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Off-shell interactions for closed-string tachyons

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Abstract: Off-shell interactions for localized closed-string tachyons in $\mathbb{C}/\mathbb{Z}_N$ superstring backgrounds are analyzed and a conjecture for the effective height of the tachyon potential is elaborated. At large-N, some of the relevant tachyons are nearly massless and their interactions can be deduced from the S-matrix. The cubic interactions between these tachyons and the massless fields are computed in a closed form using orbifold CFT techniques. The cubic interaction between nearly-massless tachyons with different charges is shown to vanish and thus condensation of one tachyon does not source the others. It is shown that to leading order in N, the quartic contact interaction vanishes and the massless exchanges completely account for the four point scattering amplitude. This indicates that it is necessary to go beyond quartic interactions or to include other fields to test the conjecture for the height of the tachyon potential.

Keywords: Superstrings and Heterotic Strings, Tachyon Condensation.

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1. Introduction

There are a number of physical situations, for example in cosmology, where it is necessary to deal with unstable and time-dependent backgrounds. It is of interest to develop calculational tools within string theory that can describe such backgrounds in an essentially stringy way.

A useful laboratory for studying unstable or time-dependent backgrounds in string theory is provided by tachyons in open string theory. These tachyons correspond to the instabilities of various unstable brane configurations and their condensation is expected to describe the decay of these unstable branes to flat space. The static as well as time dependent aspects of such decays have been analyzed quite extensively.

By comparison, tachyons in closed string theory, even though more interesting physically, have proved to be less tractable. For these tachyons, in most cases there is no natural candidate for a stable minimum of the potential where the tachyon fields can acquire an expectation value. For a closed-string tachyon with a string-scale mass it is difficult to disentangle a well-defined potential from other gravitational effects. In some cases, as in the case of thermal tachyon which signifies the onset of Hagedorn transition, the mass can be fine-tuned to be very small [1, 2], but the tachyon has cubic couplings to the dilaton and other massless scalar fields. Consequently, it sources other massless fields which considerably complicates the dynamics and quickly drives the system into strong coupling or strong curvature region [3, 4]. Similarly, for the open string tachyon in the brane-antibrane system, the mass can be tuned to zero by adjusting the distance between the brane and the antibrane [5] to be the string scale. However, at the endpoint of condensation, the distance between brane and antibrane vanishes and the tachyon eventually has string scale mass and the effective field theory breaks down.

In this paper we show that for localized tachyons in the twisted sector of the $\mathbb{C}/\mathbb{Z}_N$ orbifold theories, some of these difficulties can be circumvented. The paper is organized as follows. In section 2, we motivate our computation and elaborate arguments that lead to a conjecture for the effective height of the tachyon potential. In section 3, we review the orbifold CFT for a single complex twisted boson and the four-twist correlation function that is required to describe the scattering of four tachyons. Using factorization and symmetry arguments we calculate all three-point correlation functions needed for our purpose. In section 4, we review superstring theory on the $\mathbb{C}/\mathbb{Z}_N$ orbifold and using the CFT results compute the four-tachyon scattering amplitude as well as the gauge-invariant cubic interactions between the tachyons and the massless fields. We show that the tachyon of interest does not source other tachyons to quartic order and its dynamics can thus be studied independently of others in a consistent manner. In section 5, we fix the normalization of the four point amplitude by factorization and show that to order $1/N^2$, the four tachyon scattering is completely accounted for by the massless exchanges. We conclude in section 6 with some comments.

2. A conjecture for the height of the tachyon potential

We now pursue some of the analogies of closed-string localized tachyons and open-string tachyons with the aim of identifying a model where explicit computations are possible.

2.1 Analogies with open-string tachyons

There are three main simplifications which make the open-string tachyons tractable.

- The tachyons are localized on the worldvolume of an unstable brane. It is reasonable to assume that condensation of the tachyon corresponds to the annihilation of the brane and the system returns to empty flat space.
- Conservation of energy then implies Sen's conjecture [6, 7] that the height of the tachyon potential should equal the tension of the brane that is annihilated.
- The gravitational backreaction of D-branes can be made arbitrarily small by making the coupling very small because the tension of D-branes is inversely proportional to closed string coupling g_c whereas Newton's constant is proportional to g_c^2. This makes it possible to analyze the tachyon potential without including the backreaction of massless closed-string modes.

The twisted-sector tachyons in $\mathbb{C}/\mathbb{Z}_N$ string backgrounds are in many respects quite analogous.

- The $\mathbb{C}/\mathbb{Z}_N$ theory has the geometry of a cone with deficit angle $2\pi(1-1/N)$ and the twisted-sector tachyons are localized at the tip [8, 9]. There is considerable evidence now that condensation of these tachyons relaxes the cone to empty flat space and thus much like the open string tachyons, there is a natural candidate for the endpoint of the condensation [10].
- There is a precise conjecture for the effective height of the tachyon potential that is analogous to the Sen's conjecture in the open string case [11].
- At large-N, some of the relevant tachyons are nearly massless. Therefore, one would expect that there is a natural separation between the string scale and the scale at which the dynamics of the tachyons takes place and thus higher order stringy corrections can be controlled. The required S-matrix elements are completely computable using orbifold CFT techniques.

Note that for the $\mathbb{C}/\mathbb{Z}_N$ tachyons we can talk about the height of the tachyon potential because the background is not Lorentz invariant. Quite generally, two closed-string CFT backgrounds which are both Lorentz invariant cannot be viewed as two critical points of a scalar potential at different heights. This is because a nonzero value of a scalar potential at a critical point with flat geometry would generate a tadpole for the dilaton and the string equations of motion would not be satisfied. By contrast, the $\mathbb{C}/\mathbb{Z}_N$ backgrounds are not flat because there is a curvature singularity at the tip of the cone. It is natural to assume that the tachyon potential provides the energy source for the curvature. There is not dilaton tadpole because, for a conical geometry, the change in the Einstein-Hilbert term in the action precisely cancels the change in the height of the potential. Hence the total bulk action which generates the dilaton tadpole vanishes on the solution. There is an important boundary contribution that is nonzero and as a result there is a net change in the classical action. This reasoning leads to a sensible conjecture for the height of the tachyon potential.

2.2 A conjecture

String theory on the $\mathbb{C}/\mathbb{Z}_N$ orbifold background was first considered in [8, 12, 9] to model the physics of horizons in euclidean space. Geometrically, $\mathbb{C}/\mathbb{Z}_N$ is a cone with deficit angle $2\pi(1-\frac{1}{N})$. The tip of the cone is a fixed point of the $\mathbb{Z}_N$ orbifold symmetry and there are tachyons in the twisted sectors that are localized at the tip signifying an instability of the background.

A physical interpretation of these tachyons was provided by Adams, Polchinski, and Silverstein [10]. They argued that giving expectation values to the tachyon fields would relax the cone to flat space. The most convincing evidence for this claim comes from the geometry seen by a D-brane probe in the sub-stringy regime. In the probe theory, one can identify operators with the right quantum numbers under the quantum $\hat{\mathbb{Z}}_N$ symmetry of the orbifold[1] that correspond to turning on tachyonic vevs. By selectively turning on specific tachyons, the quiver theory of the probe can be 'annealed' to successively go from the $\mathbb{Z}_N$ orbifold to a lower $\mathbb{Z}_M$ orbifold with $M < N$ all the way to flat space. The deficit angle seen by the probe in this case changes appropriately from $2\pi(1-\frac{1}{N})$ to $2\pi(1-\frac{1}{M})$.

Giving expectation value to the tachyon field in spacetime corresponds to turning on a relevant operator on the string worldsheet. Thus the condensation of tachyons to different CFT backgrounds is closely related to the renormalization group flows on the worldsheet between different fixed points upon turning on various relevant operators. An elegant description of the RG flows is provided using the gauged nonlinear sigma model [13] and mirror symmetry [14]. The worldsheet dynamics also supports the expectation that the cone will relax finally to flat space. Various aspects of localized tachyons and related systems have been analyzed in [15]–[34]. For a recent review see [35].

These results are consistent with the assumption that in the field space of tachyons there is a potential $V(\mathbf{T})$ where we collectively denote all tachyons by $\mathbf{T}$. The $\mathbb{Z}_N$ orbifold sits at the top of this potential, the various $\mathbb{Z}_M$ orbifolds with $M < N$ are the other critical points of this tachyonic potential, and flat space is at the bottom of this potential. Such a potential can also explain why a conformal field theory exists only for special values of deficit angles. We will be concerned here with the static properties such as the effective height of the potential and not so much with dynamical details of the process of condensation.

It may seem difficult to evaluate the change in the classical action in going from the $\mathbb{Z}_N$ orbifold to the $\mathbb{Z}_M$ orbifold but we are helped by the fact that the orbifold is exactly conformal. Hence the equations of motion for the dilaton and the graviton are satisfied exactly for both backgrounds. To calculate the change in the action, let us consider the string effective action to leading order in α'

$$S = \frac{1}{2\kappa^2}\int_M \sqrt{-g}\, e^{-2\phi}\left[R + 4(\nabla_M\phi\nabla^M\phi) - 2\kappa^2\delta^2(x)(\nabla_\mu\mathbf{T}\nabla^\mu\bar{\mathbf{T}} + V(\mathbf{T}))\right] + \\ + \frac{1}{\kappa^2}\int_{\partial M}\sqrt{-g}\, e^{-2\phi}\, K\,, \tag{2.1}$$

[1]The quantum $\hat{\mathbb{Z}}_N$ symmetry is simply the selection rule that twists are conserved modulo N. The twist field in the k-th twisted sector has charge k under this symmetry.

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where K is the extrinsic curvature, $\kappa^2/8\pi$ is Newton's constant G, and $V(\mathbf{T})$ denotes the tachyon potential localized at the defect. Here $M = 0, \ldots, 9$ run over all spacetime directions, the cone is along the 8, 9 directions, and $\mu = 0, \ldots, 7$ are the directions longitudinal to the eight dimensional defect localized at the tip of the cone. The extrinsic curvature term is as usual necessary to ensure that the effective action reproduces the string equations of motion for variations $\delta\phi$ and δg that vanish at the boundary.

The action is very similar to the one for a cosmic string in four dimensions. For a cosmic string in four dimensions or equivalently for a 7-brane in ten dimensions, the deficit angle δ in the transverse two dimensions is given by $\delta = 8\pi G\rho \equiv \kappa^2\rho$ where ρ is the tension of the 7-brane. We are assuming that when the tachyon field $\mathbf{T}$ has an expectation value $\mathbf{T}_N$, its potential supplies an 7-brane source term for gravity such that the total spacetime is $\mathbf{M}_8 \times \mathbb{C}/\mathbb{Z}_N$ where $\mathbf{M}_8$ is the flat eight-dimensional Minkowski space. Einstein equations then imply $R = 2\kappa^2\delta^2(x)V(\mathbf{T})$ and a conical curvature singularity at $x = 0$. Because of this equality, there is no source term for the dilaton and as a result the dilaton equations are satisfied with a constant dilaton. We see that the bulk contribution to the action is precisely zero for the solution. The boundary has topology $\mathbb{R}^8 \times \mathbf{S}^1$. For a cone, the circle $\mathbf{S}^1$ has radius r but the angular variable will go from 0 to $2\pi/N$. The extrinsic curvature for the circle equals $1/r$ and thus the contribution to the action from the boundary term equals $2\pi A/N\kappa^2$. There is an arbitrary additive constant in the definition of the action that is determined by demanding that flat space should have vanishing action. In any case, we are concerned with only the differences and we conclude that in going from $\mathbb{C}/\mathbb{Z}_N$ to flat space the total change in action per unit area must precisely equal $\frac{2\pi}{\kappa^2}(1 - \frac{1}{N})$.

In the full string theory, we should worry about the higher order α' corrections to the effective action. These corrections are dependent on field redefinitions or equivalently on the renormalization scheme of the world-sheet sigma model. However, the total contribution of these corrections to the bulk action must nevertheless vanish for the orbifold because we know that the equations of motion of the dilaton are satisfied with a constant dilaton which implies no source terms for the dilaton in the bulk. Thus, the entire contribution to the action comes from the boundary term even when the α' corrections are taken into account and we can reliably calculate it in a scheme independent way using the conical geometry of the exact solution at the boundary.

