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Strings as Solitons & Black Holes as Strings

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Abstract

Supersymmetric closed string theories contain an infinite tower of BPS-saturated, oscillating, macroscopic strings in the perturbative spectrum. When these theories have dual formulations, this tower of states must exist nonperturbatively as solitons in the dual theories. We present a general class of exact solutions of low-energy supergravity that corresponds to all these states. After dimensional reduction they can be interpreted as supersymmetric black holes with a degeneracy related to the degeneracy of the string states. For example, in four dimensions we obtain a point-like solution which is asymptotic to a stationary, rotating, electrically-charged black hole with Regge-bounded angular momentum and with the usual ring-singularity replaced by a string source. This further supports the idea that the entropy of supersymmetric black holes can be understood in terms of counting of string states. We also discuss some applications of these solutions to string duality.

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1. Introduction

The concept of 'strong-weak' coupling duality has led to surprising new insights into the dynamics of supersymmetric four-dimensional gauge theories. Duality provides equivalent descriptions of the same physics in terms of either the 'electric' or the 'magnetic' variables. When one description is strongly-coupled and complicated, the dual description can be weakly-coupled and simple. Many nonperturbative phenomena in one description can then be understood perturbatively in the dual description. One important piece of information for relating the two descriptions is the spectrum of BPS-saturated states. These states belong to a 'short' representation of the supersymmetry algebra, and their mass is proportional to their charge. As a result, the semiclassical spectrum of BPS states is often reliable even at strong coupling [1]. Analysis of the BPS spectrum has played a crucial role in motivating duality and also in many subsequent developments. For example, in the simplest context of $N = 4$ super Yang-Mills theory, fundamental particles of the magnetic theory correspond to BPS-saturated, magnetic monopoles in the electric theory [2]. In global $N = 2$ theories the BPS-saturated states are important for a qualitative understanding of several dynamical phenomena such as confinement and chiral symmetry breaking [3,4].

There is now increasing evidence for a similar duality between various string theories. It is natural to expect that the spectrum of BPS-saturated states will be equally important for a dynamical understanding of these dualities. For example, in $N = 2$ string theories one can construct a generalized Kac-Moody Lie superalgebra in terms of BPS states, which governs the form of the perturbative prepotential [5].

By analogy with field theory, one might hope to find fundamental strings themselves as solitons in the dual string theory [6,7,8,9]. Moreover, because strings are extended, oscillating objects, one might expect a much richer spectrum of such solitons. As usual in strong-weak coupling duality this is problematic for non-BPS states, but in string theory, many of the fundamental string states are BPS-saturated and therefore should exist as

BPS solitons in a dual description. In particular, the perturbative spectrum of closed string theories contains stable, macroscopic-string states [10,11] that are BPS-saturated. For concreteness let us consider heterotic string theory on $R^9 \times S^1$ and take the radius of the S^1 to be large compared to the string scale. A macroscopic string is a winding state that wraps around this circle. In order to obtain a BPS-saturated state the right-moving oscillators must be in the ground state, but an arbitrary number of left-moving oscillators can be excited subject only to the mass-shell condition. We thus obtain an infinite tower of stable, oscillating, macroscopic strings. Even though we regard these states as macroscopic, string-like objects, one can always take the radius of the circle to be quite small. Such a dimensional reduction gives rise to an infinite tower of point-like states in one less dimension. For example, various electrically charged supersymmetric black holes and their magnetic duals can be viewed in this manner.

In this paper we present solutions of the low-energy string equations of motion that are in close correspondence with this entire tower of BPS states. We take as our starting point the fundamental straight-string solution obtained in [12]. We then generate several new solutions by using various solution-generating techniques, dimensional reduction, and duality transformations. We are thus able to study a wide variety of string-like and point-like solutions with either 'electric' or 'magnetic' charges in a unified framework. The original straight-string solution is BPS-saturated, preserves half the supersymmetries, and has the right structure of chiral fermionic zero modes expected for a Green-Schwarz superstring. All these properties are inherited by the transformed solutions in a simple way.

Another motivation for this work is the suggestion that the entropy of black holes can be understood in terms of degeneracy of massive string states [13,14,15,16,17]. In particular, one can consider a supersymmetric, nonrotating black hole with given mass and charges in toroidally compactified heterotic string theory in four dimensions and compare its entropy with the number of BPS states in the spectrum with the same quantum numbers. The degeneracy of these states can be computed reliably in string perturbation theory because their mass and charges are not renormalized. Sen has shown that this counting agrees with the entropy associated with the stretched horizon of the corresponding black holes [17]. The solutions that we describe here provide an attractive classical picture of this degeneracy in terms of the oscillations of an underlying string.

The outline of this paper is as follows. In section two, we first describe the infinite tower of BPS states in the perturbative spectrum of closed strings. We then obtain the most general charged, oscillating-string solution that corresponds to these states. We begin with the family of traveling-wave solutions constructed in [18] by applying the generating transform of [19] to the straight-string solution. We then apply the 'charged-wave' generating transform of reference [20,21]. By appropriately choosing the form of the traveling wave we obtain an asymptotically flat solution which correctly matches onto a string source. Furthermore, we show that a BPS-saturated solution is obtained only when the traveling wave and the current have the right chirality to be consistent with the corresponding state in the string spectrum. The asymptotic parameters can be readily identified with the quantum numbers of the tower of BPS string states. We then discuss the dimensional reduction of these solutions to obtain string solutions in lower dimensions by using periodic arrays. This clarifies the relation between string solutions in different dimensions.

In section three, we obtain a family of point-like solutions in four dimensions by dimensional reduction of the oscillating-string solution in five dimensions. The resulting four-dimensional solution is identical to the supersymmetric limit of a stationary, electrically-charged, rotating black hole solution obtained by Sen [22]. The asymptotic parameters of the black hole are all related to the profile of oscillation of the underlying string in a simple way. Moreover, the angular momentum is Regge-bounded as expected for a state in the string spectrum. Near the core, however, the naked ring-singularity of the four-dimensional stationary solution is replaced by the much milder singularity of a five-dimensional string source.

In section four, we briefly review the string duality transformations and discuss some properties of the infinite tower of solitonic solutions obtained by dualizing the solutions of section two. We point out that in four dimensions, by dualizing the supersymmetric black holes of section three one obtains the tower of rotating, magnetically-charged states that are required by S-duality [23].

