8] that the classical renormalization computed from the self-energy of the fields around such a state vanishes. The perturbative quantum calculation that we describe in this paper is in a complementary regime. The perturbative calculation is performed in the flat space background; it is valid in the regime when the coupling constant is sufficiently small so that the gravitational field around the state considered is small. From this point of view the calculation of the classical field energies is nonperturbative because it takes into account the classical field condensate formed around the state. Taken together, these two results in the two regimes do seem to support the expectation that the mass is not renormalized, although we cannot rule out some other nonperturbative corrections.

Even though one would like to prove the nonrenormalization for all states with $\bar{N} = 1$ but arbitrary N , we are able to prove it only in the restricted regime $1 \ll N \ll M^2$. More precisely, the restriction on N is that under $M \rightarrow \lambda M$, $\lambda \rightarrow \infty$, we have $N \rightarrow N$. So the states that we consider in this paper satisfy (1.1) together with

$$\bar{N} = 1, \quad 1 \ll N \ll M^2. \quad (1.2)$$

It may appear that these states are ‘nearly supersymmetric,’ and therefore the mass renormalization should be small anyway. However, it should be remembered that a nonsuper state belongs to a long multiplet, whereas a super state belongs to a short multiplet. So, there is no smooth limit that takes one to the other. All that is expected from supersymmetry is that when $N = \frac{1}{2}$ the mass renormalization should vanish. So, the mass-shift is

expected to be proportional to $N - \frac{1}{2}$, but, it can depend, in general, on any function of the mass M . We find a very specific dependence that the quantum correction to the mass is bounded by a quantity inversely proportional to the tree-level mass. Thus, the mass renormalization is vanishingly small even for the state that is in a long multiplet with N much larger than $\frac{1}{2}$, as long as the tree-level mass is sufficiently large.

This paper is organized as follows. In section 2, we give the form of a nonsupersymmetric, highly massive vertex operator. In section 3 we review the formal definition of the two point function in an arbitrary genus Riemann surface, mainly to set up the notations used in the following sections. In section 4, we concentrate on the one loop two-point function in detail. As is well-known, formal string amplitudes are typically divergent for on-shell external momenta and are well-defined only after proper analytic continuation. We describe the analytic continuation required for the present two-point amplitude in detail. Using the well-defined amplitude obtained this way we study its mass (M) dependence and show that it is bounded above by a finite M independent quantity. In section 5, we generalize the one loop result to higher loops. Finally, in section 6 we present the conclusions.

2. Massive Vertex Operator

In order to calculate the mass renormalization of the states with $N = n + 1/2$, $\bar{N} = 1$, we first need to find their vertex operators. For simplicity we set all Wilson lines to zero and choose the right-moving and the left-moving charges q^a , $a = 4, \dots, 9$ and $\bar{q}^{\bar{a}}$, $\bar{a} = 4, \dots, 9$ respectively to take values in a lattice $\Gamma_{6,6}$. We write the right-moving momentum as $k = (p, q)$ and the left-moving momentum as $\bar{k} = (p, \bar{q})$ where p^α , $\alpha = 0, \dots, 3$ is the momentum in the noncompact four dimensions. In the light cone gauge there are several possible states. For example, for spacetime bosons we can have

$$\bar{\alpha}_{-1}^j \psi_{-\frac{1}{2}}^{i_1} \dots \psi_{-\frac{1}{2}}^{i_{2n+1}} |k, \bar{k}\rangle, \quad \bar{\alpha}_{-1}^j \alpha_{-1}^{i_1} \psi_{-\frac{1}{2}}^{i_2} \dots \psi_{-\frac{1}{2}}^{i_{2n}} |k, \bar{k}\rangle, \quad \bar{\alpha}_{-1}^j \alpha_{-1}^{i_1} \dots \alpha_{-1}^{i_n} \psi_{-\frac{1}{2}}^{i_{n+1}} |k, \bar{k}\rangle, \quad (2.1)$$

and many other possibilities. To illustrate the main features of our nonrenormalization theorem, we shall describe the calculation only for the states $\bar{\alpha}_{-1}^j \alpha_{-1}^{i_1} \dots \alpha_{-1}^{i_n} \psi_{-\frac{1}{2}}^{i_{n+1}} |k, \bar{k}\rangle$ with totally symmetric polarization for the right-movers. We have also checked it for

many other states, but not for all possible states with $N = n + \frac{1}{2}$. However, the main considerations are sufficiently generic and we believe that the results are valid for all states subject to the constraints (1.1) and (1.2).