One can convert this prediction for the change in action into a conjecture for the height of the tachyon potential. We expect that the tachyon potential should be identified with the source of energy that is creating the curvature singularity. Let us see how it works in some detail. Note that a cone has a topology of a disk and its Euler character χ equals one. Using the Gauss-Bonnet theorem, we then conclude $\chi = \frac{1}{4\pi}\int_{\mathbb{C}/\mathbb{Z}_N} R + \frac{1}{2\pi}\int_{\mathbf{S}^1} K = 1$. This implies that $R = 4\pi(1 - \frac{1}{N})\delta^2(x)$ and we arrive at our conjecture that

$$V(\mathbf{T}_N) = \frac{2\pi}{\kappa^2}\left(1 - \frac{1}{N}\right). \tag{2.2}$$

We are thus led to a plausible picture rather analogous to the open-string tachyons in which the tachyon potential $V(\mathbf{T})$ supplies the source of energy required to create a defect and flat space is the stable supersymmetric ground state. The landscape of the tachyon fields in the closed string case is, however, more intricate. There are several tachyonic modes and many critical points corresponding to cones with different deficit angles and thus a richer set of predictions to test.

2.3 A model and a strategy for computing off-shell interactions

The potential for open-string tachyons been analyzed using a number of different approaches. It has been possible to test Sen's conjecture within open string field theory in a number of different formalisms [36]–[39] both for the bosonic and the superstring. For a recent review and a more complete list of references see [40]. Some properties of the decay process have also been analyzed exactly in boundary conformal field theory [41] and in certain toy models exactly even nonperturbatively [42, 43, 44].

It would be interesting to similarly develop methods within closed-string theory to test the conjecture above for the potential of localized tachyons. For the bosonic string, the string field theory does not have the simple cubic form as in Witten's open string field theory [45]. Nevertheless, a well-developed formalism with non-polynomial interactions is available [46]. Okawa and Zwiebach have recently applied this formalism successfully in the level-truncation approximation [47] and have found more than 70% agreement with the conjectured answer which is quite encouraging. We will be interested here in the localized tachyon in the superstring. For superstrings, there is a string field theory formalism available only for the free theory [48] but not yet for the interacting theory so we need to approach the problem differently.

Corresponding to (2.2), there is a natural object in the worldsheet RG flows that can be identified with the tachyon potential [49]. For relevant flows, however, the relation between worldsheet quantities and spacetime physics is somewhat indirect given the fact that away from the conformal point the Liouville mode of the worldsheet no longer decouples. It is desirable to see, to what extent, (2.2) can be verified directly in spacetime.

To sidestep the use of string field theory, we work instead in the limit of large-N and consider the decay process that takes $\mathbb{C}/\mathbb{Z}_N$ to $\mathbb{C}/\mathbb{Z}_k$ with $k = N - j$, for some small even integer $j = 2, 4, \ldots$ We assume that the tachyonic field T_k that connects these two critical points has a well-defined charge k under the quantum $\hat{\mathbb{Z}}_N$ symmetry with its mass given by $m_k^2 = -\frac{2(N-k)}{\alpha' N}$. This assumption is motivated from the worldsheet mirror description of this process [14] where the relevant operator that is turned on has a well-defined charge under the quantum symmetry. To justify this assumption further we will check that there are no cubic couplings between this tachyon and other nearly massless tachyons. Therefore, giving expectation value to this particular tachyon does not create a tadpole for other tachyons.

Because this tachyon is nearly massless we can consider its effective dynamics much below the string mass scale by integrating out the massive string modes and are justified in ignoring possible string scale corrections to the effective action. The simplest way to model the condensation process is to imagine an effective potential

$$\frac{2\pi}{\kappa^2}\left(1-\frac{1}{N}\right)-\left(\frac{2}{\alpha'}\right)\frac{(N-k)}{N}|T_k|^2+\frac{\lambda_k}{4}|T_k|^4\,. \tag{2.3}$$

The potential has two extrema. At $T_k = 0$ it has a maximum and its value at the maximum is given by (2.2) which supplies the energy source required at the tip of the cone $\mathbb{C}/\mathbb{Z}_N$. There is a minimum at $|T_k|^2 = \frac{4(N-k)}{\alpha' \lambda_k N}$ and the energy at this minimum is lowered. We would like to identify this minimum with the cone $\mathbb{C}/\mathbb{Z}_k$ which implies the prediction

$$\frac{1}{\lambda_k}\left(\frac{2}{\alpha'}\right)^2\left(\frac{N-k}{N}\right)^2 = \frac{2\pi}{\kappa^2}\left(\frac{1}{k}-\frac{1}{N}\right), \tag{2.4}$$

so that the energy at the minimum is exactly what is needed to create the smaller deficit angle of the cone $\mathbb{C}/\mathbb{Z}_k$. We are implicitly working at large N because we have assumed that the tachyon is nearly massless and it is meaningful to talk about its potential ignoring the α' corrections. In the large N limit, the final prediction for the quartic term becomes

$$\lambda_k = \frac{\kappa^2}{2\pi}\left(\frac{2}{\alpha'}\right)^2 (N-k)\,. \tag{2.5}$$

It would be natural to set $\alpha' = 2$ here as in most closed string calculations but we prefer to maintain α' throughout to keep track of dimensions and to allow for easy comparison with other conventions.

For the consistency for this picture it is essential that the tachyon T_k does not source any other nearly massless fields apart from the dilaton. We have already explained that the dilaton tadpole vanishes because the bulk action is zero for the cone. However, if there are tadpoles of any of the very large number of nearly massless tachyons in the system, it would ruin the simple picture above. Quantum $\hat{\mathbb{Z}}_N$ symmetry surely allows terms like $T_k T_k T_{N-2k}$ because the charge needs to be conserved only modulo N. If such a term is present then the tachyon T_{N-2k} will be sourced as soon as T_k acquires an expectation value and its equations of motion will also have to satisfied. We will then be forced to take into account the cubic and quartic interactions of all such fields. Fortunately, as we show in section 4.4, even though the cubic couplings of this type are allowed *a priori*, they actually vanish because of H-charge conservation. We can thus restrict our attention consistently to a single tachyon up to quartic order.

In what follows, we proceed with this simple ansatz. Note that $N - k = j$ is of order one and thus the required quartic term that is of order one. To extract the contact quartic term, we first need to calculate the four point tachyon scattering amplitude and subtract from it the massless exchanges.

There is a subtlety in this procedure that is worth pointing out. We are interested in the one-particle irreducible quartic interaction. To obtain it from the four particle scattering amplitude, we should subtract all one-particle reducible diagrams. Now, in string theory, an infinite number of particles of string-scale mass are exchanged along with the massless fields of supergravity and it would be impractical if we have to subtract all such exchanges. Note however, that the mass of the tachyon of interest is inversely proportional to N and there is a clear separation of energy scales. We are interested in the effective field theory at these much lower energies that are down by a factor of $1/N$ compared to the string scale. Therefore, massive string exchanges are to be integrated out. For tree-level diagrams, integrating out a massive field simply means that we *keep* all one-particle reducible diagrams in which the massive field is exchanged. This generates an effective quartic interaction in much the same way the four-fermi interaction is generated by integrating out the massive vector boson. For this reason, we do not need to subtract the exchanges of massive string modes.

The procedure would then be to compute the four-point amplitude, subtract from it all massless or nearly-massless exchanges, and then take the string scale to infinity and N to infinity keeping fixed the tachyon mass and the external momenta to focus on the energy scale of interest. To put it differently, if we subtract only the massless exchanges, we are automatically solving the equations of motion the massive fields [50]–[53]. This observation explains why the full formalism of string field theory is not needed in our case as it would be for a string scale tachyon and we can proceed consistently within effective field theory.

An analogous large-N approximation was used by Gava, Narain, and Sarmadi [54] to analyze the off-shell potential of an open string tachyon that arises in the D2-D0 system. This tachyon signals the instability of the system towards forming a lower energy bound state in which the D0-brane is dissolved into the D2-brane. These authors introduce a parameter N by considering a system of a single D2 brane with N D0-branes already dissolved in it and then introduce an additional D0-brane. The relevant tachyon is then nearly massless when N is large and one can consistently analyze the system in effective field theory in much the same way as we wish to do here. For the open string tachyon also, the mass-squared is inversely proportional to N and the quartic term turns out to be of order one. The potential then has a lower energy minimum corresponding precisely to the lowered energy of the additional D0-brane dissolved into the D2 brane.

One would hope that a similar story works for the nearly massless closed-string tachyons but there is no *a priori* way to determine the value of the quartic term without actually computing it. Unfortunately, our computation shows that the quartic term in this case is not of order one but much smaller, of order $1/N^3$. We discuss the results and implications in some detail in section 6.

It is now clear that for our purpose we require the S-matrix element for the scattering of four tachyons and the three-point couplings of these tachyons to massless or nearly massless fields. We in turn need the four-point and three-point correlation functions involving the twist fields of the bosonic and fermionic fields. The fermionic twist fields have a free field representation and their correlation function are straightforward to compute. The computations involving the bosonic twist fields are fairly involved and require the full machinery of orbifold CFT. For this reason in the next section we focus only on the CFT of a single complex twisted boson and determine the required correlators using CFT techniques and factorization.

3. Bosonic CFT on $\mathbb{C}/\mathbb{Z}_N$

The main thrust of this section will be the computation of various three-point functions involving the bosonic twist fields. These correlation functions enter into the cubic interactions of the tachyon with the untwisted massless fields carrying polarization and momenta along the cone directions. The orbifold CFT is fairly nontrivial and the correlations cannot be computed using free field theory. Fortunately, it turns out that all the three-point functions required for our purpose can be extracted from factorization of the four-twist correlation function which is already known in the literature. This fact considerably simplifies life.

Our starting point will be the four-twist correlation function which has been computed in [55, 56, 57]. We review some basic facts about the CFT of a single complex boson in the twisted and the untwisted sector in sections 3.1 and 3.2 and the relevant aspects of the four-twist correlation function in section 3.3. We then calculate the various three-point functions in section 3.4 up to a four-fold discrete ambiguity using factorization and symmetry arguments. The discrete ambiguity will be fixed later by demanding BRST invariance of the full string vertex.

3.1 Untwisted sector

We begin with a review of the Hilbert space of the untwisted sector a complex scalar X taking values in $\mathbb{C}/\mathbb{Z}_N$. The purpose of this section is to keep track of factors of N and to collect some formulas on how the states in the oscillator basis split into primaries and descendants of the conformal algebra. This will be important later for factorization using conformal blocks.

