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STRINGS ON A CONE AND BLACK HOLE ENTROPY

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ABSTRACT

String propagation on a cone with deficit angle $2\pi(1-\frac{1}{N})$ is described by constructing a non-compact orbifold of a plane by a Z_N subgroup of rotations. It is modular invariant and has tachyons in the twisted sectors that are localized to the tip of the cone. A possible connection with the quantum corrections to the black hole entropy is outlined. The entropy computed by analytically continuing in N would receive contribution only from the twisted sectors and be naturally proportional to the area of the event horizon. Evidence is presented for a new duality for these orbifolds similar to the $R \to \frac{1}{R}$ duality.

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1. Introduction

In this paper we discuss the propagation of strings on a conical space. The chief motivation for this work stems from its possible application to computing the entropy of a black hole in string theory. In field theory, an efficient way to compute the entropy is via the Euclidean path integral on a cone. The leading contribution to the entropy comes from field modes near the horizon of the black hole which in Euclidean space corresponds to the tip of a cone. In order to understand this leading contribution, the details of the geometry of a specific black hole can be ignored and it is adequate to consider field propagation in a conical space. We would like to do something similar in string theory. This work should be regarded as a step in that direction. For special values of the deficit angle of the cone, it is easy to construct the corresponding string theory as a Z_N orbifold of the theory in flat space. We describe this construction in the next section. These orbifolds can also be viewed as describing string propagation in the background of a cosmic string. We shall also see some evidence for duality in a new guise. Moreover, for large N, there are nearly massless tachyons in the spectrum that are localized to the tip of the cone. These could be useful for studying the tachyon instability in string theory.

We shall now describe some basic aspects of black-hole entropy that will be important in the following discussion. There are three distinct notions of the entropy associated with a black hole. The Bekenstein-Hawking entropy is calculated using thermodynamics and quantum field theory in a fixed, classical background of a black hole [1,2]. It is given by

$$S_{BH} = \frac{A}{4G\hbar} \tag{1.1}$$

where A is the area of the event horizon of the black hole and G is the renormalized Newton's constant. The Gibbons-Hawking entropy [3], on the other hand, is obtained by evaluating the full functional integral of quantum gravity around a saddle point which represents the Euclidean continuation of the Schwarzschild solution. Remarkably, it also gives the same expression for the entropy as the Bekenstein-Hawking formula with the renormalized Newton's constant replaced by its bare value. It is natural to ask about the quantum corrections to the leading semiclassical formula. Susskind and Uglum [4] have argued that these quantum corrections can all be absorbed into the renormalization of Newton's constant. Thus the renormalized Gibbons-Hawking entropy equals the Bekenstein-Hawking entropy.

Both these derivations do not offer any statistical interpretation of the thermodynamic black hole entropy in terms of counting of states. 't Hooft [5] has advocated that the entropy of quantum fluctuations as seen by a Schwarzschild observer should account for the black hole entropy. In field theory this quantity is ultraviolet divergent. Several authors have found a similar divergence in Rindler spacetime which approximates the geometry of a large black hole very near the horizon [4,6,7,8,9,10]. There is a simple description of the leading divergence. A fiducial observer that is stationed at a fixed radial distance from the black hole, has to accelerate with respect to the freely falling observer in order not to fall into the black hole. Very near the horizon, the fiducial observer is like a Rindler observer in flat Minkowski space. As a result, she sees a thermal bath [11] at a position-dependent proper temperature $T(z) = \frac{1}{2\pi z}$ where z is the proper distance from the horizon. Using Planck's formula for a single massless boson we get the entropy density:

$$s(z) = \frac{4}{3}\frac{\pi^2}{30}(\frac{1}{2\pi z})^3 . \tag{1.2}$$

Note that we have been able to define the entropy density because entropy is an *extensive* quantity as it should be. However, the dominant contribution comes from the region near the horizon $z = 0$ and is not extensive but proportional to the area. If we put a cutoff on the proper distance at $z = \epsilon$ (or alternatively on proper temperature) the total entropy is:

$$\begin{aligned} S &= \int_\epsilon^\infty s(z) A dz \\ &= \frac{A}{360\pi\epsilon^2} \quad , \end{aligned} \tag{1.3}$$

where A is the area in the transverse dimensions. This is in agreement with the result obtained in [5,4] by other means. Because the thermal bath is obtained by tracing over states that are not accessible to the observer in the Rindler wedge, this entropy is also the same as the 'entropy of entanglement' [8,6,10,12] or the 'geometric entropy' [7,9]. For a massive field with mass m, there will be corrections to this formula which will be down by powers of $m\epsilon$.

It is not clear how this divergent quantity can equal the black hole entropy which is finite. There are other difficulties with identifying this entropy of entanglement with the Bekenstein-Hawking entropy. For expample, the entropy of entanglement has no classsical contribution and starts at one loop whereas the black hole entropy is inversely proportional to the coupling constant. Furthermore, the entropy of entanglement depends on the species and couplings of various particles in the theory whereas the black hole entropy does not. 't Hooft has argued that it is necessary to understand the ultraviolet structure of the theory in order to address these issues. He has conjectured that these difficulties will be resolved once the correct short distance structure is known. He has further suggested that this divergence of entropy in field theory is intimately related to the puzzle of loss of information in black hole evaporation. If the entropy does have a statistical interpretation in terms of counting of states then its divergence would suggest an infinite number of states associated with a finite mass black hole. As long as the black hole has an event horizon, it can apparently store an arbitrary amount of information in terms of correlations between the outgoing radiation and the high energy modes near the horizon. When the horizon eventually disappears, the information in these correlations is irretrievably lost.