In covariant RNS formalism, in ghost number zero picture, the vertex operator for this specific state is given by

$$V(k, \zeta, \nu; \bar{k}, \bar{\zeta}, \bar{\nu}) = \frac{1}{\sqrt{(-1)^{n+1} n!}} \bar{\zeta}_A (\bar{\partial} \bar{X}^A) \exp(i\bar{k} \cdot \bar{X}) \times \quad (2.2)$$

$$\times \zeta_{A_1 A_2 \dots A_{n+1}} D^2 X^{A_1} D^2 X^{A_2} \dots D^2 X^{A_n} D X^{A_{n+1}} \exp(ik \cdot X)$$

where $D = \partial_\theta + \theta \partial_\nu$ and $X(\nu, \theta) = X(\nu) + \theta \psi(\nu)$. This is a physical state with $M = |\bar{q}|$ provided $\zeta_{A_1 A_2 \dots A_{n+1}}$ is totally symmetric, and

$$\bar{\zeta} \cdot \bar{k} = 0, \quad \zeta_{A_1 A_2 \dots A_i \dots A_j \dots A_{n+1}} \delta_{A_i}^{A_j} = 0, \quad \zeta_{A_1 A_2 \dots A_i \dots A_{n+1}} k^{A_i} = 0. \quad (2.3)$$

3. The Two Point Function

The mass-shift and the decay rate of a perturbative string state can be extracted from the real and the imaginary part of the two point function of the corresponding vertex operator. The two point function at order g is given by

$$A(\zeta_i, k_i, \bar{k}_i) = \sum_{(\alpha, \beta)} \int d\mathbf{M} d^2 \nu_1 d^2 \nu_2 \cdot Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau) \cdot \bar{Z}(\bar{\tau}) \quad (3.1)$$

$$\times \langle V(k_1, \zeta_1, \nu_1; \bar{k}_1, \bar{\zeta}_1, \bar{\nu}_1) V(k_2, \zeta_2, \nu_2; \bar{k}_2, \bar{\zeta}_2, \bar{\nu}_2) \rangle,$$

where $d\mathbf{M}$ is the measure over moduli space \mathcal{M}_g of the genus g surface. The details of the measure will not be important for our discussion but can be found in [9,10]. $Z \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (\tau)$ and $\bar{Z}(\bar{\tau})$ are the partition functions in the right and the left sector respectively with $(3g - 3)$ insertions of the b ghosts as well as $(2g - 2)$ insertions of picture-changing operators to soak up all the ghost zero modes for $g \geq 2$ [9,10]. For genus one, the volume of the conformal killing vectors is factored out. We have summed over the spin structures (α, β) . $\langle VV \rangle$ denotes the correlation function of the vertex operators for a given point in the moduli space obtained by integrating over only the matter fields.

The various correlators on a Riemann surface of genus g can be readily evaluated. The fermion propagator is given by the Szego kernel

$$\langle \psi^A(\nu_1) \psi^B(\nu_2) \rangle_{(\alpha, \beta)} = \frac{\eta^{AB}}{E(\nu_1, \nu_2 | \tau)} \frac{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z | \tau)}{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (0 | \tau)}, \quad (3.2)$$

where τ^{IJ} is the period matrix, $E(\nu_1, \nu_2 | \tau)$ is the prime form, and $\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (z | \tau)$ is the usual ϑ function with characteristics (α, β) [11]. The vector z is defined by $z^I \equiv \int_{\nu_1}^{\nu_2} \omega^I$ where $\omega^I(\nu)$, $I = 1, \dots, g$ are the holomorphic abelian differentials on the genus g surface. We note that the z^I coordinatizes the Jacobian variety of the Riemann surface, $\mathbb{C}^g / (\mathbb{Z} + \tau \mathbb{Z})$, so we can write $z^I \equiv z_1^I + iz_2^I = \sigma_1^I + \tau^{IJ} \sigma_2^J$, such that σ_1^I and σ_2^I take values over the unit interval $[0, 1]$. Moreover, $\tau^{IJ} = \tau_1^{IJ} + i\tau_2^{IJ}$, $z_2^I = \tau_2^{IJ} \sigma_2^J$. The bosonic correlators can be read off from the correlators of exponential insertions [12,10] :

$$\begin{aligned} \langle \prod_{i=1}^n e^{ik_i \cdot X(\nu_i)} e^{i\bar{k}_i \cdot \bar{X}(\bar{\nu}_i)} \rangle &= \frac{\delta(\sum k_i) \delta(\sum \bar{k}_i)}{Z^{\text{cl}}} \prod_{i < j} E(\nu_i, \nu_j | \tau)^{k_i \cdot k_j} \bar{E}(\bar{\nu}_i, \bar{\nu}_j | \bar{\tau})^{\bar{k}_i \cdot \bar{k}_j} \times \\ &\times \sum_{(l, \bar{l})} \exp \left\{ i\pi \left[(l \cdot \tau \cdot l) - (\bar{l} \cdot \bar{\tau} \cdot \bar{l}) \right] + 2\pi i \left[(l \cdot \sum k_i \int^{\nu_i} \omega) - (\bar{l} \cdot \sum \bar{k}_i \int^{\bar{\nu}_i} \bar{\omega}) \right] \right\}, \end{aligned} \quad (3.3)$$