Section five contains some concluding remarks.

2. Macroscopic Fundamental Strings

2.1. Infinite tower of Macroscopic Strings

We now describe the perturbative spectrum of macroscopic strings on $R^9 \times S^1$. The S^1 has radius R which we take to be large compared to the string scale and the macroscopic string wraps around this circle. The string states are then characterized by their winding number n and quantized momentum m/R in the compact direction. Specifically, if p_R and p_L are the right-moving and the left-moving momenta respectively in the compact direction, we have

$$p_R = (\frac{m}{2R} - \frac{nR}{2\alpha'}), \qquad p_L = (\frac{m}{2R} + \frac{nR}{2\alpha'}). \tag{2.1}$$

It is convenient to use the Green-Schwarz formalism in the light-cone gauge. Let us first discuss the heterotic string which has $N=1$ spacetime supersymmetry that is generated by right-moving worldsheet currents. The supercharge has 16 real components corresponding to a single Majorana-Weyl fermion. A BPS-saturated state preserves 8 of these supersymmetries [24], and the remaining 8 broken supercharges generate a 16-dimensional 'short' representation of the $N=1$ superalgebra. [1]

A 'short' representation in spacetime can be obtained only if all right-moving oscillators on the world-sheet are in the ground state. The left-movers can have arbitrary oscillations subject to the Virasoro constraint:

$$N_L = 1 + \alpha' \left(p_R^2 - p_L^2 - q_L^2\right) = 1 - mn - \alpha' q_L^2, \tag{2.2}$$

[1] As discussed in [25,12], the $N=1$ supersymmetry algebra in $D=9$ admits a central extension proportional to p_R.

where N_L is the number of transverse left-moving oscillators and q_L are the charges on the internal torus of the heterotic string. Here we are working at a point in the Narain moduli space where the lattice factorizes as $\Gamma_{0,16} + \Gamma_{1,1}$. The mass of this state is related to the right-moving momentum along the string

$$M^2 = 4p_R^2. \tag{2.3}$$

For type-II strings, we have $N = 2$ supersymmetry that is carried by both right-movers and left-movers. Once again, a BPS-saturated state is obtained if at least 8 supersymmetries are unbroken which can be either all right-moving or all left-moving. Let us consider the spectrum when the unbroken supercharges are all right-moving. In the worldsheet theory these states are constructed as before by tensoring the right-moving ground state with an arbitrary left-moving state subject to the mass-shell conditions:

$$\begin{aligned} N_L &= \alpha' \left(p_R^2 - p_L^2\right) = -mn, \\ M^2 &= 4p_R^2. \end{aligned} \tag{2.4}$$

There is a corresponding tower of states when the unbroken supersymmetries are all left-moving. All these states preserve one quarter of the the original 32 supersymmetries and belong to a 'short' representation of the supersymmetry algebra which is now (16×256)-dimensional. A special case is when both right-moving and left-moving oscillators are in the ground state. These states preserve one half of the original supersymmetries and belong to an 'ultra-short' representation which is (16×16)-dimensional.

Thus, the spectrum of BPS-saturated macroscopic strings splits into topological sectors labeled by the winding number n, and for every n there is an infinite tower of oscillating states with momentum m/R. We denote these states by (m, n). Of course it is necessary to specify the charge vectors and which oscillators are excited to completely label the state. T-duality exchanges winding and momenta and in the case of type-II theories takes the IIA theory to the IIB theory [26].

For each theory there is an absolutely stable macroscopic string state: $(1,1)$ for the heterotic theory and $(0,1)$ for the type-II theories. The rest of the infinite tower of BPS states are only neutrally stable. For example, the multiply-wound state $(0,n)$ can decay into n singly-wound $(0,1)$ states sitting on top of each other. However, because this tower of states exists already in the single-string Hilbert space, they should be regarded as distinct from multi-string states with the same quantum numbers [2].

This discussion can be readily generalized to compactifications to lower dimensions. The mass shell formula (2.3) remains unchanged, but the constraint formulae (2.2) and (2.4) now receive contributions from the internal conformal field theory. The nature of these BPS states for compactifications of the heterotic string theory with $N=4$ and $N=2$ supersymmetry are described in [33] and [5] respectively.

2.2. Fundamental Straight Strings

The massless bosonic fields that couple to a macroscopic string are the dilaton ϕ , the metric tensor $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and gauge field A_μ. For simplicity we shall take the gauge group to be a single $U(1)$.

In Type-II theories all these fields come from the Neveu-Schwarz Neveu-Schwarz (NS-NS) sector because, to lowest order, a fundamental string does not couple to the massless fields from the Ramond-Ramond (R-R) sector. In ten dimensions there are no gauge fields in the NS-NS sector, but after compactification they may be present in lower dimensions. As we shall see in section four, some of the solitonic strings do couple to R-R fields, but in that case, the R-R fields are related to the NS-NS fields by a duality transformation, and need not be considered separately. There can also be more general stringy solitons that couple to both the NS-NS and the R-R fields [7,9,34], however we do not discuss them in this paper.

[2] For non-perturbative solitons this is not always the case. For a discussion of related issues see, for example, [4,27,28,29,30,31,32].

In heterotic string theory these fields are present in all dimensions. In ten-dimensional heterotic string theory the gauge group can be either $SO(32)$ or $E_8 \times E_8$, but we can break it to $U(1)^{16}$ by turning on Wilson lines along the circle that the macroscopic string wraps around.

The low-energy action for these fields in D dimensions [35] is given by

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} e^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H^2 - \alpha' F^2 \right), \tag{2.5}$$

where $F = dA$ and H is the three-form antisymmetric tensor field strength

$$H = dB - 2\alpha' \omega_3^{YM}(A) + \dots , \tag{2.6}$$

with ω_3 the Chern-Simons three-form. Note that the Lorentz Chern-Simons term will play no role in our solutions.