3.1.1 States and vertex operators

States in the untwisted sector of the CFT on $\mathbb{C}/\mathbb{Z}_N$ are constructed by projecting the Hilbert space $\mathcal{H}_{\mathbb{C}}$ of CFT on the complex plane onto $\mathbb{Z}_N$ invariant states. The $\mathbb{Z}_N$ generator R satisfying $R^N = 1$ acts on $\mathcal{H}_{\mathbb{C}}$ as a unitary operator; $R^\dagger = R^{-1}$. From this one constructs the orthogonal projection operator

$$P = \frac{1}{N}\sum_{k=0}^{N-1} R^k$$

satisfying $P^2 = P$; $P^\dagger = P$. The $\mathbb{C}/\mathbb{Z}_N$ Hilbert space is then $\mathcal{H}_{\mathbb{C}/\mathbb{Z}_N} = P\mathcal{H}_{\mathbb{C}}$. Defining complexified momenta

$$p = \frac{1}{\sqrt{2}}(p_8 - ip_9) \qquad \bar{p} = \frac{1}{\sqrt{2}}(p_8 + ip_9)\,,$$

we start from momentum states $|p,\bar{p}\rangle$ in $\mathcal{H}_{\mathbb{C}}$ normalized as

$$\langle p', \bar{p}'|p,\bar{p}\rangle = (2\pi)^2\delta(p-p')\delta(\bar{p}-\bar{p}').$$

The states

$$|p,\bar{p}\rangle_N \equiv P|p,\bar{p}\rangle = \frac{1}{N}\sum_{k=0}^{N-1}|\eta^k p, \bar{\eta}^k\bar{p}\rangle$$

with $\eta = e^{2\pi i/N}$ form a continuous basis on $\mathbb{C}/\mathbb{Z}_N$ with normalization

$$_N\langle p', \bar{p}'|p,\bar{p}\rangle_N = (2\pi)^2\delta_N(\vec{p},\vec{p}') \qquad (3.1)$$

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where we have defined

$$\delta_N(\vec{p},\vec{p}') \equiv \frac{1}{N}\sum_{k=0}^{N-1}\delta(p-\eta^k p')\delta(\bar{p}-\bar{\eta}^k\bar{p}')\,.$$

The completeness relation on $\mathcal{H}_{\mathbb{C}/\mathbb{Z}_N}$ reads

$$\mathbf{1} = \int_{\mathbb{C}}\frac{dpd\bar{p}}{(2\pi)^2}|p,\bar{p}\rangle_N\,{}_N\langle p,\bar{p}|\,.$$

For an arbitrary vertex operator $\mathcal{O}$, we denote its projection onto the $\mathbb{Z}_N$ invariant subspace by $[\mathcal{O}]_N$ defined by

$$[\mathcal{O}]_N \equiv \frac{1}{N}\sum_{k=0}^{N-1} R^k\mathcal{O}R^{-k}\,. \tag{3.2}$$

The vertex operators corresponding to the states $|p,\bar{p}\rangle_N$ are $[e^{i(pX+\bar{p}\bar{X})}]_N$. Their BPZ inner product is related to the hermitean inner product (3.1) on the Hilbert space by an overall normalization constant A and a sign change on one of the momenta

$$\langle [e^{i(p'X+\bar{p}'\bar{X})}]'_N(\infty)[e^{i(pX+\bar{p}\bar{X})}]_N(0)\rangle = A(2\pi)^2\delta^2_N(\vec{p},-\vec{p}') \tag{3.3}$$

where the prime means $\mathcal{O}'(\infty) = \lim_{z\to\infty} z^{2h}\bar{z}^{2\tilde{h}}\mathcal{O}(z)$ [58]. By construction, the overall normalization A should be the same as for the CFT on $\mathbb{C}$. We will later explicitly check this from unitarity.

The full Hilbert space in the untwisted sector is built up by acting on the momentum eigenstates with creation operators $\alpha_{-\{m\}}$, $\bar{\alpha}_{-\{m\}}$, $\tilde{\alpha}_{-\{m\}}$, $\bar{\tilde{\alpha}}_{-\{m\}}$ and then taking the $\mathbb{Z}_N$ invariant combinations. A general state can be written as

$$\mathcal{O}^{\{\bar{m}\}\{\bar{\tilde{m}}\}}_{\{m\}\{\tilde{m}\}}(p,\bar{p}) = \left[\prod_i(\alpha_{-i})^{m_i}\prod_j(\bar{\alpha}_{-j})^{\bar{m}_j}\prod_k(\tilde{\alpha}_{-k})^{m_k}\prod_l(\bar{\tilde{\alpha}}_{-l})^{m_l}\cdot e^{i(pX+\bar{p}\bar{X})}\right]_N\,.$$

The *level* of a state is defined as the pair (M,N) with $M=\sum(m_i+\bar{m}_i)$, $N=\sum(\tilde{m}_i+\bar{\tilde{m}}_i)$. Of particular interest to us are the lowest level operators which are part of the vertex operators for massless string states:

$$\begin{aligned}
\text{level}(0,0):&\ \ \mathcal{O}^{00}_{00}\\
\text{level }(1,0):&\ \ \mathcal{O}^{00}_{10},\,\mathcal{O}^{10}_{00}\\
\text{level }(0,1):&\ \ \mathcal{O}^{00}_{01},\,\mathcal{O}^{01}_{00}\\
\text{level }(1,1):&\ \ \mathcal{O}^{00}_{11},\,\mathcal{O}^{11}_{00},\,\mathcal{O}^{01}_{10},\,\mathcal{O}^{10}_{01}\,.
\end{aligned} \tag{3.4}$$

Their explicit definition is

$$\begin{aligned}
\mathcal{O}^{00}_{00} &= [e^{i(pX+\bar{p}\bar{X})}]_N \\
\mathcal{O}^{00}_{10} &= [\alpha_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\,, & \mathcal{O}^{00}_{01} &= [\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{10}_{00} &= [\bar{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\,, & \mathcal{O}^{01}_{00} &= [\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{00}_{11} &= [\alpha_{-1}\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N, & \mathcal{O}^{11}_{00} &= [\bar{\alpha}_{-1}\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\\
\mathcal{O}^{01}_{10} &= [\alpha_{-1}\bar{\tilde{\alpha}}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\,, & \mathcal{O}^{10}_{01} &= [\bar{\alpha}_{-1}\tilde{\alpha}_{-1}\cdot e^{i(pX+\bar{p}\bar{X})}]_N\,.
\end{aligned}$$

For example, the vertex operator for $\mathcal{O}_{01}^{00}$ is given by

$$\mathcal{O}_{01}^{00}(p,\bar{p}) = i\sqrt{\frac{2}{\alpha'}}[\bar{\partial} X e^{i(pX+\bar{p}\bar{X})}]_N \,.$$

3.1.2 Primaries and descendants

For computing cubic vertices, We need to write the operators (3.4) in terms of primaries and descendants of the Virasoro algebra. For a primary field at level (M,N) we will use the notation $\mathcal{V}^{(M,N)}(p,\bar{p})$.[2]

The state at level $(0,0)$ is obviously a primary,

$$\mathcal{V}^{(0,0)} = \mathcal{O}_{00}^{00} = [e^{i(pX+\bar{p}\bar{X})}]_N \,. \tag{3.5}$$

At level $(1,0)$, there is a primary $\mathcal{V}^{(1,0)}$ and a descendant $L_{-1}\cdot\mathcal{V}_{(0,0)}$:

$$\begin{aligned} \mathcal{V}^{(1,0)} &= \frac{1}{\sqrt{2p\bar{p}}}\left(\bar{p}\mathcal{O}_{10}^{00} - p\mathcal{O}_{00}^{10}\right) \\ L_{-1}\cdot\mathcal{V}^{(0,0)} &= \sqrt{\frac{\alpha'}{2}}\left(\bar{p}\mathcal{O}_{10}^{00} + p\mathcal{O}_{00}^{10}\right) . \end{aligned} \tag{3.6}$$

where primary fields are delta-function normalized under the BPZ inner product as in (3.3). Similarly, at level $(0,1)$ we have:

$$\begin{aligned} \mathcal{V}^{(0,1)} &= \frac{1}{\sqrt{2p\bar{p}}}\left(p\mathcal{O}_{01}^{00} - \bar{p}\mathcal{O}_{00}^{01}\right) \\ \tilde{L}_{-1}\cdot\mathcal{V}^{(0,0)} &= \sqrt{\frac{\alpha'}{2}}\left(p\mathcal{O}_{01}^{00} + \bar{p}\mathcal{O}_{00}^{01}\right) . \end{aligned} \tag{3.7}$$

At level $(1,1)$, we find a primary and three descendants:

$$\begin{aligned} \mathcal{V}^{(1,1)} &= \frac{1}{2p\bar{p}}\left(p^2\mathcal{O}_{11}^{00} - p\bar{p}\mathcal{O}_{10}^{01} - p\bar{p}\mathcal{O}_{01}^{10} + \bar{p}^2\mathcal{O}_{00}^{11}\right) \\ \tilde{L}_{-1}L_{-1}\cdot\mathcal{V}^{(0,0)} &= \frac{\alpha'}{2}\left(p^2\mathcal{O}_{11}^{00} + p\bar{p}\mathcal{O}_{10}^{01} + p\bar{p}\mathcal{O}_{01}^{10} + \bar{p}^2\mathcal{O}_{00}^{11}\right) \\ \tilde{L}_{-1}\cdot\mathcal{V}^{(1,0)} &= \sqrt{\frac{\alpha'}{2p\bar{p}}}\left(p^2\mathcal{O}_{11}^{00} + p\bar{p}\mathcal{O}_{10}^{01} - p\bar{p}\mathcal{O}_{01}^{10} - \bar{p}^2\mathcal{O}_{00}^{11}\right) \\ L_{-1}\cdot\mathcal{V}^{(0,1)} &= \sqrt{\frac{\alpha'}{2p\bar{p}}}\left(p^2\mathcal{O}_{11}^{00} - p\bar{p}\mathcal{O}_{10}^{01} + p\bar{p}\mathcal{O}_{01}^{10} - \bar{p}^2\mathcal{O}_{00}^{11}\right) . \end{aligned} \tag{3.8}$$

3.2 Twisted sector

Twisted sector states are created by the insertion of *twist fields*. The bosonic twist field that creates a state in the k-th twisted sector is denoted by σ_k. The σ_k are primary fields of weight $h = \tilde{h} = \frac{1}{2}k/N(1-k/N)$. Their OPE's are given by

$$\begin{aligned} \partial X(z)\sigma_k(0) &\sim z^{-1+k/N}\tau_k(0) + \cdots \\ \partial\bar{X}(z)\sigma_k(0) &\sim z^{-k/N}\tau_k'(0) + \cdots \end{aligned} \tag{3.9}$$

[2]For the levels that we need to consider, there is only one primary field at each level so for our purposes there is no ambiguity in this notation.

where $\tau_k,\ \tau'_k$ are *excited twist fields*. In the presence of a twist field, the mode numbers of $\partial X,\ \partial\bar{X}$ get shifted:

$$\begin{aligned}\partial X &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \alpha_{m-k/N} z^{-m-1+k/N}\\ \partial\bar{X} &= -i\sqrt{\frac{\alpha'}{2}}\sum_m \bar{\alpha}_{m+k/N} z^{-m-1-k/N}\,.\end{aligned}\tag{3.10}$$

The state $|\sigma_k\rangle \equiv \sigma_k(0)|0\rangle$ is annihilated by all positive frequency modes. The commutation relations are

$$[\alpha_{n-k/N}, \bar{\alpha}_{m+k/N}] = \left(m + \frac{k}{N}\right)\delta_{m,-n}\,.$$

We take the BPZ inner product between twist fields to be normalized as

$$\langle \sigma'_{N-k}(\infty)\sigma_k(0)\rangle = A \tag{3.11}$$

where A is the same constant that appears in (3.3). This is just a convenient choice — a different normalization of the twist fields can be absorbed in the proportionality constant multiplying the vertex operators for strings in the twisted sectors. The latter is ultimately determined by unitarity as we discuss in section 4.

3.3 Four-twist correlation function

The four-twist amplitude is given by [55, 56, 57]

$$Z_4(z,\bar{z}) \equiv \langle \sigma'_{N-k}(\infty)\sigma_k(1)\sigma_{N-k}(z,\bar{z})\sigma_k(0)\rangle = AB|z(1-z)|^{-2\frac{k}{N}(1-\frac{k}{N})} I(z,\bar{z})^{-1} \tag{3.12}$$

where

$$\begin{aligned}I(z,\bar{z}) &= F(z)\bar{F}(1-\bar{z}) + \bar{F}(\bar{z})F(1-z)\,,\\ F(z) &= {}_2F_1\left(\frac{k}{N}, 1-\frac{k}{N}, 1, z\right).\end{aligned}$$

The overall normalization A is expressible as a functional determinant and is common to all amplitudes of the $X,\ \bar{X}$ CFT on the sphere, while B is a numerical factor which we will relate, through factorization, to the normalization of the two-twist correlator. We will frequently need the asymptotics for the hypergeometric function F:

$$\begin{aligned}F(z) &\sim 1\,, \qquad z\to 0\,,\\ F(1-z) &\sim -\frac{1}{\pi}\sin\frac{\pi k}{N}\ln\frac{z}{\delta}\,, \qquad z\to 0\,,\\ F(z) &\sim e^{\pi i k/N}\frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}z^{-k/N} - e^{-\pi i k/N}\frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}z^{-(1-k/N)}\,, \qquad z\to\infty\,,\\ F(1-z) &\sim \frac{\Gamma(1-2k/N)}{\Gamma^2(1-k/N)}z^{-k/N} + \frac{\Gamma(2k/N-1)}{\Gamma^2(k/N)}z^{-(1-k/N)}\,, \qquad z\to\infty\,,\end{aligned}\tag{3.13}$$

where δ is defined by

$$\ln\delta = 2\psi(1) - \psi\left(1-\frac{k}{N}\right) - \psi\left(\frac{k}{N}\right) \tag{3.14}$$

with $\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z)$. For $1-k/N$ small, $\ln\delta$ behaves as

$$\ln\delta = \frac{N}{N-k} + \mathcal{O}\left(\frac{N-k}{N}\right)^2 .$$

3.4 Three-point correlation functions

We are now ready to calculate various three-point functions of two twist fields and an untwisted field by factorizing the four-twist amplitude.[3] In a general CFT, the four-point amplitude can be expanded as

$$Z_4(z,\bar z) = \sum_p (C^p_{-+})^2 \mathcal{F}(p|z)\bar{\mathcal{F}}(p|\bar z) \tag{3.15}$$

where the sum runs over primary fields, C^p_{-+} are coefficients in the $\sigma_{N-k}\sigma_k$ OPE, and $\mathcal{F}(p|z)$, $\bar{\mathcal{F}}(p|\bar z)$ are the conformal blocks. The conformal blocks in turn can be expanded for small z as

$$\begin{aligned} \mathcal{F}(p|z) &= z^{h_p - k/N(1-k/N)}\left(1+\frac{1}{2}h_p z + \mathcal{O}(z^2)\right) \\ \bar{\mathcal{F}}(p|\bar z) &= \bar z^{h_p - k/N(1-k/N)}\left(1+\frac{1}{2}\tilde h_p \bar z + \mathcal{O}(\bar z^2)\right). \end{aligned} \tag{3.16}$$

For a discussion and derivation of this formula see for example [59].