with $Z^{\text{cl}} = \sum \exp \{ i\pi (l \cdot \tau \cdot l - \bar{l} \cdot \bar{\tau} \cdot \bar{l}) \}$. Here l_I^A and $\bar{l}_I^{\bar{A}}$, $I = 1, \dots, g$, $A, \bar{A} = 0, \dots, 9$ are the left and right moving momenta running through the loops: $\int_{b_I} \partial X^A = l_I^A$ and $\int_{b_I} \bar{\partial} \bar{X}^{\bar{A}} = \bar{l}_I^{\bar{A}}$ where b_I are the b -cycles of the Riemann surface. The sum over (l, \bar{l}) should be understood as an integral over the noncompact momenta l_I^α , $\alpha = 0, \dots, 3$, and a sum over the compact momenta $(l_I^a, \bar{l}_I^{\bar{a}})$ which lie on a lattice $\Gamma_{6,6}^g$:

$$\sum_{(l, \bar{l})} \equiv \int \prod_{\alpha, I} dl_I^\alpha \sum_{(l_I^a, \bar{l}_I^{\bar{a}})}. \quad (3.4)$$

To calculate the correlators involving operators like $\zeta_A \partial X^A$, we calculate the correlators with $\exp(i\zeta_A X^A)$, take the derivative $-i\partial$, and keep the term linear in ζ . To simplify this procedure it is convenient to write the polarization tensor $\zeta_{A_1 \dots A_{n+1}}$ formally as the symmetrized product of vectors: $\zeta_{A_1}^1 \zeta_{A_2}^2 \dots \zeta_{A_{n+1}}^{n+1}$. Moreover, we use point splitting for composite operators. For example, we write $:O_1 O_2(\sigma):$ as $O_1(\sigma_1) O_2(\sigma_2)$, calculate the

correlators, subtract the terms singular in $(\sigma_1 - \sigma_2)$ and then take the limit $\sigma_1 \rightarrow \sigma$ and $\sigma_2 \rightarrow \sigma$.

We would like to emphasize that the two point function for the nonsuper states is nonzero even after summing over spin structures. In this respect they are very different from super states whose two point function vanishes after summing over spin structure as a consequence of supersymmetry. To see how it works in our case, note that the two-point function (3.1) contains a term $\langle \partial\psi\psi(\nu_1)\partial\psi\psi(\nu_2) \rangle$. Using expression (3.2), the fact that the spin-structure dependent part of the fermionic partition function is proportional to $\vartheta^4 \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (0|\tau)$, and the Riemann theta identity, we see that this term is nonzero even after summing over spin-structures.

With these remarks we are now ready to evaluate the amplitude. For simplicity we discuss the one-loop case first.

4. One Loop

The nonzero part of the two point function in this case can be written as

$$A = \int \frac{d^2\tau}{\tau_2^2} d^2z B(\bar{\tau}) \sum_{(s,\bar{s})} C_{(s,\bar{s})}(\tau, \bar{\tau}, z, \bar{z}, n) \sum_{(l,\bar{l})} D_s(\tau, k, z, l) \bar{D}_{\bar{s}}(\bar{\tau}, \bar{k}, \bar{z}, \bar{l}), \quad (4.1)$$

where the sum is over $s = 0, \dots, n-1$; $\bar{s} = 0, 1$ and $B(\bar{\tau}) = \bar{\eta}^{-24}(\bar{\tau})$. The quantities $C_{(s,\bar{s})}$ and D_s are given by

$$C_{s,\bar{s}} = K_s \bar{K}_{\bar{s}} [\partial^2 \log E(z, \tau)]^{n-1-s} [\bar{\partial}^2 \log \bar{E}(\bar{z}, \bar{\tau})]^{1-\bar{s}} [E(z, \tau)]^{-k^2} [\bar{E}(\bar{z}, \bar{\tau})]^{-\bar{k}^2} \exp[\pi(k^2 + \bar{k}^2)\sigma_2 \cdot \tau_2 \cdot \sigma_2] \quad (4.2)$$

where

$$K_s = \{\zeta_{2s+1} \cdot \zeta_{2s+2}\} \dots \{\zeta_{2n+1} \cdot \zeta_{2n+2}\},$$

$$\bar{K}_0 = \bar{\zeta}_1 \cdot \bar{\zeta}_2 \quad \bar{K}_1 = 1$$

and

$$D_s = (l \cdot \zeta^1) \dots (l \cdot \zeta^{2s}) \exp\{i\pi(l \cdot \tau_1 \cdot l + 2l \cdot k z_1)\} \exp\{-\pi(l + k\sigma_2) \cdot \tau_2 \cdot (l + k\sigma_2)\}, \quad (4.3)$$

The equation for $\bar{D}_{\bar{s}}$ is similar. We have carefully checked that the delta-function contact terms coming from the contractions between holomorphic and antiholomorphic operators vanish to give this factorized form.