It is almost impossible to test these ideas within field theory, especially when one is dealing with a nonrenormalizable theory such as quantum gravity. Fortunately, string theory offers a suitable framework for addressing this question. It is a perturbatively finite theory of quantum gravity and comes with a well-defined matter content. Moreover, Susskind [13,14] has argued that string theory may also possess some of the properties required for describing black hole evaporation without information loss. It is therefore of great interest to know how the ultraviolet behavior of the entropy is controlled in string theory. One can hope that string theory will illuminate this question in important ways.

In field theory, there is a general method of computing the entropy of entanglement using Euclidean path integral over a conical space. As we shall see in section three, the analogous formula in string theory for the entropy at G loop is

$$S_G = \frac{d(N\,A_G)}{dN}|_{N=1}\,. \tag{1.4}$$

where A_G is the vacuum amplitude at G loop in string perturbation theory on a conical space with deficit angle $\delta = 2\pi(1-\frac{1}{N})$. A major obstacle in using this formula is that the conical background in general does not satisfy the equations of motion because there is a curvature singularity at the tip of the cone. Luckily, in string theory, for special values of the deficit angle $\delta = 2\pi(1-\frac{1}{N})$ with integer N, the theory manages to be on-shell at least at tree level. We would like to use this fact to obtain the entropy by analytically continuing in N. Similar suggestion has been made also in [9]. We do not yet have a complete expression for the entropy and we wish to return to it in a subsequent publication [15]. However, the entropy computed this way already appears to have several desirable features. Moreover, a number of technical issues arise in the construction of the orbifold that are interesting in their own right. With this objective in mind, in the next section, we describe the propagation of strings on a cone for integer N. In section three, we discuss its relation to the computation of entropy.

2. Strings on a Cone

We first construct the bosonic orbifold from the uncompactified string theory in twenty-six dimensions. Our spacetime will be $M_{24} \times K_N$ where M_{24} is flat spacetime and K_N is a cone with deficit angle $2\pi(1-\frac{1}{N})$. We can further compactify some of the dimensions of the M_{24} if we so desire. The details of compactification will not be important. For non-integer N, this background is not a solution to the string equations of motion because when a string encounters the curvature singularity at the tip of the cone, it would develop a kink. For integer N, however, we can have consistent propagation of strings despite the curvature singularity.

In this case, we can tile the entire plane with N copies of the cone. The configurations on the plane that are symmetric under Z_N rotations define consistent string configurations on the cone [see fig. 1]. We can then regard K_N as an orbifold of the plane R_2/Z_N [16,17]. Notice that unlike the orbifolds considered in string compactification we are interested in an orbifold of a non-compact space. If instead of R_2 we consider a compact space like a torus [18], then we cannot take N to be arbitrary and the allowed orbifold groups are very limited. Moreover, in that case, there are more than one points with a conical singularity and the orbifold only locally looks like a cone.

Fig. 1: Configurations on the plane that define consistent string configurations on the cone with deficit angle $\frac{4\pi}{3}$. The solid and the dashed lines indicate strings with zero and nonzero winding number around the tip of the cone respectively.

The Hilbert space of the orbifold is obtained by first considering the theory on the plane which is conformal and modular invariant and then projecting onto Z_N invariant states. As is well known, we also have to include the twisted string states because the winding number on the plane around the tip of the cone is conserved only modulo N [fig. 1]. It is convenient to combine the co-ordinates of the plane into a complex boson $X = \frac{X_1 + iX_2}{\sqrt{2}}$ and $\bar{X} = \frac{X_1 - iX_2}{\sqrt{2}}$. The orbifold group then acts on X by multiplication by a phase $e^{\frac{2\pi i k}{N}}$. In the untwisted sector this field has the standard mode expansion:

$$X = x + p\tau + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n}{n}e^{-2in(\tau-\sigma)} + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n}{n}e^{-2in(\tau+\sigma)}. \tag{2.1}$$

and the ground states are labelled by the momenta in the plane $|p, \bar{p}\rangle$. The spectrum before projection is the same as the twenty-six dimensional string and consists of states with some number of creation operators acting on the ground states. For states with nonzero p and $\bar{p}$, the projection onto Z_N invariant states reduces the spectrum by a factor of N. For example, for $N = 3$, the states $\alpha_{-n}|p, \bar{p}\rangle$ are projected onto $\frac{1}{3}(\alpha_{-n}|p, \bar{p}\rangle + e^{\frac{2\pi i}{3}}\alpha_{-n}|e^{-\frac{2\pi i}{3}}p, e^{\frac{2\pi i}{3}}\bar{p}\rangle + e^{\frac{4\pi i}{3}}\alpha_{-n}|e^{-\frac{4\pi i}{3}}p, e^{\frac{4\pi i}{3}}\bar{p}\rangle)$. Thus, on a Z_N cone, we still have the same set of particles in the untwisted sector as in flat space, except that the allowed combinations of momenta are reduced N-fold. The zero momentum states, however, need to be treated differently. In this case, only those combinations of creation operators that are invariant under Z_N rotations are allowed. For example, $\alpha_{-n}|0,0\rangle$ is projected out but $\alpha_{-n}\bar{\tilde{\alpha}}_{-m}|0,0\rangle$ is allowed. As we shall see, it is the zero momentum sector that mixes with the twisted sectors under modular transformations. In the twisted sectors, the boson is subject to the boundary condition $X(\sigma+\pi,\tau)=e^{\frac{2\pi ik}{N}}X(\sigma,\tau)$, k=0,..., N-1. The mode expansion and the commutation relations are given by:

$$\begin{aligned} X &= \frac{i}{2}\sum_n \frac{\alpha_{n+\frac{k}{N}}}{n+\frac{k}{N}} e^{-2i(n+\frac{k}{N})(\tau-\sigma)} + \frac{i}{2}\sum_n \frac{\tilde{\alpha}_{n-\frac{k}{N}}}{n-\frac{k}{N}} e^{-2i(n-\frac{k}{N})(\tau+\sigma)} \,, \\ [\alpha_{m+\frac{k}{N}}, \bar{\alpha}_{n-\frac{k}{N}}] &= (m+\frac{k}{N})\delta_{m+n} \,, \qquad [\tilde{\alpha}_{m+\frac{k}{N}}, \bar{\tilde{\alpha}}_{n-\frac{k}{N}}] = (m+\frac{k}{N})\delta_{m+n} \,. \end{aligned} \tag{2.2}$$

The zero point energy for the $\frac{k}{N}$ moded (complex) boson is given by $\frac{1}{2}\frac{k}{N}(1-\frac{k}{N})-\frac{1}{12}$. The 22 untwisted bosons contribute $\frac{-11}{12}$. Consequently, we have $N-1$ tachyons, one in each twisted sector with mass-squared $\frac{4k}{N}(1-\frac{k}{N})-8$. Note that there are no zero modes in the expansion (2.2). As a result, these tachyons are localized to the tip of the cone but can have momenta in the remaining 24 dimensions.

The vacuum amplitude for the cone is not very different from the one in flat space. For the bosonic string in flat space, the G-loop amplitude is given by (see [19] [20] [21])

$$A_G \sim g^{2G-2}\mathrm{L}^{26}\int_{\mathcal{M}_G}\frac{d(WP)}{\mathrm{vol}(\mathrm{ker}P_1)}\left(\frac{2\pi^2\det{}'\Delta_g}{\int d^2z\sqrt{g}}\right)^{-13}\det{}'(P_1^\dagger P_1) \tag{2.3}$$

Here $d(WP)$ is the Weil-Petersson measure over the genus-G moduli space $\mathcal{M}_G$, Δ_g is the scalar Laplacian $-\frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b$ and P_1 is the operator that maps vectors into symmetric traceless two-tensors: $(P_1v)_{ab}=\nabla_a v_b+\nabla_b v_a - g_{ab}\nabla v$. The volume factor L^{26} comes from the zero modes of the scalar Laplacian, one power for each real boson. $\det{}'$ is the determinant only over nonzero modes. In the orbifold theory, one inverse power of the determinant in (2.3) coming from one complex free boson and two powers of L coming from the zero modes, get replaced by the orbifold partition function $\mathcal{Z}(N)$ for the field X, at genus G.

It is easy to write down the one-loop partition function explicitly. The world-sheet at one loop is a torus. The metric for a torus in a given conformal class is parametrized by a complex modular parameter τ: $ds^2=|d\sigma_1+\tau d\sigma_2|^2$ with $0\le\sigma_1,\sigma_2<1$. The partition function is given by the orbifold sum

$$\mathcal{Z}(N)=\sum_{k,l}\mathcal{Z}_{k,l}(N)\ . \tag{2.4}$$

Each term $\mathcal{Z}_{k,l}$ represents the path integral of a complex boson on a torus with twisted boundary conditions. The path integral gives one inverse power of a determinant $Det\left[{a \atop b}\right](\Delta_g)$ of the scalar Laplacian on a torus subject to the boundary conditions

$$X(\sigma_1+1\,,\,\sigma_2)=e^{\frac{2\pi ik}{N}}\,X(\sigma_1\,,\,\sigma_2)\,,\quad X(\sigma_1\,,\,\sigma_2+1)=e^{\frac{2\pi il}{N}}\,X(\sigma_1\,,\,\sigma_2)\,. \tag{2.5}$$

Instead of evaluating the bosonic determinant we shall evaluate a related quantity, $Det\left[{a \atop b}\right](\bigtriangledown^z_{-\frac{1}{2}})$ which is the determinant of a chiral Dirac operator. This will also be useful later when we discuss the superstring. It can be regarded as a path integral of a complex chiral fermion with boundary condition $\psi(\sigma_1+1\,,\,\sigma_2)=-e^{2\pi ia}\psi(\sigma_1\,,\,\sigma_2)$, and $\psi(\sigma_1\,,\,\sigma_2+1)=-e^{2\pi ib}\psi(\sigma_1\,,\,\sigma_2)$. It is straightforward to evaluate this determinant in the operator formalism [22]. Writing $q=e^{2\pi i\tau}$, and using the standard relation between the path integral and the operator formalism, it is equal to the trace $\operatorname*{Tr}_{\mathcal{H}}\,(h_b\,q^{H_a})$. H_a is the Hamiltonian of a chiral, twisted fermion:

$$H_a=\sum_{n=1}^{\infty}(n-\frac{1}{2}+a)d_n^\dagger d_n+(n-\frac{1}{2}-a)\bar{d}_n^\dagger\bar{d}_n+\frac{a^2}{2}-\frac{1}{24} \tag{2.6}$$