In our case, the sum over primaries in (3.15) is in fact a discrete sum over primaries at different levels as well as an integral over 'momenta' $(p,\bar p)$. Therefore, to order $|z|^2$ we have

$$\begin{aligned} Z_4(z,\bar z) = & \int_{\mathbb{C}} \frac{dpd\bar p}{(2\pi)^2 A}|z|^{\alpha' p\bar p - 2k/N(1-k/N)} \times \\ & \times\Bigg[(C^{(0,0)}_{-+}(p,\bar p))^2 + z\left(\frac{\alpha'}{4}p\bar p(C^{(0,0)}_{-+}(p,\bar p))^2 + (C^{(1,0)}_{-+}(p,\bar p))^2\right) \\ & \quad + \bar z\left(\frac{\alpha'}{4}p\bar p(C^{(0,0)}_{-+}(p,\bar p))^2 + (C^{(0,1)}_{-+}(p,\bar p))^2\right) + \\ & \quad + |z|^2\Bigg(\left(\frac{\alpha' p\bar p}{4}\right)^2 (C^{(0,0)}_{-+}(p,\bar p))^2 + (C^{(1,1)}_{-+}(p,\bar p))^2 \\ & \qquad + \frac{\alpha' p\bar p}{4}((C^{(1,0)}_{-+}(p,\bar p))^2 + (C^{(0,1)}_{-+}(p,\bar p))^2)\Bigg) + \cdots\Bigg]. \end{aligned} \tag{3.17}$$

The coefficients $C^{(M,N)}_{-+}$ are the three-point functions

$$C^{(M,N)}_{-+}(p,\bar p) = \langle \sigma'_{N-k}(\infty)\sigma_k(1)\mathcal{V}^{(M,N)}(p,\bar p)(0)\rangle$$

that we are interested in.

[3] Some of the correlation functions have been computed independently in [47].

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Factorization implies that the above expression for Z_4 should equal the expansion of (3.12) for small z which is given by

$$
\begin{aligned}
Z_4(z,\bar z) &= \frac{AB\pi}{4\sin(\frac{\pi k}{N})}|z|^{-2k/N(1-k/N)}\left(-\frac{1}{\log\frac{|z|}{\delta}}+\frac{a(z+\bar z)}{(\log\frac{|z|}{\delta})^2}-\frac{2a^2|z|^2}{(\log\frac{|z|}{\delta})^3}+\dots\right)\\
&= \frac{\alpha' AB}{2\sin\pi k/N}\int_{\mathbb{C}} dp d\bar p |z|^{\alpha' p\bar p-2k/N(1-k/N)}\delta^{-\alpha' p\bar p}\times\\
&\quad\times\left[1+a\alpha' p\bar p(z+\bar z)+(a\alpha' p\bar p)^2|z|^2+\dots\right] \qquad (3.18)
\end{aligned}
$$

where

$$
a=\frac{1}{2}\left(\left(\frac{k}{N}\right)^2+\left(1-\frac{k}{N}\right)^2\right). \qquad (3.19)
$$

In the second line of (3.18), we have used the identity

$$
\left(\log\frac{|z|}{\delta}\right)^{-(n+1)} = \frac{(-\alpha')^{n+1}}{2\pi n!}\int_{\mathbb{C}} dp d\bar p (p\bar p)^n\left(\frac{|z|}{\delta}\right)^{\alpha' p\bar p}.
$$

Comparing 3.17 and 3.18, we can now read off various operator product coefficients. For the first coefficient we find

$$
(C^{(0,0)}_{-+}(p,\bar p))^2 = \frac{\pi^2\alpha' A^2 B}{\sin\pi k/N}\delta^{-\alpha' p\bar p}.
$$

Using the fact that

$$
C^{(0,0)}_{-+}(0,\bar 0) = A
$$

as in (3.5) and (3.11) we can determine the numerical constant B that appears in 3.12,

$$
B=\frac{\sin\frac{\pi k}{N}}{\pi^2\alpha'}. \qquad (3.20)
$$

Further comparing (3.17) and (3.18) determines the higher operator product coefficients up to signs:

$$
\begin{aligned}
C^{(0,0)}_{-+}(p,\bar p) &= A\delta^{-\alpha' p\bar p/2}\\
C^{(1,0)}_{-+}(p,\bar p) = -C^{(0,1)}_{-+}(p,\bar p) &= \epsilon_1 A\frac{\sqrt{\alpha' p\bar p}}{2}\left(1-\frac{2k}{N}\right)\delta^{-\alpha' p\bar p/2}\\
C^{(1,1)}_{-+}(p,\bar p) &= \epsilon_2 A\frac{\alpha' p\bar p}{4}\left(1-\frac{4k}{N}\left(1-\frac{k}{N}\right)\right)\delta^{-\alpha' p\bar p/2}. \qquad (3.21)
\end{aligned}
$$

where $\epsilon_{1,2}=\pm 1$ are the sign ambiguities that arise from taking square roots. In the second line, we have taken the opposite sign for the coefficients as required by worldsheet parity, which takes $\mathcal{V}_{(1,0)}\to\mathcal{V}_{(0,1)}$ and $\sigma_k\to\sigma_{N-k}$. The three-point functions involving descendants are easily calculated from the ones involving primaries using the fact that L_{-1} and $\tilde L_{-1}$ act on vertex operators as ∂ and $\bar\partial$ respectively.

These results are easily transformed to the α-oscillator basis. The operators we need were denoted by $\mathcal{O}^{\bar{m}\bar{\tilde{m}}}_{m\tilde{m}}$ in section 3.1. Using the formulas (3.5)–(3.8) to transform to the α-oscillator basis one finds the required three-point functions. Using the notation

$$D^{\bar{m}\bar{\tilde{m}}}_{m\tilde{m}}(p,\bar{p},k) \equiv \langle \sigma'_{N-k}(\infty)\sigma_k(1)\mathcal{O}^{\bar{m}\bar{\tilde{m}}}_{m\tilde{m}}(p,\bar{p})(0)\rangle$$

we find

$$\begin{aligned}
D^{00}_{00} &= A\delta^{-\alpha' p\bar{p}/2} \\
D^{00}_{10} &= \sqrt{\frac{\alpha'}{2}} A\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1+\epsilon_1) - \frac{\epsilon_1 k}{N}\right) \\
D^{10}_{00} &= \sqrt{\frac{\alpha'}{2}} Ap\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1-\epsilon_1) + \frac{\epsilon_1 k}{N}\right) \\
D^{00}_{01} &= \sqrt{\frac{\alpha'}{2}} A\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1-\epsilon_1) + \frac{\epsilon_1 k}{N}\right) \\
D^{01}_{00} &= \sqrt{\frac{\alpha'}{2}} Ap\delta^{-\alpha' p\bar{p}/2}\left(\frac{1}{2}(1+\epsilon_1) - \frac{\epsilon_1 k}{N}\right) \\
D^{00}_{11} &= \frac{\alpha'}{8} A\bar{p}^2\delta^{-\alpha' p\bar{p}/2}\left(\epsilon_2\left(1-\frac{4k}{N}\left(1-\frac{k}{N}\right)\right)+1\right) \\
D^{11}_{00} &= \frac{\alpha'}{8} Ap^2\delta^{-\alpha' p\bar{p}/2}\left(\epsilon_2\left(1-\frac{4k}{N}\left(1-\frac{k}{N}\right)\right)+1\right) \\
D^{01}_{10} &= \frac{\alpha'}{8} Ap\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(-\epsilon_2\left(1-\frac{4k}{N}\left(1-\frac{k}{N}\right)\right)+1+2\epsilon_1\left(1-\frac{2k}{N}\right)\right) \\
D^{10}_{01} &= \frac{\alpha'}{8} Ap\bar{p}\delta^{-\alpha' p\bar{p}/2}\left(-\epsilon_2\left(1-\frac{4k}{N}\left(1-\frac{k}{N}\right)\right)+1-2\epsilon_1\left(1-\frac{2k}{N}\right)\right).
\end{aligned} \tag{3.22}$$

We later argue that the correct choice for our purpose is $\epsilon_1 = \epsilon_2 = -1$ which makes the total cubic string vertex BRST-invariant.

Note that this method to calculate the three-point function from factorization works only as long as there is only one primary field at each level. At level $(2,0)$, for example, one finds that there are two primaries and their correlators cannot be determined from factorization only. Computation of these higher correlators is substantially more difficult but fortunately we do not require them here. We need to subtract only massless exchanges and the higher level primary fields do not enter into the massless vertex operators. Equipped with the four and three point correlation functions we are thus ready to discuss the tachyon interactions.

4. Strings on $\mathbb{C}/\mathbb{Z}_N$

We now consider type-IIA/B strings on $\mathbf{M}^8 \times \mathbb{C}/\mathbb{Z}_N$ with $\mathbf{M}^8$ representing 7+1 dimensional Minkowski space. After reviewing the conventions and the vertex operators, we write the four point scattering amplitude. We then write the gauge invariant cubic interaction between the tachyon and the massless fields and finally determine the one-particle irreducible effective quartic interaction.

4.1 Conventions

We use the coordinates $X^M = (X^\mu, X, \bar{X})$ where $\mu = 0, \ldots, 7$ and $X = \frac{1}{\sqrt{2}}(X^8 + iX^9)$. Similarly, the world-sheet fermions are $\psi^M = (\psi^\mu, \psi, \bar{\psi})$ with $\psi = \frac{1}{\sqrt{2}}(\psi^8 + i\psi^9)$. Our metric signature is $(-++\cdots+)$. The orbifold $\mathbb{C}/\mathbb{Z}_N$ represents a cone with opening angle $2\pi/N$. The $\mathbb{Z}_N$ generator is given by

$$R = \exp\left(2\pi i \frac{N+1}{N} J_{89}\right).$$

where J_{89} generates rotations in the X^8, X^9 plane. We take N to be odd so that $R^N = 1$ on spacetime fermions and the bulk tachyon is projected out by GSO projection [10].

In the untwisted sector, one has to project onto $\mathbb{Z}_N$ invariant states, for which we use the notation $[\ldots]_N$:

$$[\mathcal{O}]_N \equiv \frac{1}{N} \sum_{k=0}^{N-1} R^k \mathcal{O} (R^{-1})^k \,. \tag{4.1}$$

The vertex operators in the sector twisted by R^k will contain bosonic twist fields σ_k and fermionic ones s_k. Their role is to create branch cuts in the OPEs with X, $\bar{X}$ and ψ, $\bar{\psi}$ respectively. In the presence of a twist field, mode numbers get shifted by an amount k/N.