4.1. Analytic Continuation

We note that the two-point function, as given by the integral (4.1), diverges at the boundary of the moduli space for on-shell values of the momentum k . To obtain a well-defined amplitude one needs to analytically continue the external momenta to an unphysical region where all modular integrals are convergent and then analytically continue back to the physical region. Such an analytic continuation relevant to the four point superstring amplitude of massless states at one loop has been done in [13,14]. We describe below the details of the analytic continuation in the present case. As mentioned in the introduction, we will later use the well-defined amplitude obtained this way to study its M -dependence.

It is easy to see that divergences of the above integral can only come from the boundary of the moduli space ($\tau_2 \rightarrow \infty$). The integrand in eqn. (4.1) at the large τ_2 limit is

$$A_\infty = \int \frac{d^2\tau}{\tau_2^2} d^2z \exp(+2i\pi\bar{\tau}) [\exp\{-i\pi z_1(\bar{k}^2 - k^2)\}] \\ [\exp\{-\pi\sigma_2\tau_2(k^2 + \bar{k}^2)\}] \exp[\pi\tau_2\sigma_2^2(k^2 + \bar{k}^2)] D_{n-1}\bar{D}_1 \quad (4.4)$$

where

$$D_{n-1}\bar{D}_1 = \sum_{l,\bar{l}} \left(l \cdot \zeta^1 \dots l \cdot \zeta^{2(n-1)} \right) (\bar{l} \cdot \bar{\zeta}^1 \bar{l} \cdot \bar{\zeta}^2) \exp\{i\pi\tau_1(l - \bar{l}^2)\} \\ \exp\{2i\pi(\sigma_1 + \sigma_2\tau_1)(l \cdot k - \bar{l} \cdot \bar{k})\} \exp\{-2\pi\tau_2(l + k\sigma_2)^2\} \exp\{\pi\tau_2(l^2 - \bar{l}^2)\} \\ \exp\{2\pi\tau_2\sigma_2(l \cdot k - \bar{l} \cdot \bar{k})\} \exp\{\pi\tau_2\sigma_2^2(k^2 - \bar{k}^2)\}. \quad (4.5)$$

Since the two point function (4.1) is invariant under the modular transformation $\tau_1 \rightarrow \tau_1 + 1$ for all τ_2 , its asymptotic form as $\tau_2 \rightarrow \infty$ must also be invariant under it. This implies

$$+2 + (l^2 - \bar{l}^2) - \sigma_2[2n + 2(l \cdot k - \bar{l} \cdot \bar{k})] = 2r \quad (4.6)$$

where r is an integer. Furthermore, since this relation should be valid for arbitrary σ_2 , the term in square bracket has to be zero¹. Now, doing the τ_1 integration in (4.4), we get

$$A_\infty = \int \frac{d\tau_2}{\tau_2^2} d^2z \exp\{2\pi\tau_2\sigma_2(1 - \sigma_2)(-k^2)\} \sum_{l,\bar{l}} \delta_{(l^2 - \bar{l}^2), -2} \\ \delta_{(l \cdot k - \bar{l} \cdot \bar{k}), n} \left(l \cdot \zeta^1 \dots l \cdot \zeta^{2(n-1)} \right) (\bar{l} \cdot \bar{\zeta}^1 \bar{l} \cdot \bar{\zeta}^2) \exp\{-2\pi\tau_2(l + k\sigma_2)^2\}. \quad (4.7)$$

¹ The same conclusion can be arrived at if one performs the σ_1 integration first.

Clearly, the integral is divergent if we impose the on-shell condition on the integrand. We define the above integral by analytically continuing in the complex plane of $s = -p_\mu p^\mu$. The integral is then convergent in the region

$$\text{Re}(-k^2) = \text{Re}(-p^2 - q^2) < 0 \quad (4.8)$$

The result can be reevaluated at the on-shell values of the momenta and it is well-defined.

4.2. Bound on the two point function

It is easy to see from (4.1) that the quantities $C_{(s,\bar{s})}$ do not depend on M , but only on n , so in the limit $n \ll M^2$ considered in (1.2), any mass dependence, if at all, comes from the functions D_s and $\bar{D}_{\bar{s}}$. We now show that the amplitude (4.1) is actually bounded above by an M -independent quantity.

It is convenient to divide A into two parts:

$$A = A_1 + A_2 \quad (4.9)$$

where A_1 and A_2 correspond to replacing $B(\bar{\tau})$ in (4.1) with $\exp(2i\pi\bar{\tau})$ and $\{(\bar{\eta}(\bar{\tau}))^{-24} - \exp(2i\pi\bar{\tau})\}$ respectively. A_1 contains the (unphysical) tachyonic divergence, which is removed by the integration over τ_1 . A_2 does not contain any such divergence and the modulus of the integrand in A_2 is itself finite over the entire moduli space.