The fermionic oscillators satisfy the canonical anticommutation relations $\{d_n^\dagger,d_m\}=\delta_{mn}$ and $\{\bar{d}_n^\dagger,\bar{d}_m\}=\delta_{mn}$, and $\mathcal{H}$ is the usual Fock space representation of these commutations. The group Z_N acts on this Fock space through $hdh^{-1}=-e^{-2\pi ib}d$, $h\bar{d}h^{-1}=-e^{2\pi ib}\bar{d}$. The trace equals (up to an arbitrary phase)

$$e^{2\pi iab}q^{\frac{a^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}+a}e^{2\pi ib})\,(1+q^{n-\frac{1}{2}-a}e^{-2\pi ib})\ . \tag{2.7}$$

Using the product representation of the theta function $\vartheta\left[{a \atop b}\right](\tau)$ with characteristics [23] , we see that

$$Det\left[{a \atop b}\right](\bigtriangledown^z_{-\frac{1}{2}})=\operatorname*{Tr}_{\mathcal{H}}\,(h_b\,q^{H_a})=\frac{\vartheta\left[{a \atop b}\right](0|\tau)}{\eta(\tau)}\ , \tag{2.8}$$

where $\eta(\tau)$ is the Dedekind η function.

Now we return to the problem at hand. Up to zero modes, the chiral boson determinant is the inverse of the chiral fermion determinant. The orbifold sum (2.4) becomes

$$\mathcal{Z}(N)=\sum_{k,l=0}^{N-1}\left|\frac{\eta(\tau)}{\vartheta\left[{\frac{k}{N}+\frac{1}{2} \atop \frac{l}{N}+\frac{1}{2}}\right]}\right|^2 \tag{2.9}$$

The bosonic zero modes can give a divergence for some of these terms such as $\mathcal{Z}_{0,0}$, and should be treated more carefully. For the moment, we shall continue to treat them in a somewhat cavalier manner.

The Weil-Petersson measure for the torus is $\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}$ and vol(kerP_1) = Imτ. Using the standard expressions for the other determinants [21] we obtain

$$A_1(N) \sim \text{Area} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \quad |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\mathrm{Im}\tau|^{-11} \quad \frac{1}{N}\,\mathcal{Z}(N) \tag{2.10}$$

The factor of $\frac{1}{N}$ in this formula comes from the fact that the operator that projects onto Z_N invariant states is $\frac{1}{N}(1 + h + h^2 + \ldots + h^{N-1})$. The area here refers to the volume L^{24} of the transverse dimensions. The integration is over the genus-one moduli space $\mathcal{M}_1$ which is the fundamental domain of the modular group

$$|\tau| > 1\,, \; -\frac{1}{2} < \mathrm{Re}\tau < \frac{1}{2}\,, \; \mathrm{Im}\tau > 0\,. \tag{2.11}$$

The modular group $SL(2,\mathbb{Z})/Z_2$ for the torus is the group of disconnected diffeomorphisms. The co-ordinate transformations $(\sigma_1\,,\sigma_2) \rightarrow (d\sigma_1 + b\sigma_2\,, c\sigma_2 + a\sigma_1)$ with a, b, c, d integers and $ad - bc = 1$, transform the metric into a conformally inequivalent metric parametrized by a new modular parameter

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}\,, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon\, SL(2,\mathbb{Z}). \tag{2.12}$$

We have to divide the $SL(2,\mathbb{Z})$ by Z_2 because the elements $\{\mathbf{1}, -\mathbf{1}\}$ leave τ unchanged. In order that the theory does not suffer from global diffeomorphism anomaly, it is necessary that the integrand in the amplitude (2.10) be invariant under the action of the modular group. The modular group is generated by the elements $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -\frac{1}{\tau}$. Under these transformations the theta functions transform as

$$\begin{aligned} T: \quad & \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) \;\rightarrow\; e^{-\pi i a^2 - \pi i a}\vartheta\begin{bmatrix} a \\ a+b+\frac{1}{2} \end{bmatrix}(\tau) \\ S: \quad & \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) \;\rightarrow\; (-i\tau)^{\frac{1}{2}} e^{2\pi i a b}\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(\tau) \end{aligned} \;. \tag{2.13}$$

Moreover,

$$\begin{aligned} \vartheta\Big[{a+m \atop b+n}\Big](\tau) &= e^{2\pi i n a}\vartheta\Big[{a \atop b}\Big](\tau) \\ \vartheta\Big[{-a \atop -b}\Big](\tau) &= \vartheta\Big[{a \atop b}\Big](\tau) \end{aligned} \;. \tag{2.14}$$

Using these properties it is easy to check that (2.10) is modular invariant, and the modular integration can be restricted to the fundamental domain (2.11) .