We also need to know the source terms for various fields in the presence of a macroscopic string. These couplings can be calculated directly by vertex-operator calculations as in [10]. Equivalently, one can use the sigma model action as in [12]:

$$S_\sigma = -\frac{1}{4\pi\alpha'} \int d^2\sigma \, (\sqrt{\gamma}\gamma^{mn} \partial_m X^\mu \partial_n X^\nu g_{\mu\nu} + \epsilon^{mn} \partial_m X^\mu \partial_n X^\nu B_{\mu\nu}) + S_A, \tag{2.7}$$

where X^μ are the spacetime coordinates of the string, and ϵ_{mn} is the antisymmetric tensor-density on the worldsheet with $\epsilon_{01} = 1$. The gauge field coupling given by S_A in (2.7) is somewhat delicate due to world-sheet anomalies in the left-moving currents; this leads to an important quantum modification of the current and source terms which will be discussed in §2.6. At this order, there is no explicit source term for the dilaton, but the dilaton couples to the string implicitly through its coupling to other fields.

The equations of motion that follow from the combined action are

$$
\begin{aligned}
&4\nabla^2\phi - 4\left(\nabla\phi\right)^2 + R - \frac{1}{12}H^2 - \alpha' F^2 = 0\\
&\nabla_\mu\left(e^{-2\phi}H^{\mu\nu\rho}\right) = \frac{\kappa^2}{\pi\alpha'\sqrt{-g}}\int d^2\sigma\epsilon^{mn}\partial_m X^\nu \partial_n X^\rho \delta^{(D)}\left(x - X(\sigma)\right)\\
&R^{\mu\nu} + 2\nabla^\mu\nabla^\nu\phi - \frac{1}{4}H^{\mu\rho\sigma}H^\nu{}_{\rho\sigma} - 2\alpha' F^{\mu\rho}F^\nu{}_\rho\\
&\qquad\qquad = -\frac{\kappa^2 e^{2\phi}}{2\pi\alpha'\sqrt{-g}}\int d^2\sigma\sqrt{\gamma}\gamma^{mn}\partial_m X^\mu \partial_n X^\nu \delta^{(D)}\left(x - X(\sigma)\right)\\
&\nabla_\mu\left(e^{-2\phi}F^{\mu\nu}\right) + \frac{1}{2}e^{-2\phi}H^{\nu\rho\sigma}F_{\rho\sigma}\\
&\qquad\qquad = \frac{\kappa^2}{4\pi\alpha'^2\sqrt{-g}}\int d^2\sigma J^m \partial_m X^\nu \delta^{(D)}\left(x - X(\sigma)\right).\\
&\qquad\qquad + \frac{\kappa^2}{2\pi\alpha'\sqrt{-g}}\int d^2\sigma\epsilon^{mn}\partial_m X^\nu \partial_n X^\mu A_\mu(x)\delta^{(D)}\left(x - X(\sigma)\right).
\end{aligned}
\tag{2.8}
$$

After taking into account world-sheet anomalies, we shall show in §2.6 that J^m is the consistent anomaly current and that the full gauge field source term (the right-hand side of the last equation) becomes the covariant anomaly current.

The fundamental string solution [12] of these equations of motion is given by

$$
\begin{aligned}
ds^2 &= -e^{2\phi}du dv + d\vec{x}\cdot d\vec{x},\\
B_{uv} &= \frac{1}{2}(e^{2\phi} - 1),\\
e^{-2\phi} &= 1 + \frac{Q}{r^{D-4}},\\
A_\mu &= 0,
\end{aligned}
\tag{2.9}
$$

where $u = x^0 - x^{D-1}$ and $v = x^0 + x^{D-1}$ are the lightcone coordinates along the string worldsheet, x^i (with $i = 1, ..., (D-2)$) are the transverse coordinates, r is the radial distance in the transverse space, and $x^{D-1} \equiv x^{D-1} + 2\pi R$. By changing the sign of the axion field B, one can reverse the orientation of the string. The parameter Q is proportional to the ADM-mass per unit length. It was shown in [12] that this solution satisfies a Bogomol'nyi bound, admits a covariantly constant spinor and preserves half the supersymmetries. There is no momentum in the compact direction, and the central charge in the supersymmetry algebra in $D-1$ dimensions is proportional to the winding number. This solution can be identified with the field configuration around a neutral macroscopic string state $(0, n)$ by choosing the ADM mass per unit length to be $n/(2\pi\alpha')$.

2.3. Neutral Oscillating Strings

We would like to obtain a solution that carries charge as well as arbitrary left-moving oscillations. When the charge and the oscillations are small, we could proceed by quantizing the collective coordinates of the straight-string solution discussed in §2.1; this would be analogous to constructing a dyon with a small electric charge by quantizing the collective coordinates around a neutral static magnetic monopole solution. However, for large charge and oscillations, one would like to directly construct a charged oscillating string-solution analogous to the time-dependent Julia-Zee dyon solution in field theory.

In this subsection we discuss the neutral oscillating string. To this end, we shall first review the solution-generating transform of [19]. This transform was used in [18] to construct a general class of macroscopic string solutions with traveling waves. However, as we shall see, only a very small sub-class of these solutions can be identified with the tower of macroscopic string states that we are interested in. To make this identification we require that the solution be asymptotically flat and supersymmetric. We further require that it match onto appropriate string sources. We apply these criteria, in this and the following subsections, to obtain the solution that corresponds to an arbitrary (m, n) state.

The solution generating transform of [19] can be applied to any solution of the equations of motion (2.8) which admits a null Killing vector K^μ that is hypersurface-orthogonal. This means that the vector K and the corresponding one-form satisfy the conditions

$$K^2 = 0, \qquad L_K \Psi = 0, \qquad dK = K \wedge dM, \tag{2.10}$$

where Ψ refers to all fields, L_K is the Lie-derivative along K, and M is a function that is determined by the solution. The straight-string solution (2.9) admits two such Killing vectors $\partial/\partial v$ and $\partial/\partial u$ with $M = -2\phi$, so the generating technique can be used to construct waves that are right-moving and left-moving respectively. For a left-moving wave the resulting solution is given by

$$\begin{aligned} ds^2 =& - e^{2\phi}\left(dudv - T(v,\vec{x})dv^2\right) + d\vec{x}\cdot d\vec{x} \\ B_{uv} =& \frac{1}{2}(e^{2\phi} - 1) \\ e^{-2\phi} =& 1 + \frac{Q}{r^{D-4}} \qquad \partial^2 T(v,\vec{x}) = 0 \end{aligned} \tag{2.11}$$

where ∂^2 is the Laplacian operator in the flat transverse space. It is easy to see that the equations of motion for ϕ and $B_{\mu\nu}$ are unchanged with the new fields. It was shown in [19] that the Einstein equations are also unchanged if T is a solution of the flat Laplacian. The wave travels at the speed of light because the solution still admits $\partial/\partial u$ as a null Killing vector. Replacing $T(v,\vec{x})dv^2$ with $T(u,\vec{x})du^2$ one obtains the right-moving traveling-wave solutions.