Now, clearly

$$|A| \leq |A_1| + |A_2| \quad (4.10)$$

We will first show that $|A_2|$ is bounded above by a finite M -independent quantity. Using the fact that the absolute value of a sum is less than the sum of the absolute values, we have

$$|A_2| \leq A' \equiv \int \frac{d^2\tau}{\tau_2^2} d^2z |B(\bar{\tau})| \sum_{(s,\bar{s})} |C_{(n-1-s,1-\bar{s})}(\tau, \bar{\tau}, z, \bar{z}, n)| \sum_{(l,\bar{l})} |D_s(\tau, k, z, l)| |\bar{D}_{\bar{s}}(\bar{\tau}, \bar{k}, \bar{z}, \bar{l})| \quad (4.11)$$

Now, note that, because τ_2 is a positive definite, A' is finite for all values of τ_2 . Furthermore, it depends on the external charges (k, \bar{k}) only through the quantities $(k\sigma_2^I, \bar{k}\sigma_2^I)$ which can always be brought inside the unit cell by a lattice shift. In other words, $(k\sigma_2, \bar{k}\sigma_2) =$

$(L, \bar{L}) + (\delta k, \delta \bar{k})$, where (L, \bar{L}) is a lattice vector and $(\delta k, \delta \bar{k})$ lies inside the unit cell of the lattice. The summation variables (l, \bar{l}) can be suitably shifted to replace $(k\sigma_2, \bar{k}\sigma_2)$ by $(\delta k, \delta \bar{k})$ in the exponents of $|D\bar{D}|$ in (4.11). The extra terms involving $L \cdot \zeta$ and $\bar{L} \cdot \bar{\zeta}$ that appear in (4.11) in the process depend on (k, \bar{k}) . However, since $k \cdot \zeta = 0$ we have the condition that $L \cdot \zeta = -\delta k \cdot \zeta$ (and similarly for the barred quantities). As a result, the entire quantity A' depends only on $(\delta k, \delta \bar{k})$. Now, when $M \rightarrow \lambda M, \lambda \rightarrow \infty$, some of the components of the charges (k, \bar{k}) must go to infinity to satisfy the conditions (1.1) and (1.2). However, $(\delta k, \delta \bar{k})$ varies inside a bounded domain (the unit cell) whose limits are independent of M , and A' is finite for all $(\delta k, \delta \bar{k})$. Therefore, we conclude that A' and, in turn $|A_2|$, is bounded above by a finite quantity independent of M .

Now, we are left with the task of showing that $|A_1|$ is also bounded by an M independent finite quantity. For A_1 , we will perform the τ_1 integration so that the unphysical tachyonic divergence is removed. Let us first consider $A_{1\infty}$ which includes only the leading term in the large τ_2 expansion of $C_{s,\bar{s}}$ (eqn. (4.2)). Doing the τ_1 integration gives eqn. (4.7). Now, since the first exponential, with the analytically continued k defined is less than one, we have

$$A_{1\infty} \leq \int \frac{d\tau_2}{\tau_2^2} d^2 z \sum_{l, \bar{l}} \delta_{(l^2 - \bar{l}^2), -2} \delta_{(l \cdot k - \bar{l} \cdot \bar{k}), n} \left(l \cdot \zeta^1 \dots l \cdot \zeta^{2(n-1)} \right) \left(\bar{l} \cdot \bar{\zeta}^1 \bar{l} \cdot \bar{\zeta}^2 \right) \exp\{-2\pi\tau_2(l + k\sigma_2)^2\} \quad (4.12)$$

The Kroenecker delta's restrict the lattice sum to a subset, but since the terms appearing in the sum are all positive, the restricted sum is less than or equal to the unrestricted sum, that is,

$$A_{1\infty} \leq \int \frac{d\tau_2}{\tau_2^2} d^2 z \sum_{l, \bar{l}} \left(l \cdot \zeta^1 \dots l \cdot \zeta^{2(n-1)} \right) \left(\bar{l} \cdot \bar{\zeta}^1 \bar{l} \cdot \bar{\zeta}^2 \right) \exp\{-2\pi\tau_2(l + k\sigma_2)^2\}. \quad (4.13)$$

Just as in the case of A' (4.11), we can now shift the summation variable (l, \bar{l}) of the lattice to show that (4.13) is bounded above by a finite M independent quantity.

The above arguments can also be made for contributions to A_1 coming from the subleading terms in the large τ_2 expansion of $C_{s,\bar{s}}$. One can again show that the modulus of each of these terms is bounded above by a finite M -independent quantity. Since there

are an infinite number of terms, one of course needs to show that the series converges. This is easy to show using the fact the higher order terms in the large- τ_2 expansion are exponentially damped.

To summarize, we have first described the analytic continuation in the non-compact momentum plane to make the two point amplitude well-defined. Then, we separated the amplitude into two terms mainly to handle the unphysical tachyonic divergence and showed that the modulus of each of the terms, and hence the modulus of the one-loop two-point function $|A|$, is bounded above by a finite M -independent quantity.