The bosonic twisted sector has been discussed in detail in the previous section. The fermionic twist fields are denoted by s_k. Their OPEs are

$$\begin{aligned} \psi(z) s_k(0) &\sim z^{k/N} t_k + \cdots \\ \bar{\psi}(z) s_k(0) &\sim z^{-k/N} t'_k + \cdots \,. \end{aligned} \tag{4.2}$$

The mode expansions in the NS sector are

$$\begin{aligned} \psi(z) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_{r-k/N} z^{-r-\frac{1}{2}+k/N} \\ \bar{\psi}(z) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \bar{\psi}_{r+k/N} z^{-r-\frac{1}{2}-k/N} \end{aligned} \tag{4.3}$$

with commutation relations

$$\{\psi_{r-k/N}, \bar{\psi}_{s+k/N}\} = \delta_{r,-s}.$$

These twist fields have a free field representation in terms of bosonized fields H, $\tilde{H}$. The latter are defined by

$$\begin{aligned} \psi &= e^{iH}\,, & \bar{\psi} &= e^{-iH} \\ \tilde{\psi} &= e^{i\tilde{H}}\,, & \bar{\tilde{\psi}} &= e^{-i\tilde{H}} \end{aligned} \tag{4.4}$$

The twist fields are then represented by

$$s_k = e^{i\frac{k}{N}H}\,, \qquad \tilde{s}_k = e^{-i\frac{k}{N}\tilde{H}}\,. \tag{4.5}$$

In this representation, the only computationally nontrivial CFT correlators are the ones involving the bosonic twist fields σ_k that were calculated in the previous section. General amplitudes are restricted by *quantum symmetry* $\hat{\mathbb{Z}}_N$ and *charge conservation* for the $H, \tilde{H}$ fields.

4.2 Tachyon spectrum and vertex operators

Let us review the GSO projection and the tachyonic spectrum [8, 10]. The vertex operator for the k-th twisted sector ground state, in the -1 picture, is

$$\sigma_k e^{i\frac{k}{N}(H-\tilde{H})} e^{-\phi-\tilde{\phi}} .$$

For k odd, this ground state is not projected out by the GSO projection and the lowest lying mode is a tachyon with mass $m^2 = -2/\alpha'(1-\frac{k}{N})$. Its vertex operator is

$$T_k^{(-1,-1)}(z,\bar{z},p) = g_c' e^{ip\cdot X} c\tilde{c}\sigma_k e^{i\frac{k}{N}(H-\tilde{H})} e^{-\phi-\tilde{\phi}}(z,\bar{z}) \qquad k \text{ odd} . \tag{4.6}$$

The normalization constant g_c' will be determined in terms of the closed string coupling g_c from factorization and the requirement that the tachyon vertex operator represents a canonically normalized field in $7+1$ dimensions.

For k even, the ground state is projected out by GSO projection and the tachyon is an excited state $\bar{\psi}_{-\frac{1}{2}+k/N}\tilde{\bar{\psi}}_{-\frac{1}{2}+k/N}|0,p\rangle$ with vertex operator

$$T_k^{(-1,-1)}(z,\bar{z},p) = g_c' e^{ip\cdot X} c\tilde{c}\sigma_k e^{i(k/N-1)(H-\tilde{H})} e^{-\phi-\tilde{\phi}}(z,\bar{z}) \qquad k \text{ even} .$$

Its mass-shell condition is $\frac{-\alpha' p^2}{4} = -\frac{1}{2}\frac{k}{N}$. This vertex operator is the complex conjugate of T_{N-k} and we denote it by $\bar{T}_{N-k}$. Henceforth, we take k to be odd and describe all tachyons by the vertex operators T_k, $\bar{T}_k$. The most marginal tachyon has $k = N-2$ and $m^2 = -4/N\alpha'$.

We will also need the tachyon vertex operators in the 0 picture

$$T_k^{(0,0)}(z,\bar{z},p) = -\frac{\alpha'}{2} g_c' e^{ip\cdot X} p\cdot\psi \; p\cdot\tilde{\psi} c\tilde{c}\sigma_k e^{i\frac{k}{N}(H-\tilde{H})}(z,\bar{z}) + \cdots .$$

The omission stands for terms that arise from the part of the picture-changing operator involving the $\mathbb{C}/\mathbb{Z}_N$ fields. These terms have different $(H,\tilde{H})$ charges from the one displayed. In the amplitudes we consider it is possible to choose pictures such that the omitted terms do not contribute because of H-charge conservation.

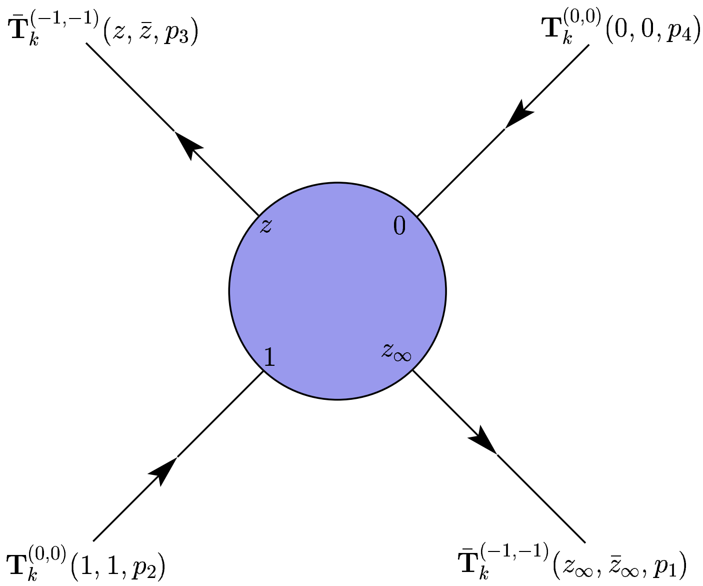
4.3 Scattering amplitude for four tachyons

We are now ready to write down the four-tachyon scattering amplitude. Using (3.12) it is given by

$$\begin{aligned} V_4 &= \int d^2z \langle \bar{T}_k^{(-1,-1)}(z_\infty,\bar{z}_\infty,p_1) T_k^{(0,0)}(1,1,p_2) \bar{T}_k^{(-1,-1)}(z,\bar{z},p_3) T_k^{(0,0)}(0,0,p_4)\rangle \\ &= \frac{g_c'^4 C}{\pi^2\alpha'} \sin\left(\frac{\pi k}{N}\right) \Delta(\alpha' p_1\cdot p_3/2)^2 \int d^2z |1-z|^{-\frac{\alpha' t}{2}-2} |z|^{-\frac{\alpha' s}{2}-2} I(z,\bar{z})^{-1} , \end{aligned} \tag{4.7}$$

where C represents the product of A with similar functional determinants from the X^μ, the fermions and the ghosts. The assignment of pictures and worldsheet positions of the vertex operators are shown in figure 1. The Mandelstam variables s, t, u are defined as

$$s = -(p_1+p_2)^2 , \qquad t = -(p_2+p_3)^2 , \qquad u = -(p_1+p_3)^2$$



$$\bar{T}_k^{(-1,-1)}(z,\bar{z},p) = g'_c \sigma_{N-k} e^{-i\frac{k}{N}(H-\tilde{H})} c\tilde{c} e^{-\phi-\tilde{\phi}} e^{ip\cdot X(z,\bar{z})}$$
$$T_k^{(0,0)}(z,\bar{z},p) = -\tfrac{\alpha'}{2} g'_c \sigma_k e^{i\frac{k}{N}(H-\tilde{H})} c\tilde{c} p\cdot\psi(z,\bar{z}) p\cdot\tilde{\psi}(z,\bar{z}) e^{ip\cdot X(z,\bar{z})} + \ldots$$

Figure 1: Four-tachyon scattering amplitude.

and the symbol Δ to denotes the 8-dimensional momentum-conserving delta-function

$$\Delta \equiv i(2\pi)^8 \delta^8 \left(\sum p_i\right).$$

All momenta are incoming and the arrows indicate the flow of quantum $\hat{\mathbb{Z}}_N$ charge.

The asymptotics of $I(z,\bar{z})$ as $z \to 0$ or $z \to 1$ imply that V_4 does not have the usual pole structure in the s and t channels. This is a consequence of the noncompactness of the cone. In the s and t channels, the exchanged states are untwisted states which have momentum along the cone directions $(X, \bar{X})$ and a continuous mass spectrum from eight-dimensional point of view. These states can contribute because there is no translation invariance in those directions and hence momentum is not conserved. As a result, the poles are replaced by softer logarithmic divergences. The leading s-channel behavior comes from integrating near $z = 0$,

$$V_4^s \approx -\frac{{g'_c}^4 C}{2\pi\alpha'} \Delta \left(\alpha' p_1 \cdot \frac{p_3}{2}\right)^2 \int_0 d^2z \frac{|z|^{-\frac{\alpha' s}{2}-2}}{\log\frac{|z|}{\delta}}. \tag{4.8}$$

Integrating near $z = 1$ gives the leading t-channel contribution

$$V_4^t \approx -\frac{{g'_c}^4 C}{2\pi\alpha'} \Delta \left(\alpha' p_1 \cdot \frac{p_3}{2}\right)^2 \int_0 d^2z \frac{|z|^{-\frac{\alpha' t}{2}-2}}{\log\frac{|z|}{\delta}}. \tag{4.9}$$

In the u channel, which comes from the $z \to \infty$ region of the integral, the exchanged states are localized twisted sector states and there one gets a sum over pole terms. There are no

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massless (or nearly massless) poles in the u-channel; the first contribution comes from a massive exchange

$$V_4^u \approx -\frac{2{g'_c}^4 C}{\pi\alpha'} \tan\left(\frac{\pi k}{N}\right) \Delta\left(\alpha' p_1 \cdot p_3/2\right)^2 \frac{\Gamma^4(\frac{k}{N})}{\Gamma^2(\frac{2k}{N}-1)} \frac{1}{\alpha' u/2 + 2(2-3k/N)}\,. \tag{4.10}$$

4.4 Cubic couplings to other twisted states

We now show that, at least in the quartic approximation to the tachyon potential, a constant vev for one of the nearly-marginal tachyons does not generate a tadpole for any of the other nearly-massless fields in the twisted sectors. It is therefore consistent to neglect these states in the analysis to quartic order.

The couplings between twisted sector states are severely restricted by quantum $\hat{\mathbb{Z}}_N$ symmetry and H-charge conservation. Let us start with the three-point couplings. Possible tadpoles come from couplings $\langle \bar{T}_k^{(-1,-1)} \bar{T}_k^{(-1,-1)} \Phi^{(0,0)} \rangle$ where the superscript denotes the picture. Quantum symmetry and H-charge conservation imply that the zero-picture state $\Phi^{(0,0)}$ has to be proportional to $\sigma_{2k-N} e^{i2k/N(H-\tilde{H})}$. The lowest state with these quantum numbers has vertex operator

$$\Phi^{(0,0)} = e^{ip_\mu X^\mu} \sigma_{2k-N} e^{i2k/N(H-\tilde{H})} c\tilde{c}$$

which has mass-squared $m^2 = 4/\alpha'(-2+3k/N)$ which is of order of the string scale if k/N is close to one for example when $k = N-2$. This is consistent with what we find in equation (4.10). It shows that the lowest lying exchanged state in the u-channel has precisely this mass, and there are no poles from exchanging tachyonic or nearly massless fields.

There are four-point couplings of the form $\langle \bar{T}_k^{(-1,-1)} T_k^{(0,0)} \bar{T}_k^{(-1,-1)} \Phi^{(0,0)} \rangle$ which could also source other nearly-massless tachyons. Again using quantum symmetry and H-charge conservation one sees that $\Phi^{(0,0)}$ is either equal to $T_k^{(0,0)}$ or proportional to $\sigma_k e^{i(k/N+1)(H-\tilde{H})}$. The lowest mass state with the latter quantum numbers has vertex operator

$$\Phi^{(0,0)} = e^{ip_\mu X^\mu} \sigma_k e^{i(k/N+1)(H-\tilde{H})} c\tilde{c}$$

with $m^2 = 2/\alpha'(-1+3k/N)$. This is again a massive state with string scale mass for $k/N \approx 1$.

It seems possible to generalize this argument for at least a large class of higher point functions but we restrict ourselves only up to quartic order.