The solution for $T(v,\vec{x})$ can be decomposed as a power series in r using $(D-2)$-dimensional spherical harmonics Y_ℓ,

$$T(v,\vec{x}) = \sum_{\ell\geq 0}\left[a_\ell(v)r^\ell + b_\ell(v)r^{-D+4-\ell}\right]Y_\ell. \tag{2.12}$$

Terms that go as r^β with $\beta = 0$ can be removed by a change of coordinates. The remaining terms in this expansion have a direct physical interpretation. Terms with $\beta \geq 2$ and $\beta \leq -D+3$ can be interpreted as gravitational plane-fronted waves superposed on the string background. Note that the former are not asymptotically flat and while the latter are, they do not contribute to the asymptotic quantities such as the ADM mass and, moreover, they do not match onto string sources. The most relevant term for our purpose is the one with $\beta = 1$. We shall see in §2.6 that this is the only term that matches onto a string source corresponding to an oscillating string. The term with $\beta = -D+4$ is also interesting even though it does not match onto a classical string source; asymptotically it corresponds to a momentum wave without oscillations. Keeping only these two terms, the general form of T is given by

$$T(v,\vec{x}) = \vec{f}(v)\cdot\vec{x} + p(v)r^{-D+4}. \tag{2.13}$$

In these coordinates, the solution does not appear asymptotically flat because T does not vanish at large r. To remedy this we perform the following coordinate transformation:

$$\begin{aligned} v &= v' \\ u &= u' - 2\dot{\vec{F}}\cdot\vec{x}' + 2\dot{\vec{F}}\cdot\vec{F} - \int^{v'}\dot{F}^2 dv \\ \vec{x} &= \vec{x}' - \vec{F} \end{aligned} \tag{2.14}$$

where the dot indicates a derivative with respect to v, $\vec{f}(v) = -2\ddot{\vec{F}}$, and $\dot{F}^2 = \dot{\vec{F}} \cdot \dot{\vec{F}}$. The metric and the other fields in the new coordinates are

$$\begin{aligned} ds^2 = -e^{2\phi} du\, dv + & \left[e^{2\phi}\, p(v)\, r^{-D+4} - \left(e^{2\phi} - 1\right) \dot{F}^2\right] dv^2 \\ & + 2\left(e^{2\phi} - 1\right) \dot{\vec{F}} \cdot d\vec{x} dv + d\vec{x} \cdot d\vec{x}, \\ B_{uv} = & \frac{1}{2}(e^{2\phi} - 1), \\ B_{vi} = & \dot{F}_i (e^{2\phi} - 1), \\ e^{-2\phi} = & 1 + \frac{Q}{|\vec{x} - \vec{F}|^{D-4}}, \end{aligned} \tag{2.15}$$

where we have suppressed the primes on the new variables. The metric is now asymptotically flat and one can easily compute a mini ADM 'stress-energy' tensor giving the mass and the momentum per unit length, $\pi^{0,\mu}_{ADM}$, together with the momentum flow along the string, $\pi^{D-1,\mu}_{ADM}$. We find

$$\begin{pmatrix} \pi^{0,\mu}_{ADM} \\ \pi^{D-1,\mu}_{ADM} \end{pmatrix} = k \begin{pmatrix} Q + Q\dot{F}^2 + p & Q\dot{F}^i & -Q\dot{F}^2 - p \\ -Q\dot{F}^2 - p & -Q\dot{F}^i & -Q + Q\dot{F}^2 + p \end{pmatrix}, \tag{2.16}$$

where $k = \frac{(D-4)\omega_{D-3}}{2\kappa^2}$ with ω_{D-3} being the volume of the sphere S^{D-3}.

We shall see in §2.6 that the neutral string solutions with $p = 0$ map onto a string source with a traveling wave with profile $\vec{F}$. Consequently, they can be identified with the neutral states (m, n) [3]. The ADM formulae are consistent with this interpretation because the momentum is negative and the mass is modified in accordance with the constraints and mass-shell conditions.

As in [12] there also exist multi-string solutions. In general, we have a solution of the form (2.11) provided $\partial^2 e^{-2\phi(\vec{x})} = 0$ and $\partial^2 T(v, \vec{x}) = 0$. A general multi-string solution is given by,

$$\begin{aligned} e^{-2\phi(\vec{x})} &= 1 + \sum_i \frac{Q}{|\vec{x} - \vec{x}_i|^{D-4}}, \\ T(v, \vec{x}) &= \vec{f}(v) \cdot \vec{x} + \sum_i \frac{p_i(v)}{|\vec{x} - \vec{x}_i|^{D-4}}. \end{aligned} \tag{2.17}$$

[3] The state $(1, 1)$ in heterotic string theory is an exception because it has positive momentum and no oscillations. We would like to identify this state with the solution with $\vec{F} = 0$ and $p < 0$, as will be discussed in §2.6.

Making a coordinate transformation of the form (2.14), we find

$$e^{-2\phi} = 1 + \sum_i \frac{Q}{\left|\vec{x} - \vec{x}_i - \vec{F}(v)\right|^{D-4}} \,. \tag{2.18}$$

Clearly, the constants $\vec{x}_i$ determine the locations of the strings, and we find that all the strings carry the same profile of oscillation $\vec{F}$, though in general carry different momentum profiles $p_i(v)$. These multi-string solutions will be useful later when we discuss dimensional reduction using periodic arrays of strings.