5. Generalization to Higher Loops

We now extend the result of the previous section to higher genus amplitudes. The two-point function is given by

$$A = \int d\mathbf{M} d^2\nu_1 d^2\nu_2 B(\bar{\tau}) f(\tau, \nu_1, \nu_2) \sum_{(s, \bar{s})} C_{(s, \bar{s})}(\tau, \bar{\tau}, \nu_1, \nu_2, \bar{\nu}_1, \bar{\nu}_2, n) \sum_{(l, \bar{l})} D_s(\tau, k, \nu_1, \nu_2, l) \bar{D}_{\bar{s}}(\bar{\tau}, \bar{k}, \bar{\nu}_1, \bar{\nu}_2, \bar{l}), \quad (5.1)$$

where C , D and \bar{D} are given by:

$$C_{s, \bar{s}} = K_s \bar{K}_{\bar{s}} [\partial_{\nu_1} \partial_{\nu_2} \log E(\nu_1, \nu_2 | \tau)]^{n-1-s} [\bar{\partial}_{\bar{\nu}_1} \bar{\partial}_{\bar{\nu}_2} \log \bar{E}(\bar{\nu}_1, \bar{\nu}_2 | \bar{\tau})]^{1-\bar{s}} [E(\nu_1, \nu_2 | \tau)]^{-k^2} [\bar{E}(\bar{\nu}_1, \bar{\nu}_2 | \bar{\tau})]^{-\bar{k}^2} \exp[\pi(k^2 + \bar{k}^2) \sigma_2 \cdot \tau_2 \cdot \sigma_2] \quad (5.2)$$

with $K_s, \bar{K}_{\bar{s}}$ being the contraction of the polarization tensors as defined in the one loop case and

$$D_s = \{l \cdot \zeta^1 \omega(\nu_1)\} \{l \cdot \zeta^2 \omega(\nu_2)\} \dots \{l \cdot \zeta^{2s-1} \omega(\nu_1)\} \{l \cdot \zeta^{2s} \omega(\nu_2)\} \exp\{i\pi(l \cdot \tau_1 \cdot l + 2l \cdot k z_1)\} \exp\{-\pi(l + k\sigma_2) \cdot \tau_2 \cdot (l + k\sigma_2)\}. \quad (5.3)$$

The equation for $\bar{D}_{\bar{s}}$ is similar. $B(\bar{\tau}) = \det^{-12}(-\nabla^2)$ is the genus g determinant of 24 left-moving bosons. The function $f(\tau, \nu_1, \nu_2)$ is the product of the right-moving bosonic determinants and the spin-structure-summed two-point function $\langle \partial_{\nu_1} \psi \psi \partial_{\nu_2} \psi \psi \rangle$. For our purpose we need to know only the asymptotic properties of B and f at the boundary of the moduli space which we will mention shortly.

5.1. Analytic Continuation

As in the one-loop case, the integral expression (5.1) is naively divergent at the boundary of the moduli space and we need to define the amplitude through analytic continuation. For consistency, the analytic continuation employed in eq. (4.8) should work here too. To see that it does, we need to consider in somewhat detail the boundary of the moduli space of the higher genus Riemann surfaces. This is a rather well-studied subject [15,16,17]. For our purposes, we can regard the boundary of the moduli space as consisting of two distinct kind of degenerations of the Riemann surface:

1. *Pinching along a homologically trivial cycle:*

Under this, a genus $g_1 + g_2$ Riemann surface splits into two disjoint Riemann surfaces (having one puncture each) of genera g_1 and g_2 . It is not difficult to see that such a pinching factorizes the present two-point function in such a way that one of the factors always equals one-point function of a tachyon at some loop. Since the latter quantity is zero, the contribution of this kind of degeneration is zero. We shall therefore consider degeneration along only the homologically nontrivial cycles henceforth.

2. *Pinching along a homologically nontrivial cycle:*

Under this kind of a degeneration a genus $g + 1$ surface reduces to a genus g surface with two punctures. We need to know how the various factors in (5.1) behave under this kind of a degeneration. The behaviour of C, D, \bar{D} can be read off from the behaviour of the period matrix τ and the abelian differentials ω^I . The latter are given as follows. Let $\tau^{IJ}, I, J = 1, 2, \dots, g$ be the period matrix and $\omega^I, I = 1, 2, \dots, g$ the abelian differentials of the genus g Riemann surface obtained after the degeneration. Also, let a and b be the two punctures. Then near the boundary of the moduli space corresponding to this degeneration, the period matrix and abelian differentials of the original genus $g + 1$ Riemann surface is given by

$$\begin{aligned} \tau &\rightarrow \begin{bmatrix} \tau_{IJ} & \int_a^b \omega_I \\ \int_a^b \omega_J & \tau_{00} \end{bmatrix}, \\ \omega_0(\nu_1) &\rightarrow \partial_{\nu_1} \log \left[\frac{E(a, \nu_1)}{E(b, \nu_1)} \right]. \end{aligned} \tag{5.4}$$

where

$$\tau_{00} \rightarrow \frac{-i}{\pi} \left(\frac{\log t}{2} - \log E(a, b) \right) + O(t) \text{ as } t \rightarrow 0.$$