4.5 Cubic coupling to untwisted massless fields

We now calculate the cubic vertex for two tachyons and one massless field from the untwisted sector. The vertex operator for a massless state with polarization tensor e_{MN} in the zero picture is

$$H^{(0,0)}(z,\bar{z},p,e) \equiv -g_c e_{MN} \frac{2}{\alpha'} \left[\left(i\partial X^M + \frac{\alpha'}{2} p\cdot\psi\psi^M \right) \left(i\bar{\partial} X^N + \frac{\alpha'}{2} p\cdot\tilde{\psi}\tilde{\psi}^N \right) c\tilde{c} e^{ip\cdot X}(z,\bar{z}) \right]_N . \tag{4.11}$$

Gravitons are described by a symmetric, traceless polarization tensor and B-field fluctuations correspond to an antisymmetric polarization tensor. The dilaton vertex operator requires a bit more care. In the (-1) picture, it is given by [60]

$$\frac{1}{\sqrt{8}}\left[(\psi\cdot\tilde{\psi}e^{-\phi-\tilde{\phi}}-\partial\xi\tilde{\eta}e^{-2\phi}-\partial\tilde{\xi}\eta e^{-2\tilde{\phi}})c\tilde{c}e^{ip\cdot X}\right]_N .$$

Applying the picture changing operator to this, we find that the zero-picture dilaton vertex operator is given by (4.11) with $e_{MN}=\frac{1}{\sqrt{8}}\eta_{MN}$ plus terms with either ϕ-charge different from zero or ghost number different from one. Such terms don't contribute to the three-point amplitude.

Using the three-point functions from (3.22) with $\epsilon_1=\epsilon_2=-1$ one finds the cubic coupling

$$\begin{aligned} V_3(p_1,p_2;p_3,e) &= \langle T_k^{(-1,-1)}(z_\infty,\bar{z}_\infty,p_1)\bar{T}_k^{(-1,-1)}(1,1,p_2)H^{(0,0)}(0,0,p_3,e)\rangle \\ &= -\frac{\alpha' {g'_c}^2 g_c C}{2}\delta^{\frac{-\alpha' p_3\bar{p}_3}{2}}\Delta\Big(e_{\mu\nu}p_2^\mu p_2^\nu - e_{\mu X}p_2^\mu\bar{p}_3 - e_{\bar{X}\mu}p_2^\mu p_3 + e_{\bar{X}X}p_3\bar{p}_3\Big)\,. \end{aligned} \tag{4.12}$$

Note that this expression is not symmetric in the polarization indices; this means that there is a coupling to the B-field as well as to the graviton and dilaton. One way to motivate the choice $\epsilon_1=\epsilon_2=-1$ in (3.22) is that it is the only one that leads to a BRST-invariant amplitude; indeed one easily checks that (4.12) is invariant under

$$e_{MN}\to e_{MN}+p_{3M}a_N+p_{3N}b_M \tag{4.13}$$

upon using $p_1^2=p_2^2=-m^2$, $p_3^2=0$. In (4.12), the graviton is in the transverse-traceless gauge. In order to compute the graviton exchange diagram we would like to know the correct vertex to use in the harmonic gauge $p^M e_{(MN)}-\frac{1}{2}p_N e_M^M=0$ for the graviton. In this gauge there is more residual gauge invariance and one would typically expect to have to add terms proportional to $p^M e_{(MN)}$ and e_M^M until the vertex is invariant under the larger set of gauge transformations. In our case however we saw that (4.12) is already invariant under (4.13) without imposing $a\cdot p_3=b\cdot p_3=0$. Hence (4.12) is the correct vertex to use in the harmonic gauge. Another proof of the validity of (4.12) in the harmonic gauge will be given in [61].

The final form of the cubic coupling (4.12) shows that the tachyon has a gaussian form factor in its coupling to the massless untwisted fields. At large-N, the width of the gaussian in position space scales as $\sqrt{N}$. The coupling is thus not point-like but spread over a very large radius of order $\sqrt{N}$ times the string length. Because the opening angle of the cone is also getting smaller as $1/N$, the total area over which this interaction takes place is still of order one in string units.

5. Unitarity and massless exchanges

In this section, we first determine the over-all normalization from unitarity in section 5.1, and then compute the massless exchange diagrams in 5.2. By subtracting these exchanges from the four tachyon scattering amplitude we can extract a possible quartic contact term.

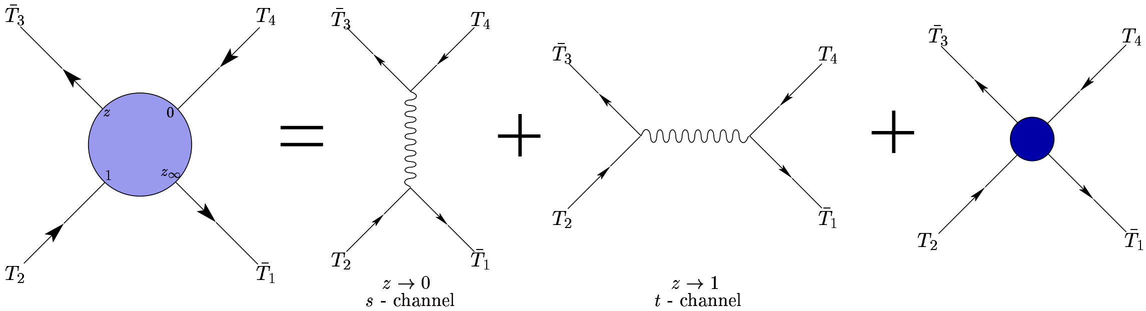

Figure 2: Factorization and the quartic contact interaction.

The Feynman diagram is shown in figure 2. We find that in the u-channel only massive particles of string-scale mass are exchanged consistent with (4.10). Hence we need to subtract only the s and t channel exchanges. As explained in 2.3, the quartic contact term on the right hand side of figure 2 is the effective quartic term at low energy and includes the exchanges of particles of string scale mass.

5.1 Determination of normalization constants

So far we have introduced a number of undetermined normalization constants:

C: overall normalization of the path integral on the sphere.

g_c: normalization of the graviton vertex operator or the 'closed string coupling'.

g'_c: normalization of the tachyon vertex operator.

Note that the constant A that was introduced for the $X\bar{X}$ CFT in (3.11) is absorbed in C along with other functional determinants and the constant B is already determined in (3.20).

Unitarity of the S-matrix allows us to express all constants in terms of α' and g_c. The latter is in turn proportional to the gravitational constant κ. We now work out these relations keeping track of possible factors of N.

The constant C can be expressed in terms of α' and g_c by factoring the four-tachyon amplitude on graviton exchange. From (4.12) we calculate the contribution to the 4-point function coming from the exchange of longitudinal gravitons,

$$V_4^{exch} = -i \int \frac{d^8 p^\mu}{(2\pi)^8} \int_{\mathbb{C}} \frac{dp d\bar{p}}{(2\pi)^2} \frac{V^3_{\mu\nu}(p_1, p_2; p) V^{3\mu\nu}(p_3, p_4; p)}{p^\mu p_\mu + 2p\bar{p}} \tag{5.1}$$

$$= -{g'_c}^4 g_c^2 C^2 \frac{1}{16\pi^2} \Delta(\alpha' p_1 \cdot p_3/2)^2 \int_0 d^2 z \frac{|z|^{-\frac{\alpha' s}{2} - 2}}{\log \frac{|z|}{\delta}} \,. \tag{5.2}$$

In writing the momentum space propagator in the first line, we have assumed a specific normalization for the spacetime field created by the vertex operator with normalization g_c. The normalization we used is appropriate for a canonically normalized field on the covering space $\mathbf{M}^8 \times \mathbb{C}$ which is periodic under $x \to e^{2\pi i/N}x$. The details are explained in appendix A. Comparing (5.2) with (4.8) we find the overall normalization

$$C = \frac{8\pi}{\alpha' g_c^2}\,. \tag{5.3}$$

This is the familiar flat-space value as it should be by construction.

The normalization of the tachyon vertex operator g_c' can be determined in terms of g_c by factoring the 2 graviton-2 tachyon amplitude on the pole coming from tachyon exchange:

$$W_4 \equiv \int d^2z \langle T_k^{(0,0)}(z_\infty, \bar{z}_\infty, p_1) H^{(-1,-1)}(1,1,p_2,e_2) \bar{T}_k^{(0,0)}(z,\bar{z},p_3) H^{(-1,-1)}(0,0,p_4,e_4)\rangle\,. \tag{5.4}$$

For simplicity we can take the graviton to have polarization along the longitudinal X^μ directions. For $z \to 0$, there is a pole at $s = -2/\alpha'(1-k/N)$ coming from tachyon exchange. We can find the coefficient at the pole from the OPE of $\bar{T}_k^{(0,0)}$ with $H^{(-1,-1)}$. From the 3-point function (3.22) we know that

$$\sigma_{N-k}(z)[e^{i(px+\bar{p}\bar{x})}]_N(0) \sim |z|^{-\alpha' p\bar{p}} \delta^{-\alpha' p\bar{p}/2} \sigma_{N-k}\,.$$

Using this we find the OPE

$$\begin{aligned} \bar{T}_k^{(0,0)}(z,\bar{z},p_3) H^{(-1,-1)}(0,0,p_4,e_4) \sim\ & \frac{\alpha'}{2} g_c g_c' e_{4\mu\nu} p_3^\mu p_3^\nu |z|^{-s-m^2-2} \delta^{-\alpha' p_4 \bar{p}_4} \times \\ & \times e^{i(p_3+p_4)_\mu X^\mu} \sigma_{N-k} e^{-i/N(H-\tilde{H})} c\tilde{c} e^{-\phi-\tilde{\phi}}(0)\,. \end{aligned}$$

Substituting in (5.4) and integrating around $z=0$ gives the pole term

$$W_4 \sim \frac{-2\pi\alpha' g_c^2 {g_c'}^2 C \delta^{-\alpha'/2(p_3\bar{p}_3+p_4\bar{p}_4)}(e_{2\mu\nu}p_1^\mu p_1^\nu)(e_{4\rho\sigma}p_3^\rho p_3^\sigma)}{s+2/\alpha'(1-k/N)} \Delta\,.$$

Comparing with (4.12) we get

$${g_c'}^2 C = \frac{8\pi}{\alpha'}$$

and hence

$$g_c' = g_c\,. \tag{5.5}$$

An extra check of these relations is provided in appendix B where we compute the first massive exchange in the u-channel and find agreement with (4.10).

We have not yet determined the proportionality constant between the vertex operator normalization g_c and the gravitational coupling κ. To compare with the prediction (2.2) we should use κ which is the cubic coupling for gravitons canonically normalized on $\mathbf{M}^8 \times \mathbb{C}$. Hence, it is related to the closed string coupling as usual by

$$\kappa = 2\pi g_c\,. \tag{5.6}$$

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5.2 Massless exchange diagrams

Having obtained the cubic vertex for two tachyons and a massless field in (4.12), we can calculate the contribution of massless exchange diagrams to the 4-tachyon amplitude. These diagrams will contain integrals over the momentum along the cone; they will be of the form

$$I_n(s) \equiv \int_{\mathbb{C}} \frac{dp d\bar{p}}{(2\pi)^2} \frac{(p\bar{p})^{n-1}\delta^{-\alpha' p\bar{p}}}{-s+2p\bar{p}}$$
$$= -\frac{(-\alpha')^{1-n}(n-1)!}{16\pi^2} \int_{D_1} d^2z \frac{|z|^{-\frac{\alpha' s}{2}-2}}{(\log\frac{|z|}{\delta})^n}\,. \tag{5.7}$$

The second form is useful for comparing with the string amplitude (4.7). The domain D_1 is the unit disc.