2.4. Charged Oscillating Strings

Charged solutions can be obtained by applying the solution generating transform of [21]. The basic idea behind this transform is as follows [20,36]. In string theory, the left-moving and the right-moving coordinates can be rotated independently, except for the rotation of zero modes. Consider string fields that are independent of p left-moving and q right-moving compact coordinates of an internal torus as well as the time coordinate t. Because the rotation of zero modes is unimportant for the coordinates that the fields do not depend on the string field equations of motion and their low-energy limit are expected to possess an $O(1,p)_L \times O(1,q)_R$ symmetry. In the low-energy theory these rotations correspond to nonlinear field transformations described in [20]. Now, if a solution of the equations of motion is not invariant under this symmetry, then one obtains a new 'twisted' solution. Some of the symmetries, corresponding to rotations of the internal coordinates, change the parameters of the internal torus. Thus, if we want to keep the parameters of the internal torus fixed, then the moduli space of solutions is $(O(1,p)/O(p)) \times (O(1,q)/O(q))$.

The fundamental string solution is independent of time and also of the internal coordinate corresponding to the $U(1)$ gauge group which we take to be left-moving. So, we can twist (2.9) using an $O(1,1)_L$ rotation to obtain a one parameter family of charged strings carrying left-moving current. It turns out that the resulting solution [21,24] admits a null, hypersurface-orthogonal, Killing vector $\partial/\partial v$. So we can apply the the generating technique of the previous subsection to obtain the most general solution representing a charged string with traveling waves:

$$
\begin{aligned}
ds^2 =& - e^{2\phi}\left(dudv - W(v,\vec{x})dv^2\right) + d\vec{x}\cdot d\vec{x} \\
B_{uv} =& \frac{1}{2}(e^{2\phi}-1) \\
A_v =& \frac{q(v)}{r^{D-4}+Q} \\
e^{-2\phi} =& 1+\frac{Q}{r^{D-4}} \\
W =& \vec{f}(v)\cdot\vec{x} + \frac{p(v)}{r^{D-4}} + \frac{2\alpha' q(v)^2}{Q\left(r^{D-4}+Q\right)}.
\end{aligned}
\tag{2.19}
$$

This solution represents a charged, superconducting string with a left-moving electric current as well as left-moving oscillations. The current is equal in magnitude to the charge per unit length and is proportional to q. We have also generalized the charge to a charge wave $q(v)$. The most general solution of this form has $A_v = N(v,\vec{x})e^{2\phi}$ with $\partial^2 N(v,\vec{x}) = 0$, but we shall assume that the string is only a monopole source and drop the higher spherical harmonics in the full solution for A_v.

One can obtain an asymptotically flat spacetime by a coordinate transformation exactly as in the previous subsection. The ADM stress-energy is then given by

$$
\begin{pmatrix} \pi^{0,\mu}_{ADM} \\ \pi^{D-1,\mu}_{ADM} \end{pmatrix} = k \begin{pmatrix} Q + Q\dot{F}^2 + \frac{2\alpha' q^2}{Q} + p & -Q\dot{F}^2 - \frac{2\alpha' q^2}{Q} - p & Q\dot{F}^i \\ -Q\dot{F}^2 - \frac{2\alpha' q^2}{Q} - p & -Q + Q\dot{F}^2 + \frac{2\alpha' q^2}{Q} + p & -Q\dot{F}^i \end{pmatrix}. \tag{2.20}
$$

In heterotic string theory in ten dimensions we expect macroscopic string solutions with left-moving oscillations and only left-moving charges; these are clearly included in our construction. In toroidal compactifications of the heterotic and type II strings we expect solutions with both left-moving and right-moving charges. We expect the most general solution can be obtained from our solutions by a combination of dimensional reduction and boosting. We leave the details of this to future work.

2.5. Supersymmetry

We have already discussed that the BPS states in the first-quantized string spectrum are invariant under half or one quarter of the spacetime supersymmetries. If the spacetime solutions we have described are to be identified with such string states we must be able to show that they too preserve the same fraction of the spacetime field supersymmetries. We now show this by explicit calculation, restricting to ten dimensions for simplicity.

Let us first consider the heterotic string. In a given background of bosonic fields, the supersymmetry transformation laws of the fermion fields are

$$\begin{aligned} \delta\lambda_- &= \left(\gamma^\mu\partial_\mu\phi - \frac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\epsilon_+ \\ \delta\psi_{+\mu} &= \left(\partial_\mu + \frac{1}{4}\Omega_{-\mu}{}^{ab}\Gamma_{ab}\right)\epsilon_+ \\ \delta\chi &= F_{\mu\nu}\gamma^{\mu\nu}\epsilon_+, \end{aligned} \tag{2.21}$$

where λ_- is the dilatino, $\psi_{+\mu}$ is the gravitino, χ is the gaugino, ϵ_+ is the supersymmetry parameter and we have introduced the generalized spin connection $\Omega_{-\mu} = \omega_\mu - H_\mu/2$. All spinors are Majorana-Weyl and the subscript denotes their chirality. Also γ^μ, Γ^a are gamma matrices with spacetime and tangent space indices, respectively. The general charged string with left-moving oscillations is given by (2.19); substituting into (2.21) we obtain

$$\begin{aligned} \delta\lambda_- &= -\frac{1}{2}\partial_i\phi\Gamma^i\Gamma^-\Gamma^+\epsilon_+ \\ \delta\psi_{+v} &= \left[\partial_v - \frac{1}{8}e^\phi\left(\partial_i W + 2W\partial_i\phi\right)\Gamma^i\Gamma^+\right]\epsilon_+ \\ \delta\psi_{+u} &= \left[\partial_u + \frac{1}{4}e^\phi\partial_i\phi\Gamma^i\Gamma^+\right]\epsilon_+ \\ \delta\psi_{+i} &= \left[\partial_i - \frac{1}{2}\partial_i\phi - \frac{1}{4}\partial_i\phi\Gamma^-\Gamma^+\right]\epsilon_+ \\ \delta\chi &= e^{-\phi}\partial_i A_u\Gamma^i\Gamma^+\epsilon_+, \end{aligned} \tag{2.22}$$

where $\Gamma^\pm = \Gamma^0 \pm \Gamma^1$. It is easy to see that the variations vanish when ϵ is of the form

$$\epsilon_+ = e^{\phi/2}\epsilon_{+0}, \qquad \partial_\mu\epsilon_{+0} = 0, \qquad \Gamma^+\epsilon_{+0} = 0. \tag{2.23}$$

Therefore, the solution admits a Killing spinor field of a particular helicity along the worldsheet, and half the supersymmetries are preserved.