Keeping in mind that there is an unphysical tachyon in the $\bar{L}_0 = 1$ spectrum of the left-moving bosonic string (prior to the $\text{Re } \tau$ integration), we can infer the following asymptotic form of $B(\bar{\tau})$:

$$B(\bar{\tau}) = (\bar{q}_0)^{-2} B_1(\bar{q}_0, \bar{q}_I) + B_2(\bar{q}_0, \bar{q}_I) \quad (5.5)$$

where B_2 is finite over the entire moduli space and B_1 is finite at the specific degeneration of the Riemann surface under consideration. Since there is no such divergence from the right sector, the asymptotic form of f is:

$$f(\tau, \nu_1, \nu_2) = [1 + O(q_0^2)] f_1(q_I, \nu_1, \nu_2) \quad (5.6)$$

which is finite over the entire moduli space. Here $q_I = \exp[i\pi\tau_{II}]$ and similarly for the barred terms.

Combining all this, the asymptotic behaviour of A at this degeneration is given by

$$A_\infty = \int d\mathcal{M} d^2\tau_{00} \int d^2\nu_1 d^2\nu_2 (\bar{q}_0)^{-2} [\exp\{-i\pi z_1^0(\bar{k}^2 - k^2)\}] \quad (5.7)$$

$$[\exp\{-\pi\sigma_2^0 \text{Im}\tau_{00}(k^2 + \bar{k}^2)\}] \exp[\pi\sigma_2^0 \text{Im}\tau_{00} \sigma_2^0(k^2 + \bar{k}^2)] D_{n-1} \bar{D}_1$$

where $d\mathcal{M}$ is a measure factor whose explicit form is not important for our purpose except that it is independent of $\text{Re } \tau_{00}$. We have explicitly verified this fact in the two-loop case [15]. For higher loops the measure factors in terms of the period matrix are quite complicated for explicit verification. However, from the form of the measure in the Fenchel Nielsen parametrization [10] we believe that it is true also at higher loops.

Going through steps similar to the ones in the one-loop case (i.e imposing modular invariance $\text{Re } \tau_{00} \rightarrow \text{Re } \tau_{00} + 1$ and then doing the $\text{Re } \tau_{00}$ integration), we arrive at

$$A_\infty = \int d\mathcal{M} d\text{Im}\tau_{00} \int d^2\nu_1 d^2\nu_2 \exp\{2\pi \text{Im}\tau_{00} \sigma_2^0(1 - \sigma_2^0)(-k^2)\} \sum_{l, \bar{l}} \delta_{(l_0^2 - \bar{l}_0^2), -2}$$

$$\delta_{(l_0 \cdot k - \bar{l}_0 \cdot \bar{k}), n} \{l \cdot \zeta^1 \omega(\nu_1)\} \{l \cdot \zeta^2 \omega(\nu_2)\} \dots \{l \cdot \zeta^{2s-1} \omega(\nu_1)\} \{l \cdot \zeta^{2s} \omega(\nu_2)\} \quad (5.8)$$

$$\{\bar{l} \cdot \bar{\zeta}^1 \omega(\bar{\nu}_1)\} \{\bar{l} \cdot \bar{\zeta}^2 \omega(\bar{\nu}_2)\} \dots \{\bar{l} \cdot \bar{\zeta}^{s-1} \omega(\bar{\nu}_1)\} \{\bar{l} \cdot \bar{\zeta}^s \omega(\bar{\nu}_2)\}$$

$$\exp\{-2\pi \text{Im}\tau_{00}(l^0 + k\sigma_2^0)^2\} \mathcal{L}_g \bar{\mathcal{L}}_g$$

where

$$\mathcal{L}_g = \exp\{i\pi(l^I \text{Re}\tau_{IJ} l^J + 2l^I \cdot k z_1^I)\} \exp\{-\pi(l + k\sigma_2)^I \cdot \text{Im}\tau_{IJ} \cdot (l + k\sigma_2)^J\} . \quad (5.9)$$

and its conjugate is $\bar{\mathcal{L}}_g$.

We see, therefore, that the amplitude is well-defined under the same analytic continuation (4.8) as used in the one-loop case.

5.2. Bound on the two-point function

We now prove M -independence of the two-point function in higher loops.

Consider the moduli space after pinching any one non-zero homology cycle labeled by ‘0’. As in the one-loop case, we substitute (5.5) for $B(\bar{\tau})$ in (5.1), and divide A into two parts:

$$A = A_1 + A_2, \quad (5.10)$$

Here A_1 and A_2 correspond to the two terms $(\bar{q}_0)^{-2}B_1(\bar{q}_0, \bar{q}_I)$ and $B_2(q_0, q_I)$ in (5.5) respectively.

Since $B_2(\bar{q}_0, \bar{q}_I)$ is finite, the integrand of A_2 is finite over the entire moduli space. We can show that the modulus of A_2 is bounded above by a finite M independent quantity following the steps presented for one-loop.