Let us now move on to the superstring. For simplicity we consider the nonchiral type IIA superstring moving on $M_8 \times K_N$. It will be easiest to use the Green-Schwarz formalism. We fix the light-cone gauge by using two of the directions in M_8 and obtain the remaining theory as an orbifold. Before modding out by the orbifold group, we have the supersymmetric sigma model in flat space

$$\mathcal{L} = -\frac{1}{\pi}\partial_+ X^i \partial_- X^i - \frac{i}{\pi} S^a \partial_+ S^a - \frac{i}{\pi} \tilde{S}^{\dot{a}} \partial_- \tilde{S}^{\dot{a}} \ , \tag{2.15}$$

where the coordinates X^i transforms as a vector $\mathbf{8}_V$ of $SO(8)$ and S^a and $\tilde{S}^{\dot{a}}$ transform as spinors of $SO(8)$, $\mathbf{8}_L$ and $\mathbf{8}_R$ respectively. The orbifold group Z_N is a subgroup of planar rotations, so we shall use the decomposition $SO(8) \to SO(6) \times SO(2)$, or equivalently $SO(8) \to SU(4) \times U(1)$. The vector and the spinor representations then decompose as follows

$$\begin{aligned} \mathbf{8}_V &\to \mathbf{6}(0) + \mathbf{1}(1) + \mathbf{1}(-1) \\ \mathbf{8}_L &\to \mathbf{4}(\frac{1}{2}) + \bar{\mathbf{4}}(-\frac{1}{2}) \\ \mathbf{8}_R &\to \mathbf{4}(-\frac{1}{2}) + \bar{\mathbf{4}}(\frac{1}{2}) \end{aligned} \quad . \tag{2.16}$$

Here the numbers in the parentheses are the $U(1)$ charges. We have one boson X with charge 1, eight fermions $S^m, \tilde{S}^m$ with charge $\frac{1}{2}$, and their complex conjugates. The index m transforms in the $\mathbf{4}$ of $SU(4)$. The one-loop vacuum amplitude for the superstring is quite similar to (2.10) ,

$$A_1(N) \sim \text{Area} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\,\tau)^2} \quad |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\text{Im}\tau|^{-3} \quad \frac{1}{N}\, \mathcal{Z}(N) \ . \tag{2.17}$$

The area here refers to the volume L^8 of the transverse dimensions and the first factor in the integrand comes from the six real boson that are neutral under the Z_N rotations. Before discussing the orbifold partition function $\mathcal{Z}(N)$, we should point out an important subtlety for the superstring that has to do with the fact that fermions have half integer spin. A rotation through 2π does not bring a spacetime fermion back to itself; which means we really have to embed our Z_N not into $SO(2)$ but into a double cover of $SO(2)$. As a result, it will turn out that we must distinguish between even and odd N. Let us first consider the odd N theories. In this case we have a sum similar to the bosonic case

$$\mathcal{Z}(N_{odd}) = \sum_{k,l=1}^{N} \mathcal{Z}_{k,l}(N) \,. \tag{2.18}$$

Each term $Z_{k,l}$ is a partition function for the fields $X, S^m, \tilde{S}^m$ and their complex conjugates with twisted boundary conditions

$$\begin{array}{c} S^m(\sigma_1+1\,,\,\sigma_2) = e^{\frac{2\pi i k}{N}}\, S^m(\sigma_1\,,\,\sigma_2)\,, \quad \tilde{S}^m(\sigma_1+1\,,\,\sigma_2) = e^{-\frac{2\pi i k}{N}}\, S^m(\sigma_1\,,\,\sigma_2)\,, \\ X(\sigma_1+1\,,\,\sigma_2) = e^{\frac{4\pi i k}{N}}\, X(\sigma_1\,,\,\sigma_2) \end{array} \,, \tag{2.19}$$

and similarly in the σ_2 direction. Altogether, we have four fermionic and one bosonic determinants. Using the formula (2.8) for the determinants we obtain,

$$\mathcal{Z}(N_{odd}) = \sum_{k,l=1}^{N} \left| \frac{\vartheta^4 \begin{bmatrix} \frac{k}{N}+\frac{1}{2} \\ \frac{l}{N}+\frac{1}{2} \end{bmatrix}}{\eta^3 \vartheta \begin{bmatrix} \frac{2k}{N}+\frac{1}{2} \\ \frac{2l}{N}+\frac{1}{2} \end{bmatrix}} \right|^2 . \tag{2.20}$$

We can repeat the analysis for even N with minor modification and obtain

$$\mathcal{Z}(N_{even}) = \frac{1}{4} \sum_{k,l=1}^{2N} \left| \frac{\vartheta^4 \begin{bmatrix} \frac{k}{2N}+\frac{1}{2} \\ \frac{l}{2N}+\frac{1}{2} \end{bmatrix}}{\eta^3 \vartheta \begin{bmatrix} \frac{k}{N}+\frac{1}{2} \\ \frac{l}{N}+\frac{1}{2} \end{bmatrix}} \right|^2 . \tag{2.21}$$

This formula would look identical to (2.20) in terms of new variable $N' = 2N$, however, N is chosen so that $\delta = 2\pi(1-\frac{1}{N})$. It is straightforward to check for modular invariance.