Let us now consider a heterotic string with right-moving oscillations and zero charge which is given by (2.11) with $T(v,\vec{x})dv^2$ replaced with $T(u,\vec{x})du^2$. For this case $\delta\lambda$ and $\delta\psi_i$ transform as before but the remaining components transform slightly differently:

$$\begin{aligned} \delta\psi_{+v} &= \partial_v \epsilon_+ \\ \delta\psi_{+u} &= \left[\partial_u - \frac{1}{8}e^{\phi}\left(\partial_i T \Gamma^i \Gamma^- - 2\partial_i\phi \Gamma^i\Gamma^+\right)\right]\epsilon_+. \end{aligned} \tag{2.24}$$

One can easily verify that there are no nonzero spinors for which these variations vanish and hence all supersymmetries are broken. So, in accordance with the states in the perturbative spectrum of the heterotic string, a BPS-saturated, partially supersymmetric solution exists only if the oscillations are left-moving.

In the case of type-II strings we have two supersymmetry charges, and the analysis for each of them parallels the discussion in the preceding paragraphs. In IIA theory in ten dimensions, if we set the R-R fields to zero, we can put the supersymmetry transformations in the form

$$\begin{aligned} \delta\lambda &= \left[\partial_\mu\phi\gamma^\mu\Gamma_{11} + \frac{1}{6}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right]\eta \\ \delta\psi_\mu &= \left[\partial_\mu + \frac{1}{4}\left({\omega_\mu}^{ab} + {H_\mu}^{ab}\Gamma_{11}\right)\Gamma_{ab}\right]\eta, \end{aligned} \tag{2.25}$$

where $\Gamma_{11} = \Gamma^0 \ldots \Gamma^9$ and the $d = 10$ Majorana spinor η can be decomposed into two Majorana-Weyl spinors of opposite helicity, $\eta = \epsilon_+ + \epsilon_-$. If we substitute the uncharged solutions with either left-moving traveling waves, (2.11), or right-moving traveling waves, (2.11) with $T(v,\vec{x})dv^2$ replaced by $T(u,\vec{x})du^2$, we again find Killing spinors of the form

$$\epsilon_\pm = e^{\phi/2}\epsilon_{\pm 0}, \qquad \partial_\mu\epsilon_{\pm 0} = 0 \tag{2.26}$$

with the following helicity conditions. If $T = 0$, then there is no traveling wave and both right-moving and left-moving oscillators are in the ground state. In this case we have

$$\Gamma^-\epsilon_+ = 0 \qquad \Gamma^+\epsilon_- = 0, \tag{2.27}$$

and half of the $N = 2$ supersymmetries are preserved. If $T \neq 0$ then we have a traveling wave solution moving in either direction and we obtain

$$\begin{aligned} \text{left-moving } &\Rightarrow \epsilon_+ = 0, \quad \Gamma^+ \epsilon_- = 0, \\ \text{right-moving } &\Rightarrow \Gamma^- \epsilon_+ = 0, \quad \epsilon_- = 0. \end{aligned} \tag{2.28}$$

In each case only one quarter of the $N = 2$ supersymmetries are preserved.

A similar story unfolds for the type IIB string. Setting the R-R fields to zero, the supersymmetry transformations can be written as

$$\begin{aligned} \delta\lambda &= \left[\partial_\mu \phi \gamma^\mu \eta^* - \frac{1}{6} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \eta \right] \\ \delta\psi_\mu &= \left[\partial_\mu \eta + \frac{1}{4} \omega_\mu{}^{ab} \Gamma_{ab} \eta - \frac{1}{4} H_\mu{}^{ab} \Gamma_{ab} \eta^* \right], \end{aligned} \tag{2.29}$$

where η is a Weyl spinor and η^* its complex conjugate. Substituting the general form of the solution with left-moving or right-moving traveling waves we obtain Killing spinor fields of the form (2.26), with the helicity conditions

$$\begin{aligned} \text{fundamental string } &\Rightarrow \Gamma^+ \epsilon^1_+ = 0, \quad \Gamma^- \epsilon^2_+ = 0 \\ \text{left-moving } &\Rightarrow \Gamma^+ \epsilon^1_+ = 0, \quad \epsilon^2_+ = 0 \\ \text{right-moving } &\Rightarrow \epsilon^1_+ = 0, \quad \Gamma^- \epsilon^2_+ = 0 \end{aligned} \tag{2.30}$$

where we have decomposed η into its real and imaginary Majorana-Weyl components $\eta = \epsilon^1_+ + i\epsilon^2_+$. Just as in the type IIA case the fundamental string solution preserves half the supersymmetries, the traveling wave solutions, one quarter.

In summary, the supersymmetry of the spacetime solutions exactly mirrors that of the corresponding BPS states in the perturbative spectrum [4].

[4] The matching of fermion zero modes for Type II strings requires one to take into account a peculiar chirality flip between static gauge and light-cone gauge which is exactly analogous to that discussed in [37]

2.6. Sources

In order to make a clear identification of our solutions with the underlying fundamental string states we must show that near the core they match onto appropriate string sources which satisfy both the string equations of motion and the Virasoro constraints. To do this in an unambiguous way we shall take the radius of the macroscopic string to be much larger than both the string scale and the scale of compactification. For the neutral solutions this means we can ignore the contribution to the world-sheet stress tensor from the internal conformal field theory and from normal ordering. Therefore, we need to solve only the classical Virasoro constraints for D large dimensions. For charged strings on the other hand we are forced to include one-loop anomaly effects in order to get a consistent picture. This is forced by the close connection between the Chern-Simons couplings in the low-energy field theory and sigma-model anomalies in the string world-sheet action [38]. We first discuss the source terms for the neutral strings and then the charged strings.