Dilaton exchange. The vertex is

$$V_3^{\rm dil}(p_1,p_2;p_3) = -\frac{\kappa}{\sqrt{2}}(p_3\bar{p}_3 - m^2)\delta^{-\alpha' p_3\bar{p}_3/2}$$

and the dilaton propagator is given by:

$$-\frac{i}{p^M p_M}\,.$$

Hence we find the exchange amplitude

$$V_4^{\text{exch, dil}} = \frac{4\kappa^2}{8}\Delta(m^4 I_1 - 2m^2 I_2 + I_3)\,.$$

B-field exchange. The vertex is

$$V_3^B(p_1,p_2;p_3,e) = 2\kappa\Delta\left(e_{[\mu x]}p_2^\mu \bar{p}_3 + e_{[\bar{x}\mu]}p_2^\mu p_3 + e_{[x\bar{x}]}p_3\bar{p}_3\right)\delta^{-\alpha' p_3\bar{p}_3/2}\,.$$

The propagator is, in the Feynman gauge $p^M e_{[MN]} = 0$,

$$-\frac{i}{p^M p_M}\left(\frac{1}{2}\eta_{MR}\eta_{NS} - \frac{1}{2}\eta_{MS}\eta_{NR}\right).$$

This gives the exchange amplitude

$$V_4^{\text{exch, }B} = -4\kappa^2\Delta(p_2\cdot p_4 I_2 + \frac{1}{2}I_3)\,.$$

Graviton exchange. The vertex is given by

$$V_3^{\rm grav}(p_1,p_2;p_3,e) = -2\kappa\Delta\Big(e_{(\mu\nu)}p_2^\mu p_2^\nu - e_{(\mu x)}p_2^\mu\bar{p}_3 - e_{(\bar{x}\mu)}p_2^\mu p_3 + \frac{1}{2}e_{(\bar{x}x)}p_3\bar{p}_3\Big)\delta^{-\alpha' p_3\bar{p}_3/2}\,. \tag{5.8}$$

The propagator in the harmonic gauge $p^M e_{(MN)} - \frac{1}{2}p_N e^M_M = 0$ reads

$$-\frac{i}{p^M p_M}\left(\frac{1}{2}\eta_{MR}\eta_{NS} + \frac{1}{2}\eta_{MS}\eta_{NR} - \frac{1}{8}\eta_{MN}\eta_{RS}\right).$$

This gives the exchange amplitude

$$V_4^{\text{exch, grav}} = 4\kappa^2\Delta\left(\left((p_2\cdot p_4)^2 - \frac{m^4}{8}\right)I_1 + \frac{m^2}{4}I_2 + p_2\cdot p_4 I_2 + \frac{1}{2}I_3 - \frac{1}{8}I_3\right). \tag{5.9}$$

5.3 Subtractions and quartic term for the tachyon

Summing these contributions, we see that many terms cancel and we are left with

$$V_4^{\text{exch, total}} = 4\kappa^2 \Delta (p_1 \cdot p_3)^2 I_1(s) \,. \tag{5.10}$$

A similar term comes from the massless t-channel exchanges. These contributions yield precisely the asymptotics of the string amplitude (4.8) without extra terms finite at zero momentum. Such terms, if present, would have contributed to the quartic contact term for the tachyon. In fact, for the open string system studied in [54], the massless subtractions do yield such extra terms and, in that case, they give the leading contribution to the quartic tachyon potential.

The quartic tachyon coupling is given by integral (4.7) with the massless exchanges subtracted. The coefficient in front of the integral reads

$$\frac{g_c'^4 C\alpha'}{4\pi^2} \sin\frac{\pi k}{N} (p_1 \cdot p_3)^2 = \frac{\kappa^2}{2\pi^3} \sin\frac{\pi k}{N} (p_1 \cdot p_3)^2$$

which is of order $1/N^3$. We shall now show that the remaining integral is of order one. It is given by

$$J = \int_{\mathbb{C}} d^2 z |1-z|^{-\frac{\alpha' t}{2}-2} |z|^{-\frac{\alpha' s}{2}-2} I(z,\bar{z})^{-1} + \frac{\pi}{2\sin(\pi k/N)} \int_{D_1} \frac{|z|^{-\frac{\alpha' s}{2}-2} + |z|^{-\frac{\alpha' t}{2}-2}}{\ln\frac{|z|}{\delta}} \,.$$

The second term comes from subtracting the massless exchanges (5.10). In evaluating the first term in J for large-N, one should be careful because the hypergeometric function I does not converge uniformly. The function I approaches the value 2 everywhere except in the points $z=0,1$ (see (3.13)). In evaluating the integral numerically, one finds numerical convergence problems in the regions around $z=0,1$. A similar situation was encountered in [62]. We therefore split the integral in three parts: we cut out two small discs of radius ϵ around $z=0,1$ where we approximate the integrand by its asymptotics (3.13) which we can integrate analytically. The integral over the remainder of the complex plane will be easy to evaluate numerically using Mathematica. After summing the three integrals and subtracting the massless exchanges we take ϵ to zero.

Let us start with the integral near zero with the s-channel exchanges subtracted. The result is an integral over the unit disc with a small disc around $z=0$ removed:

$$\begin{aligned} J_{z=0} &= \frac{2\pi^2}{\sin(\pi k/N)} \int_\epsilon^1 dr \frac{r^{-\frac{\alpha' s}{2}-1}}{\ln\frac{|z|}{\delta}} \\ &= \frac{2\pi^2}{\sin(\pi k/N)} \delta^{-\frac{\alpha' s}{2}} \left(E_1\left(\frac{\alpha' s}{2}\ln\frac{\epsilon}{\delta}\right) - E_1\left(-\frac{\alpha' s}{2}\ln\delta\right)\right) \\ &\approx 2\pi \ln\epsilon + \mathcal{O}\left(\frac{1}{N^2}\right) \end{aligned} \tag{5.11}$$

Here, E_1 is the exponential integral $E_1(x) = \int_x^\infty dt e^{-t}/t$ and, in the last line, we have displayed the leading term at large-N. The integral around $z=1$ with the t-channel

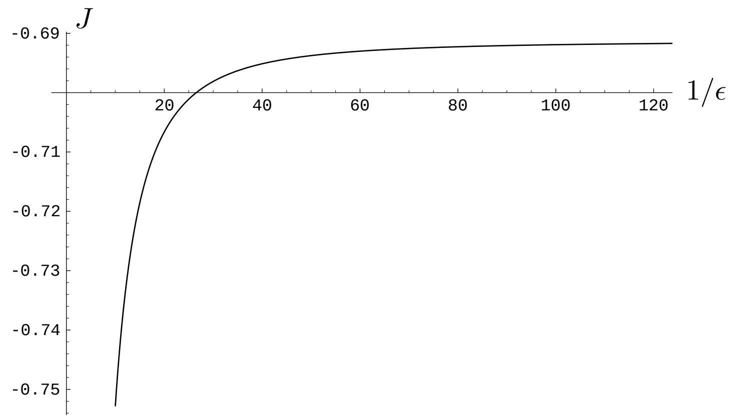

Figure 3: The integral J for $s, t = 0$ and $N = \infty$ as a function of $1/\epsilon$.

exchanges subtracted has the same leading behavior. So the leading term for J is

$$J \approx 4\pi \ln \epsilon + \frac{1}{2} \int_{\mathbb{C}\backslash \text{discs}} |1 - z|^{-2} |z|^{-2} \, .$$

The integral runs over the complex plane with small discs around $z = 0, 1$ removed. We have used that, in this integration region, the function I uniformly approaches the value of 2 at large N. We have also taken $s = t = 0$. The result of the numerical integration is plotted as a function of $1/\epsilon$ in figure 3. As ϵ goes to zero, J converges to

$$J \approx -0.691 \, .$$

6. Conclusions and comments

We have seen that the system of localized tachyons in $\mathbb{C}/\mathbb{Z}_N$ backgrounds provides a tractable system to study aspects of off-shell closed string theory. Condensation of these tachyons connects all $\mathbb{C}/\mathbb{Z}_N$ backgrounds to each other and to flat space. There is a well-defined conjecture for the height of the potential that is rather analogous to the open-string case. Moreover, considerable computational control is possible using orbifold CFT techniques.

We have analyzed the system in the large-N approximation where it is possible to read off the off-shell action from the S-matrix. We have been able to compute all three-point correlation functions required to describe the interaction of the tachyons with massless untwisted fields. Motivated by a simple model of the tachyon potential we compute the quartic contact term. If the quartic contact term is of order one, then the minimum can occur very close to the origin and higher point interactions can be consistently ignored. Our calculation however yields a quartic term that is much too small and goes instead as $1/N^3$. This implies that our simple model is not valid for describing the potential.

There are a number of possible ways to get around this problem. One possibility is that by going beyond quartic order one can find the new minimum of the potential of the desired depth. It is not clear however how one can obtain a very shallow minimum if higher order terms are important. Another more likely possibility is that the direction that we have chosen in the field space of tachyons is not the correct one for finding the minimum. Our choice of this specific tachyon was guided by the analysis of [13, 14] which indicates that to go from $\mathbb{C}/\mathbb{Z}_N$ orbifold to the $\mathbb{C}/\mathbb{Z}_k$ orbifold by RG-flow, one needs to turn on a specific relevant operator of definite charge k under the quantum symmetry. Our tachyon corresponds precisely to this relevant operator near the conformal point. This assumption is further supported by our finding in section 4.4 that turning on the tachyon of charge k does not source other tachyons and thus is a consistent approximation.

It is possible however that other excited tachyons are also involved in this process. There are a number excited states in the twisted sectors that are nearly massless or massless. The analysis of [14] is not sensitive to excited tachyons because it deals with only the chiral primaries. In particular, the analysis of [10] shows that to go from $\mathbb{C}/\mathbb{Z}_N$ to $\mathbb{C}/\mathbb{Z}_k$, the operator that is turned on in the quiver theory does not have definite charge under the quantum symmetry. It would be important to understand how these two pictures — the D-probe analysis and the RG-flows — are consistent with each other. This might help in identifying all the fields that are involved in the condensation [61].

A computation of the height of the potential for the $\mathbb{C}/\mathbb{Z}_N$ tachyons was attempted earlier in [63] where a large-N approximation was made inside the integral over the world-sheet coordinate z. As we have seen, this approximation is, however, not uniform over the region of integration and breaks down near $z = 0, 1$. As a result one obtains poles for massless exchanges instead of the correct softer logarithmic behavior that we have found. Moreover, the subtractions made in [63] were based on a postulated effective action which differs substantially from the actual cubic interactions that result from our computations.

Acknowledgments

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A. Feynman rules on the cone

In order to compare string amplitudes with the momentum space Feynman rules of a field theory on $\mathbb{C}/\mathbb{Z}_N$, it is convenient to mimic the construction of section 3 and work with fields ϕ defined on $\mathbb{C}$ and with periodicity

$$\phi(\eta x, \bar{\eta}\bar{x}) = \phi(x, \bar{x}) \tag{A.1}$$

with $\eta = e^{2\pi i/N}$. Suitable basis states on the cone are

$$\Psi_{p,\bar{p}}(x,\bar{x}) = \frac{1}{N}\sum_{k=0}^{N-1} \exp i(\eta^k px + \bar{\eta}^k \bar{p}\bar{x})\,. \tag{A.2}$$

These are normalized just like the basis states of the CFT (3.1):

$$\int_{\mathbb{C}} dx d\bar{x} \Psi^*_{p',\bar{p}'} \Psi_{p,\bar{p}} = (2\pi)^2 \delta_N(\vec{p},\vec{p}')\,.$$

Momentum space Feynman rules are obtained by expanding fields in this basis:

$$\phi(x,\bar{x}) = \int_{\mathbb{C}} \frac{dp d\bar{p}}{(2\pi)^2} \tilde{\phi}(p,\bar{p}) \Psi_{p,\bar{p}}(x,\bar{x})\,.$$

In writing the factorization (5.1) we have assumed that the string vertex operators create spacetime fields with propagator $p^\mu p_\mu + 2p\bar{p}$. Hence we should compare the string amplitudes with Feynman rules for spacetime fields ϕ with canonically normalized kinetic term

$$\begin{aligned} S_{\rm kin} &= \int d^8x \int_{\mathbb{C}} dx d\bar{x} \phi(\partial^\mu \partial_\mu + 2\partial\bar{\partial})\phi\,, \\ &= \int d^8 p^\mu \int_{\mathbb{C}} \frac{dp d\bar{p}}{(2\pi)^2} \tilde{\phi}(p^\mu p_\mu + 2p\bar{p})\tilde{\phi}\,. \end{aligned} \tag{A.3}$$

B. Consistency check: massive exchange

We saw that there is a nonvanishing coupling $\langle \bar{T}_k^{(-1,-1)} \bar{T}_k^{(-1,-1)} \Phi^{(0,0)} \rangle$ with $\Phi^{(0,0)}$ a state with mass $m^2 = 4/\alpha'(-2 + 3k/N)$. We shall now show that its exchange diagram is in agreement with the string result (4.10).