There is one more modular invariant combination for odd N given by,

$$\hat{\mathcal{Z}}(N_{odd}) = \frac{1}{4} \sum_{k,l=1}^{2N} \left| \frac{\vartheta^4 \begin{bmatrix} \frac{k}{2N}+\frac{1}{2} \\ \frac{l}{2N}+\frac{1}{2} \end{bmatrix}}{\eta^3 \vartheta \begin{bmatrix} \frac{k}{N}+\frac{1}{2} \\ \frac{l}{N}+\frac{1}{2} \end{bmatrix}} \right|^2 . \tag{2.22}$$

It may seem a little disconcerting that there are more than one ways of constructing the orbifold. After all, if we wish to use this construction for computing the entropy, we would like to get a unique answer for each theory. Fortunately, there is a good explanation for this non-uniqueness. The Green-Schwarz superstring has a Z_2 symmetry, $(-1)^F$ where F is the spacetime fermion number. This symmetry is obviously a subgroup of the double cover of $SO(2)$, $(-1)^F = e^{2\pi i J_{12}}$. The orbifold with respect to this Z_2 changes the spectrum drastically. In the untwisted sector the projection onto Z_2 invariant states removes all fermions. The twisted sector adds more particles including a tachyon. Moreover, the number of bosonic zero modes is the same in the twisted sectors because the bosonic coordinates are neutral under this Z_2. This means that the states in the twisted sector move over all space and are not restricted to the tip of the cone. We should really regard this theory as a different theory (vacuum). The orbifolds in (2.22) should then be regarded as the orbifolds not of flat space but of this different underlying theory. It is to be exptected that the entropy of black holes would be different in these two cases because, after all, the patricle spectrum of the two theories is completely different.

As an aside, we note that in all these orbifold models, supersymmetry is completely broken. This is not surprising. In order to have an unbroken supersymmetry, we must have a covariantly constant spinor on the cone, which means that the cone must have $SU(1)$ holonomy. But $SU(1)$ holonomy is no holonomy at all, and the only manifold with this holonomy is the plane. As a result, there are no unbroken supersymmetries on a cone. Even though supersymmetry is completely broken, in some cases drastically, the equivalence between the Green-Schwarz string and the Neveu-Schwarz-Ramond string still continues to hold. Let us recall the following Riemann theta identity:

$$2\prod_{i=1}^{4}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(x_i|\tau) = \prod_{i=1}^{4}\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(y_i|\tau) - \prod_{i=1}^{4}\vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(y_i|\tau) - \prod_{i=1}^{4}\vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(y_i|\tau) + \prod_{i=1}^{4}\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(y_i|\tau) \tag{2.23}$$

where $y_1 = \frac{1}{2}(x_1+x_2+x_3+x_4)$, $y_2 = \frac{1}{2}(x_1-x_2-x_3+x_4)$, $y_3 = \frac{1}{2}(x_1+x_2-x_3-x_4)$ and $y_4 = \frac{1}{2}(x_1 - x_2 + x_3 - x_4)$ and $\vartheta\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right](z|\tau) = e^{2\pi i a(z+b)} q^{\frac{a^2}{2}} \vartheta(z + a\tau + b|\tau)$. With the use of this identity we can write each term in the sum for $\mathcal{Z}(N)$ as the modulus-squared of sum of four terms. These four terms correspond to the four spin structures on the left and the right of the NSR superstring. The simplest example is the $\hat{Z}_1$ orbifold above. With $N = 1$ in (2.22) we get,

$$\hat{\mathcal{Z}}(1) = \frac{\left|\vartheta^4\left[\begin{smallmatrix}0\\ 0\end{smallmatrix}\right]\right|^2 + \left|\vartheta^4\left[\begin{smallmatrix}0\\ \frac{1}{2}\end{smallmatrix}\right]\right|^2 + \left|\vartheta^4\left[\begin{smallmatrix}\frac{1}{2}\\ 0\end{smallmatrix}\right]\right|^2 + \left|\vartheta^4\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\right|^2}{4\left|\eta^3\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right]\right|^2} \,. \tag{2.24}$$

Thus, in this case, we see the equivalence of the two formalisms simply by reinterpreting the orbifold sum in the GS formalism as the sum over spin structures in the NSR formalism with a modular invariant combination diagonal in spin structures. The corresponding NSR string has been discussed in [25]. It has a very different projection than the usual GSO projection and there are no fermions in the spectrum. Remarkably, the GS and the NSR formalism are equivalent even after this rather extreme breaking of supersymmetry. This points to a deep connection between the two formalisms that goes beyond supersymmetry.