Near the core, it is convenient to use the coordinates used in (2.11) even though they are not asymptotically flat. On the worldsheet we fix conformal gauge and use $\sigma^{\pm} = \tau \pm \sigma$. For the spacetime fields we take the ansatz (2.11), and for the string coordinates we take

$$U = U(\sigma^+, \sigma^-) \qquad V = V(\sigma^+, \sigma^-) \qquad \vec{X} = 0. \tag{2.31}$$

The equations of motion for the string coordinates are

$$\begin{aligned} g_{\mu\nu}\partial_m\left(\sqrt{\gamma}\gamma^{mn}\partial_m X^\nu\right) + \Gamma_{\mu\nu\rho}\left(\sqrt{\gamma}\gamma^{mn}\partial_m X^\nu \partial_n X^\rho\right) \\ + \frac{1}{2}H_{\mu\nu\rho}\left(\epsilon^{mn}\partial_m X^\nu \partial_n X^\rho\right) = 0, \end{aligned} \tag{2.32}$$

where $\Gamma_{\mu\nu\rho}$ is the Christoffel symbol for $g_{\mu\nu}$. With our ansatz, the u, v and i components become

$$\begin{aligned} e^{2\phi}\left(\partial_+\partial_- U - 2T\partial_+\partial_- V - \partial_v T \partial_+ V \partial_- V\right) = 0 \\ e^{2\phi}\partial_+\partial_- V = 0 \\ \partial_i\left(e^{2\phi}T\right)\partial_+ V\partial_- V - \partial_i\left(e^{2\phi}\right)\partial_+ U \partial_- V = 0 \end{aligned} \tag{2.33}$$

respectively. Similarly, the Virasoro constraints are given by

$$
\begin{aligned}
T_{++} &= -e^{2\phi}\left(\partial_+ V \partial_+ U - T \partial_+ V \partial_+ V\right) = 0 \\
T_{--} &= -e^{2\phi}\left(\partial_- V \partial_- U - T \partial_- V \partial_- V\right) = 0
\end{aligned}
\tag{2.34}
$$

Finally, the equations of motion for the spacetime fields reduce to

$$
\begin{aligned}
&\partial^2 e^{-2\phi} = -\frac{\kappa^2}{\pi\alpha'} \int d\sigma^+ d\sigma^- \left(\partial_+ V \partial_- U - \partial_- V \partial_+ U\right) \delta^D(\vec{x} - \vec{X}) \\
&\partial^2 e^{-2\phi} = -\frac{\kappa^2}{\pi\alpha'} \int d\sigma^+ d\sigma^- \left(\partial_+ V \partial_- U + \partial_- V \partial_+ U\right) \delta^D(\vec{x} - \vec{X}) \\
&0 = \int d\sigma^+ d\sigma^- \partial_+ V \partial_- V \delta^D(\vec{x} - \vec{X}) \\
&T \partial^2 e^{-2\phi} + e^{-2\phi} \partial^2 T = -\frac{\kappa^2}{\pi\alpha'} \int d\sigma^+ d\sigma^- \partial_+ U \partial_- U \delta^D(\vec{x} - \vec{X})
\end{aligned}
\tag{2.35}
$$

coming from the vu component of the $H_{\mu\nu\rho}$ equation, the vu, vv and uu components of the Einstein equation, respectively.

There is an important subtlety in solving these equations because the sources are not ordinary functions but are Dirac delta-functions in the transverse space. Do such distributional sources lead to unique spacetime solutions? One can regard the Dirac delta-function as a limit of a smooth function. Now, there are many different smooth functions which, in a limit, represent a delta-function. Geroch and Traschen [39] show that in general, due to the non-linear nature of general relativity, the limiting spacetimes are in fact different depending on the function one uses to represent the delta function. However, for metrics whose curvature tensor (or in our case the left-hand side of (2.35)) itself is a distribution, the limiting spacetimes are the same. Therefore, if we want a distributional string source to uniquely determine a spacetime solution, the left-hand sides of (2.35) must also be distributions.

Another subtlety is that for all the solutions that we consider, $e^{2\phi} = 0$ at $r = 0$. As a result, (2.33) can be satisfied without constraining the string coordinates at all provided the string is infinitely thin and lies precisely at $r = 0$. However, if we replace the delta-function in (2.35) with a smooth distribution then the equations must be solved even slightly away from $r = 0$ where $e^{2\phi}$ is nonzero. This is what we shall do in the following.

We are now ready to determine which of the solutions described earlier match onto sources. Let us first discuss the straight-string solution with $T = 0$. In this case, the left-hand sides of (2.35) are indeed distributions. As shown in [12], we can satisfy all equations of motion and constraints if we choose $V = V(\sigma^+)$ and $U = U(\sigma^-)$. Using the residual conformal invariance we can write the solution in the form

$$U = 2Rn\sigma^-, \qquad V = 2Rn\sigma^+. \tag{2.36}$$

Here we have chosen the normalization such that the string wraps n times around the circle of radius R as σ goes from 0 to π. The constant Q in the spacetime solution is then determined by the source to be

$$Q = \frac{n\kappa^2}{\pi\alpha'(D-4)\omega_{D-3}}, \tag{2.37}$$

which is such that the ADM-mass per unit length is equal to n times the string tension $n/(2\pi\alpha')$.

For the traveling wave solution (2.11), we find that the left-hand side of the uu component of the Einstein equation is not a distribution if T diverges at $r = 0$. Hence neither the solutions with $\beta \leq -D + 3$ (see the discussion after (2.12)), nor the solutions with nonzero p ($\beta = -D + 4$) can be interpreted as arising from delta-function sources. If we further require that the solution be asymptotically flat, then T is completely determined to be $\vec{f}(v) \cdot \vec{x}$. Using this form of T the string and source equations can be easily solved. Because $T = 0$ at the source we can solve the Virasoro constraints (2.34) by taking

$$U = (2Rn + a)\sigma^-, \qquad V = 2Rn\sigma^+. \tag{2.38}$$

where $a \equiv \frac{1}{\pi}\int_0^{2\pi Rn} \dot{F}^2$ is the zero mode of $\dot{F}^2$. We shall shortly see that this is a convenient choice. The constant Q in the spacetime solution is again given by (2.37). Thus, the oscillating string solutions are the only asymptotically flat solutions that correctly match onto a string source.