A_1 contains the unphysical tachyonic divergences. When only cycle ‘0’ is pinched and no other then we can deal with the tachyonic divergence as in the one-loop case. At higher loops, however, we have to take into account simultaneous pinching of more than one non-trivial homology cycles. For example, if an additional cycle labeled by ‘1’ is pinched along with ‘0’ then we have additional divergences. In this case $B_1(\bar{\tau})$ can be written as

$$(\bar{q}_0)^{-2}B_1(\bar{\tau}) = (\bar{q}_0)^{-2}B_1^{(1)}(\bar{q}_0, \bar{q}_I) + (\bar{q}_1)^{-2}B_1^{(2)}(\bar{q}_0, \bar{q}_I) + (\bar{q}_0)^{-2}(\bar{q}_1)^{-2}B_1^{(3)}(\bar{q}_0, \bar{q}_I), \quad (5.11)$$

where the B coefficients on the right have no singularities at either of the two degenerations. Correspondingly A_1 now has three possible divergent terms

$$A_1 = A_1^{(1)} + A_1^{(2)} + A_1^{(3)}. \quad (5.12)$$

It is now evident that $A_1^{(1)}$ and $A_1^{(2)}$ are similar to the one loop A_1 which can be handled by integrating the real part of the appropriate period matrix element. So, these two terms are again bounded by some finite M independent quantity.

The non-trivial term different from the one loop is $A_1^{(3)}$. We have to integrate over the real part of both the period matrix elements which dominate in the simultaneous pinching of the two homologically non-zero cycles. This will lead to a restricted lattice sum which can again be handled in exactly the same fashion as for the one loop.

It is straightforward to extend the procedure of separating A_1 (5.12) into various terms under simultaneous pinching of arbitrary number of non-zero homology cycles at every order in string loop and showing that each of the terms is finite and bounded above by an M independent finite quantity.

6. Conclusions

The real part of A equals δM^2 and the imaginary part equals, by the optical theorem, ΓM , where Γ is the decay rate. Since $|A|$ is bounded by an M independent constant, it follows that both the mass correction δM as well as the width Γ are vanishingly small if the tree-level mass is sufficiently large:

$$\delta M \leq O(1/M); \quad \Gamma \leq O(1/M). \quad (6.1)$$

One immediate consequence of our result is that it partially answers our original question of why the degeneracy of even the nonsuper states agrees with the Sen entropy of the associated black holes. We have proved here a perturbative nonrenormalization valid to all loops. The agreement with the entropy suggests that it should be valid even nonperturbatively. There are a number of related issues that are currently under investigation. Besides the entropy, the gyromagnetic ratios of the nonsuper states are also known to be in agreement with those of the associated black holes [6,5]. We therefore expect a similar nonrenormalization of the gyromagnetic ratio as well. The perturbative heterotic state considered here, which has winding and momentum along the internal torus is dual to a D-string in Type-I theory. It is interesting to know if a similar nonrenormalization holds for the D-string. In particular, the large mass limit considered here appears to be related to the large- N limit in the gauge theory on the D-string. More generally, one expects a similar nonrenormalization for a host of magnetically charged as well as dyonic states that are S-dual to the purely electrically charged states discussed here.

In this paper we have discussed black holes with zero area. For black holes with nonzero area, both in the supersymmetric as well as nearly supersymmetric cases, there is already an impressive agreement between the degeneracy of the string states and the Bekenstein-Hawking entropy [18,19,20]. Furthermore, the emission and absorption properties of these black holes also match those of the corresponding string states [21,22,23,24,25]. Indeed, the entropy-matching seems to work even for a few nonsuper states [7,26,27]. It would be interesting to see if these states satisfy mass-nonrenormalization of the kind presented here. Previous discussions of nonrenormalization in the context of nearly supersymmetric black holes with nonzero area can be found in [28,29].

We end with a few comments and speculations. Our results are in accord with the general correspondence, pointed out in [30], between perturbative string states and black holes. By this correspondence the two degeneracies match at a specific value of the coupling where the quantum correction to the spectrum is becoming appreciable. However, if there is no renormalization of the spectrum, as in our case, the degeneracies can be compared at all values of the coupling. The nonrenormalization that we have found is quite surprising because it is not a consequence of any apparent symmetry. It may be that there is a hidden gauge symmetry of string theory that is responsible for this nonrenormalization.

Our results indicate that there is an infinite tower of very massive nonsuper states with $\overline{N} = 1$ and arbitrary N , which is very similar to the infinite tower of super states with $N = \frac{1}{2}$ and arbitrary \overline{N} . The spectrum of super states has important application to the dynamics of the theory. For instance, in $N = 2$ string theories one can construct a generalized Kac-Moody Lie superalgebra in terms of super states which governs the form of the perturbative superpotential [31]. It would be interesting to see if the nonsuper states discussed here or their duals can be used to obtain additional insight into the dynamics.

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