The spectrum of the Z_N orbifold in the sector twisted by $\eta = \frac{k}{N}$ is easily obtained. We shall use the light-cone gauge and describe only the low lying states assuming that η is small. Let us first consider the GS formalism. The ground state has energy $-\frac{\eta}{2}$ and is tachyonic. In the right moving sector, we have the oscillator modes $S^m_{n-\frac{\eta}{2}}$, $S^{\bar{m}}_{n+\frac{\eta}{2}}$, $\alpha_{n-\eta}$, and $\bar{\alpha}_{n+\eta}$. Acting on the vacuum with various powers of $S^m_{-\frac{\eta}{2}}$, we generate a sixteen dimensional representation quite similar to the gauge supermultiplet in flat space. It decomposes in terms of $SU(4)$ representations $\mathbf{1}, \mathbf{4}, \mathbf{6}, \bar{\mathbf{4}}, \mathbf{1}$ with masses $-\frac{\eta}{2}, 0, \frac{\eta}{2}, \eta, \frac{3\eta}{2}$ respectively. The representations $\mathbf{4}$ and $\bar{\mathbf{4}}$ are fermions and the remaining states are bosons. We get a similar representation on the left and the low lying spectrum is the tensor product of the two, keeping only the Z_N invariant states. In addition, there are more states that are obtained by acting with various powers of $\alpha_{-\eta}$ on these states. In the NSR formalism the analysis is somewhat different. The eight worldsheet fermions ψ^i transform as the vector of $SO(8)$ exactly like the bosons . As a result, only two get twisted and the remaining six are untwisted. In the NS sector, the ground state energy is $-\frac{1}{2} + \frac{\eta}{2}$. The low energy fermion creation operators are $\psi_{-\frac{1}{2}-\eta}$ and $\bar{\psi}_{-\frac{1}{2}+\eta}$ coming from the two twisted fermions, and six $\psi^i_{-\frac{1}{2}}$ coming from the untwisted fermions. The ground state gets projected out by the GSO projection. At the next level, we have one of the creation operators above acting on the vacuum. This gives one state with energy $-\frac{\eta}{2}$, six states with energy $\frac{\eta}{2}$ and one state with energy $\frac{3\eta}{2}$. All of these are spacetime bosons. As usual, the spacetime fermions come from the Ramond sector. The ground state energy is zero in the Ramond sector. The six untwisted fermions have zero modes that form the Clifford algebrea of $SO(6)$ and have an eight-dimensional spinor representation, that splits as $\mathbf{4}$ and $\bar{\mathbf{4}}$. The $\bar{\mathbf{4}}$ gets projected out by the GSO projection and we are left with four fermions with mass 0 in **4**. The next excited state is obtained by acting on the vacuum with a creation operator with energy η coming from the twisted fermions. The GSO projection removes the **4** at this level and we obtain four fermions in $\bar{\mathbf{4}}$ with energy η. The bosonic oscillators in the NSR string are the same as in the GS string, so the low lying spectrum matches exactly with the one obtained from the GS formalism.

3. Black Hole Entropy in String Theory

The entropy is given by,

$$S = -\beta \frac{\partial(\log Z)}{\partial \beta} + \log Z. \tag{3.1}$$

In order to compute this we need to vary the Rindler temperature away from 2π which corresponds to flat space. This introduces a conical defect with the deficit angle δ which is related to the inverse temperature by $\beta = 2\pi(1 - \frac{\delta}{2\pi}) = \frac{2\pi}{N}$. Treating N as a continuous variable, we see that $S = \frac{d(N \log Z)}{dN}|_{N=1}$. In string perturbation theory at G loop, the spacetime partition function Z_G and the worldsheet vacuum amplitude A_G are related by $\log Z_G = A_G$ [26]. This gives us the desired formula for the entropy at G loop:

$$S_G = \frac{d(N A_G)}{dN}|_{N=1} . \tag{3.2}$$

Substituting (2.17) into (3.2) we obtain the final expression for the entropy in the bosonic string at one loop:

$$S_1 \sim \text{Area} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\,\tau)^2} \quad |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\text{Im}\tau|^{-11}\, \mathcal{Z}'(1) \quad . \tag{3.3}$$

Similarly for the type IIA supersting we obtain

$$S_1 \sim \text{Area} \int_{\mathcal{M}_1} \frac{d^2\tau}{(\text{Im}\,\tau)^2} \quad |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\text{Im}\tau|^{-3}\, \mathcal{Z}'(1) \quad . \tag{3.4}$$

Here $\mathcal{Z}'(1)$ for each theory is the first derivative of the orbifold partition function evaluated at $N = 1$. It is quite satisfying that the entropy comes out proportional to the area. Before making this assertion, we must clarify one point that we have so far glossed over. In the orbifold sum (2.18) the term $\mathcal{Z}_{0,0}$ is divergent. This is because the theta function vanishes due to the contribution of the bosonic zero modes to the corresonding determinant. If we treat the determinant carefully, the zero modes would turn the area in (2.17) into volume for this particular term in the sum. Modular invariance would still hold because $\mathcal{Z}_{0,0}$ does not mix with other terms under modular transformations and is invariant by itself. More importantly, it is independent of N and as we see from (3.3) , it does not contribute to the entropy. Thus, the entropy will be proportional to the area and not the volume.

Before proceeding further let us see what we expect to find. The Schwarzschild observer near the horizon sees a hot thermal bath. We can thus view the orbifolds as describing the Euclidean path integral for strings in a thermal ensemble at Rindler temperature $\frac{N}{2\pi}$. The proper temperature is position dependent and diverges at the tip of the cone. As a result, we expect the strings to undergo a Hagedorn phase transition well known in string theory. Correspondingly, we expcect to find an infrared instability in the form of tachyons. The tachyons that we find in the twisted sector are closely related to tachyons coming from the winding modes of the string around the Euclidean time direction [27] which signal the Hagedorn transition in strings at finite temperature. After the phase transition, a tachyon condensate will be formed. This can explain the tree level contribution to the entropy as coming from the latent heat of this phase transition [28]. Furthermore this condensate will be confined very close to the horizon and spread only in the transverse directions. It raises the exciting possibility that we can understand the dependence of the Bekenstein-Hawking entropy on area as well as the coupling constant [15]. Unfortunately, at the moment, the Hagedorn transition is not very well understood. Moreover, once we include interactions we also have to worry about the Jeans instability [29]. It is not clear how to properly take these effects into account. However, we are really interested only in the entropy at a special value of the temperature and we hope that at least some of the features of the entropy will be accesible without having to understand all the consequences of the Hagedorn transition. For example, it would be nice to see if the entropy can be rendered finite in the infrared by adding a tree level contribution.