First, we need the three-twist correlator (for $k/N > 1/2$)

$$C_{-,-,++} \equiv \langle \sigma'_{N-k}(\infty) \sigma_{N-k}(1) \sigma_{2N-2k}(0) \rangle\,.$$

This can be obtained from the $z \to \infty$ factorization limit of (3.12) which has intermediate states in the twisted sector. Using the asymptotics (3.13), the result is

$$|C_{-,-,++}| = A\sqrt{\frac{2}{\alpha'}} \frac{\sqrt{|\tan \pi k/N|}}{2\pi} \frac{\Gamma^2(k/N)}{|\Gamma(2k/N - 1)|}\,. \tag{B.1}$$

This agrees with the finite volume result (formula (4.47) in [55]) upon taking $\alpha' = 2$ and the momentum space volume $V_\Lambda = 1/4\pi^2$.

In the $(-1,-1)$ picture, the vertex operator for the massive state is

$$\Phi^{(-1,-1)} = \frac{N}{2k - N} \alpha_{-\frac{2k-N}{N}} \bar{\tilde{\alpha}}_{-\frac{2k-N}{N}} \cdot \sigma_{2k-N} e^{i\frac{2k-N}{N}(H - \tilde{H})} e^{ip^\mu X_\mu} c\tilde{c} e^{-\phi - \tilde{\phi}}\,.$$

One can check that this is indeed a physical state. In the $(0,0)$ picture, the relevant part of the vertex operator is

$$\Phi^{(0,0)} = \frac{2k - N}{N} \sigma_{2k-N} e^{i\frac{2k}{N}(H - \tilde{H})} e^{ip^\mu X_\mu} c\tilde{c} + \cdots\,.$$

Using the bosonic amplitude (B.1), we find for the three-point amplitude

$$\begin{aligned} M_3 &\equiv \langle \bar{T}_k^{(-1,-1)}{}'(\infty, p_1) \bar{T}_k^{(-1,-1)}(1, p_2) \mathcal{V}^{(0,0)}(0, p_3) \rangle \\ &= g_c'^3 C \sqrt{\frac{2}{\alpha'}} \frac{2k-N}{N} \Delta \frac{\sqrt{|\tan \pi k/N|}}{2\pi} \frac{\Gamma^2(k/N)}{|\Gamma(2k/N-1)|} . \end{aligned} \tag{B.2}$$

Note that one cannot decide from the on-shell 3-point function whether the coupling is derivative or nonderivative. The exchange diagram reads

$$A_4^{\text{massive exch}} = -g_c'^6 C^2 (\frac{2k-N}{N})^2 \frac{|\tan \pi k/N|}{4\pi^2} \frac{\Gamma^4(k/N)}{\Gamma^2(2k/N-1)} \Delta \frac{2/\alpha'}{u - 4/\alpha'(-2+3k/N)} . \tag{B.3}$$

This agrees with the string amplitude (4.10) at the u-channel pole; we also see from (4.10) that the three-point coupling is in fact a derivative one.

References

[1] B. Sathiapalan, *Duality in statistical mechanics and string theory*, *Phys. Rev. Lett.* **58** (1987) 1597.

[2] Y.I. Kogan, *Vortices on the world sheet and string's critical dynamics*, *Sov. Phys. JETP Lett.* **45** (1987) 709.

[3] J.J. Atick and E. Witten, *The hagedorn transition and the number of degrees of freedom of string theory*, *Nucl. Phys.* **B 310** (1988) 291.

[4] M. Dine, E. Gorbatov, I.R. Klebanov and M. Krasnitz, *Closed string tachyons and their implications for non- supersymmetric strings*, `hep-th/0303076`.

[5] I. Pesando, *On the effective potential of the Dp Dp-bar system in type-II theories*, *Mod. Phys. Lett.* **A 14** (1999) 1545 [`hep-th/9902181`].

[6] A. Sen, *Tachyon condensation on the brane antibrane system*, *J. High Energy Phys.* **08** (1998) 012 [`hep-th/9805170`].

[7] A. Sen, *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys.* **A 14** (1999) 4061 [`hep-th/9902105`].

[8] A. Dabholkar, *Strings on a cone and black hole entropy*, *Nucl. Phys.* **B 439** (1995) 650 [`hep-th/9408098`].

[9] D.A. Lowe and A. Strominger, *Strings near a rindler or black hole horizon*, *Phys. Rev.* **D 51** (1995) 1793 [`hep-th/9410215`].

[10] A. Adams, J. Polchinski and E. Silverstein, *Don't panic! Closed string tachyons in ALE space-times*, *J. High Energy Phys.* **10** (2001) 029 [`hep-th/0108075`].

[11] A. Dabholkar, *Tachyon condensation and black hole entropy*, *Phys. Rev. Lett.* **88** (2002) 091301 [`hep-th/0111004`].

[12] A. Dabholkar, *Quantum corrections to black hole entropy in string theory*, *Phys. Lett.* **B 347** (1995) 222 [`hep-th/9409158`].

[13] J.A. Harvey, D. Kutasov, E.J. Martinec and G. Moore, *Localized tachyons and RG flows*, `hep-th/0111154`.

[14] C. Vafa, *Mirror symmetry and closed string tachyon condensation*, hep-th/0111051.

[15] A. Dabholkar, *On condensation of closed-string tachyons*, *Nucl. Phys.* **B 639** (2002) 331 [hep-th/0109019].

[16] Y. Michishita and P. Yi, *D-brane probe and closed string tachyons*, *Phys. Rev.* **D 65** (2002) 086006 [hep-th/0111199].

[17] S.P. De Alwis and A.T. Flournoy, *Closed string tachyons and semi-classical instabilities*, *Phys. Rev.* **D 66** (2002) 026005 [hep-th/0201185].

[18] S.-J. Sin, *Tachyon mass, c-function and counting localized degrees of freedom*, *Nucl. Phys.* **B 637** (2002) 395 [hep-th/0202097].

[19] A. Basu, *Localized tachyons and the $g_c l$ conjecture*, *J. High Energy Phys.* **07** (2002) 011 [hep-th/0204247].

[20] T. Sarkar, *Brane probes, toric geometry and closed string tachyons*, *Nucl. Phys.* **B 648** (2003) 497 [hep-th/0206109].

[21] T. Suyama, *Closed string tachyons and RG flows*, *J. High Energy Phys.* **10** (2002) 051 [hep-th/0210054].

[22] E.J. Martinec, *Defects, decay and dissipated states*, hep-th/0210231.

[23] A. Armoni, E. Lopez and A.M. Uranga, *Closed strings tachyons and non-commutative instabilities*, *J. High Energy Phys.* **02** (2003) 020 [hep-th/0301099].

[24] Y.-H. He, *Closed string tachyons, non-supersymmetric orbifolds and generalised McKay correspondence*, *Adv. Theor. Math. Phys.* **7** (2003) 121 [hep-th/0301162].

[25] B.C. Da Cunha and E.J. Martinec, *Closed string tachyon condensation and worldsheet inflation*, *Phys. Rev.* **D 68** (2003) 063502 [hep-th/0303087].

[26] S. Minwalla and T. Takayanagi, *Evolution of D-branes under closed string tachyon condensation*, *J. High Energy Phys.* **09** (2003) 011 [hep-th/0307248].

[27] S.-J. Sin, *Comments on the fate of unstable orbifolds*, *Phys. Lett.* **B 578** (2004) 215 [hep-th/0308028].

[28] S. Groot Nibbelink and M. Laidlaw, *Stringy profiles of gauge field tadpoles near orbifold singularities, i. Heterotic string calculations*, *J. High Energy Phys.* **01** (2004) 004 [hep-th/0311013].

[29] S. Lee and S.-J. Sin, *Localized tachyon condensation and G-parity conservation*, *J. High Energy Phys.* **01** (2004) 054 [hep-th/0312175].

[30] G. Moore and A. Parnachev, *Localized tachyons and the quantum McKay correspondence*, hep-th/0403016.

[31] S.-g. Lee and S.-J. Sin, *Chiral rings and GSO projection in orbifolds*, *Phys. Rev.* **D 69** (2004) 026003 [hep-th/0308029].

[32] M. Headrick, *Decay of C/Z_n: exact supergravity solutions*, *J. High Energy Phys.* **03** (2004) 025 [hep-th/0312213].

[33] J.R. David, M. Gutperle, M. Headrick and S. Minwalla, *Closed string tachyon condensation on twisted circles*, *J. High Energy Phys.* **02** (2002) 041 [hep-th/0111212].

[34] M. Gutperle, M. Headrick, S. Minwalla and V. Schomerus, *Space-time energy decreases under world-sheet RG flow*, *J. High Energy Phys.* **01** (2003) 073 [hep-th/0211063].

[35] M. Headrick, S. Minwalla and T. Takayanagi, *Closed string tachyon condensation: an overview*, to appear.

[36] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, *J. High Energy Phys.* **03** (2000) 002 [hep-th/9912249].

[37] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, *Nucl. Phys.* **B 587** (2000) 147 [hep-th/0002211].

[38] A.A. Gerasimov and S.L. Shatashvili, *On exact tachyon potential in open string field theory*, *J. High Energy Phys.* **10** (2000) 034 [hep-th/0009103].

[39] D. Kutasov, M. Marino and G.W. Moore, *Some exact results on tachyon condensation in string field theory*, *J. High Energy Phys.* **10** (2000) 045 [hep-th/0009148].

[40] W. Taylor and B. Zwiebach, *D-branes, tachyons and string field theory*, hep-th/0311017.

[41] A. Sen, *Rolling tachyon*, *J. High Energy Phys.* **04** (2002) 048 [hep-th/0203211].

[42] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *J. High Energy Phys.* **12** (2003) 054 [hep-th/0304224].

[43] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *J. High Energy Phys.* **07** (2003) 045 [hep-th/0305159].

[44] J. McGreevy, J. Teschner and H. Verlinde, *Classical and quantum D-branes in 2D string theory*, *J. High Energy Phys.* **01** (2004) 039 [hep-th/0305194].

[45] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[46] B. Zwiebach, *Closed string field theory: quantum action and the B-V master equation*, *Nucl. Phys.* **B 390** (1993) 33 [hep-th/9206084].

[47] Y. Okawa and B. Zwiebach, *Twisted tachyon condensation in closed string field theory*, *J. High Energy Phys.* **03** (2004) 056 [hep-th/0403051].

[48] N. Berkovits, *Manifest electromagnetic duality in closed superstring field theory*, *Phys. Lett.* **B 388** (1996) 743 [hep-th/9607070].

[49] A. Dabholkar and C. Vafa, *tt* geometry and closed string tachyon potential*, *J. High Energy Phys.* **02** (2002) 008 [hep-th/0111155].

[50] K. Bardakci, *Dual models and spontaneous symmetry breaking*, *Nucl. Phys.* **B 68** (1974) 331.

[51] K. Bardakci and M.B. Halpern, *Explicit spontaneous breakdown in a dual model*, *Phys. Rev.* **D 10** (1974) 4230.

[52] K. Bardakci and M.B. Halpern, *Explicit spontaneous breakdown in a dual model, 2. N point functions*, *Nucl. Phys.* **B 96** (1975) 285.

[53] K. Bardakci, *Spontaneous symmetry breakdown in the standard dual string model*, *Nucl. Phys.* **B 133** (1978) 297.

[54] E. Gava, K.S. Narain and M.H. Sarmadi, *On the bound states of p- and $(p + 2)$-branes*, *Nucl. Phys.* **B 504** (1997) 214 [hep-th/9704006].

[55] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[56] S. Hamidi and C. Vafa, *Interactions on orbifolds*, *Nucl. Phys.* **B 279** (1987) 465.

[57] M. Bershadsky and A. Radul, *Conformal field theories with additional Z_n symmetry*, *Int. J. Mod. Phys.* **A 2** (1987) 165.

[58] J. Polchinski, *String theory, vol. 1. An introduction to the bosonic string*, Cambridge University Press, Cambridge 1998.

[59] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*, Springer, New York 1997.

[60] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[61] A. Adams, A. Dabholkar, M. Headrick and J. Raeymaekers, work in progress.

[62] I.R. Klebanov and E. Witten, *Proton decay in intersecting D-brane models*, *Nucl. Phys.* **B 664** (2003) 3 [hep-th/0304079].

[63] S. Sarkar and B. Sathiapalan, *Closed string tachyons on C/Z_n*, hep-th/0309029.