In order to complete this computation we need an analytic expression for the sum $\mathcal{Z}(N)$ so that we can take its derivative. Several comments are in order here. First, all terms in the sum are analytic functions of N so we can expect that the sum will also be analytic. Second, modular invariance of the vacuum amplitude holds only for integer N. In fact, for non-integer N, the sum may not have any interpretation as a partition function of some string theory. However, for our entropy computation we do not really require that the theory be well defined for arbitrary N. All that is needed is that the first derivative of the vacuum amplitude at $N = 1$ be well-defined and modular invariant. This certainly seems possible, especially if we think of the entropy as the counting of states of a given finite theory in flat space as seen by the Rindler observer. Finally, even if we do perform the sum, there is a high degree of non-uniquness because we can always add to the sum any function that vanishes for integer N *e.g.* $\sin(\pi N)$. We can fix this non-uniqueness if we know that the partition function does not have an essential singularity at $N = \infty$. This requires some physical input about the theory. We can guess the correct ananlytic continuation by comparing our answers with strings in a thermal ensemble discussed in [15].

It is interesting to take the large N limit of our formula (2.18) . Putting $z = \frac{k}{N}\tau + \frac{l}{N}$ in (2.18) and taking N to infinity, we see that

$$\mathcal{Z}(N)|_{N\to\infty} = \frac{N^2}{\mathrm{Im}\tau}\int d^2z \, \exp(-\frac{\pi(z-\bar{z})^2}{2\mathrm{Im}\tau}) \, \Big|\frac{\eta(\tau)}{\vartheta_{11}(z|\tau)}\Big|^2 \tag{3.5}$$

where ϑ_{11} is the theta function with half characteristics. The integration is over a torus with modular parameter τ which is parametrized with a flat metric as a parallelogram with corners at 0, 1, τ and $\tau+1$. The integrand is doubly periodic over this region and is a well defined function on the torus. This expression is clearly analytic in N and moreover is modular invariant even for non-integer N. Encouragingly, $\mathcal{Z}(N)$ does not have an essential singularity but only a second order pole at infinite N. Another interesting feature of (3.5) is that the integral is logarithmically divergent. The theta function has a zero at $z = 0$, $\vartheta_{11}(z|\tau) \sim 2\pi z\eta^3(\tau)$. As a result the leading contribution to (3.5) is

$$\mathcal{Z}(N)|_{N\to\infty} \sim N^2 \log N \, |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\mathrm{Im}\tau|^{-1} \, . \tag{3.6}$$

This expression is strikingly similar to the partition function in R_2 ($N = 1$) after properly taking into account the bosonic zero modes

$$\mathcal{Z}(N)|_{N=1} \sim L^2 \, |\eta^2(\tau)\bar{\eta}^2(\bar{\tau})\mathrm{Im}\tau|^{-1} \, . \tag{3.7}$$

We regard this as evidence for some kind of duality similar to the $R \to \frac{1}{R}$ duality. As N becomes large, the space is becoming smaller and one might think that the number of states is also becoming smaller. However, more and more twisted states come down and become almost massless as we take N to infinity. These states combine to give a partition function very similar to the partition function of the original theory. So in some sense we still have as many states as we started with.

4. Discussion

We have seen that string theory offers a suitable framework for testing the conjecture by ‘t Hooft and Susskind that the Bekenstein-Hawking entropy should be understood in terms of the entropy of fluctuations near the horizon. With this objective in mind, we have described the construction of string propagation on a cone. We have obtained an expression for the entropy that is proportional to the area of the event horizon. In order to complete this computation we need to perform the finite sum for $\mathcal{Z}(N)$ which is currently under investigation [15]. The presence of tachyons in these models also deserves attention. Tachyons signify a vacuum instability and it is very important to understand their role in string theory. For example, it has long been thought that the bosonic string represents a metastable point in the space of vacua and in the proper non-perturbative formulation we would see the theory relax into one of the stable ground states. This proposition is of course too difficult to test in flat place because the usual tachyon moves over all space and we do not even know any candidates for a nearby ground states. In our case, we have a whole family of theories which can have a large number of tachyons localized to the tip of the cone. If a tachyon condensate is formed, it will most likely change the value of the deficit angle *i.e.*the value of N. For large N, some of the tachyons are nearly massless and it may be possible to understand their condensation as a perturbation of the K_N conformal field theory with a nearly marginal operator.

The existence of tachyons in the twisted sectors along with the results of [15] indicates that there will be a Hagedorn transition close to horizon. It seems possible to understand the dependence of the black hole entropy on the area of the event horizon and also the coupling constant if a condensate is formed. If the entropy comes from the fundamental degrees of freedom of string theory beyond the Hagedorn transition, then quite possibly, it is also independent of the low energy particle spectrum. If this conjecture turns out to be correct, we may be able to learn something about the Hagedorn transition from its relation to the black hole entropy and vice versa.

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