As it stands, the solution (2.38) appears to correspond not to an oscillating string at all, but rather to a straight string lying at $r = 0$. However, if we transform to the primed coordinates (2.14) in which the metric is manifestly asymptotically flat, we obtain

$$\begin{aligned} V' &= 2Rn\sigma^+ \\ U' &= (2Rn + a)\sigma^- + \int^{V'} \dot{F}^2 \\ \vec{X}' &= \vec{F}(V'), \end{aligned} \tag{2.39}$$

which is indeed an oscillating string with profile $\vec{F}(V')$. One can directly check that the Virasoro constraints are satisfied using the metric (2.15) at the source.

The identification of this solution with the spacetime fields around an oscillating string state can be further clarified by taking $\kappa^2 \to 0$ so that the string is very weakly coupled to the spacetime fields. This implies that $e^{2\phi} \to 1$ and uniquely defines a flat-space limit. One can then directly verify that the string configuration (2.39) satisfies the Virasoro constraints in the limiting flat space and it is clearly that of an oscillating string. The momentum p^μ and winding vector n^μ of the string (2.39) in flat space can be straightforwardly determined and we find in the $({X'}^0, \vec{X}', {X'}^{D-1})$ coordinates that $n^\mu = (0, \vec{0}, n)$ and $p^\mu = (2\alpha')^{-1}(2nR + a, \vec{0}, -a)$, where we have used the fact that $\vec{F}$ has no zero mode. The momentum in the compact direction is defined as m/R and, by definition, the classical number oscillator is given by $N_L = -nm$. For the string in flat space we thus have,

$$m = -\frac{Ra}{2\alpha'}, \qquad N_L = \frac{nRa}{2\alpha'}, \qquad a = \frac{1}{\pi}\int_0^{2\pi Rn} dv \dot{F}^2. \tag{2.40}$$

Note that both the momentum in the compact direction and N_L are determined by the zero mode of $\dot{F}^2$, as one expects.

The ADM momentum of our full string solution, (2.16), is consistent with the momentum of the above string source in flat space. This confirms that our spacetime solution really does correspond to that of an oscillating string and that the momentum is not renormalized. On this latter point lets us also note that the calculation of the winding vector in flat space is equally valid for the source (2.39) in the full solution. What about the momentum vector? Starting from the string source in (2.7), the momentum conjugate to X^μ is given by:

$$p_\mu = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma (g_{\mu\nu} \dot{X}^\nu + B_{\mu\nu} X'^\nu). \tag{2.41}$$

After substituting the quantities in (2.15) evaluated at the string source, we find a cancellation between various term and we are left with the momentum vector in flat space. Note that in flat space, p_μ comes from the $g_{\mu\nu}\dot{X}^\nu$ term, while in the full solution it comes from the $B_{\mu\nu}X'^\nu$ term. This again reveals that the momentum is not renormalized and is perhaps analogous to the cancellation of perturbative contributions to the energy density for straight string solutions originally found in [11].

At this point we have provided substantial evidence that the neutral traveling wave solutions with $p = 0$ should be identified with the neutral (m, n) state. We have worked in the regime where N_L is large and the radius of the circle is large compared to the string scale so that the string source can be treated classically. In this limit, an oscillating string solution can be thought of as a coherent superposition of large numbers of first-quantized states and should not be sensitive to normal ordering effects such as the one in the Virasoro constraint in (2.2).

The solutions with $p \neq 0, F = 0$ have not played a role in our discussion so far because they do not map onto classical source terms. However, they are important for describing low-lying states. For example, the state $(1, 1)$ with $N_L = 0$ in the heterotic string theory has positive momentum and no oscillations. Therefore, it cannot be identified with a solution with $p = 0, F \neq 0$ which asymptotically always has negative momentum. However, as advocated in [40], it can be identified with the string solution with $F = 0$, $p = -Q/2R^2$ and Q given by (2.37). Using the ADM mass formula (2.16) and the calculations in this section, we see that asymptotically it corresponds to a straight string with momentum

$1/R$ in the compact direction and mass $(2R-1/R)$. Of course, for the low lying states, α' corrections could be important and can modify the classical source. Further evidence for this identification would be provided if one could show that this solution can be mapped onto some kind of "quantum string source." Such a calculation was attempted in [40] but there are subtleties in matching to the correct string states which were overlooked, so this is still an open question. Nevertheless, the comparison of the classical scattering of these spacetime solutions with scattering of $(1,1)$ states in string theory carried out in [40] seems to support this identification.

Now let us discuss how the string solutions with left-moving charges and traveling waves map onto source terms. We expect that the charged strings can be mapped onto sources coming from left-moving internal coordinates.

In order to do this correctly we must take into account the relation between the spacetime Chern-Simons couplings in the low-energy effective action and the world-sheet chiral anomalies which arise for purely left-moving currents. When we couple to external gauge or gravitational fields the Chern-Simons terms lead to a current inflow which is precisely balanced by the anomalies of the world-sheet theory [41] so that overall charge and energy-momentum are conserved. Because we include the Chern-Simons terms in our low-energy effective action, we must also include one-loop world-sheet anomalies in order to have a consistent α' expansion.

In discussing this we shall make use of the formalism developed by Naculich [42]. There it was shown that at one loop the total gauge current on the world-sheet is afflicted by the covariant anomaly rather than the consistent anomaly. The point is that the Chern-Simons coupling in (2.5) leads to an extra term in the gauge current (the last term in (2.8)) which is precisely what is needed to turn the total gauge current into the covariant anomaly current when J^m is the consistent anomaly current.

It was also shown in [42] how to incorporate these quantum effects in terms of classical chiral world-sheet bosons. This is ideal for our needs since, as in the neutral case, we would like to solve for both the spacetime and the worldsheet fields. The last term in (2.7) is then given by

$$S_A = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\sqrt{\gamma}\gamma^{mn}(\partial_m Y - \sqrt{2\alpha'}A_m)(\partial_n Y - \sqrt{2\alpha'}A_n) - \sqrt{2\alpha'}Y\epsilon^{mn}F_{mn}\right) \tag{2.42}$$

where Y is a left-moving internal coordinate satisfying the constraint

$$\partial_- Y = \sqrt{2\alpha'}A_-. \tag{2.43}$$

and we have introduced the notation $A_m = \partial_m X^\mu A_\mu$ and $F_{mn} = \partial_m X^\mu \partial_n X^\nu F_{\mu\nu}$. Varying this action with respect to the spacetime gauge field leads to (2.8) where J^m is given by the consistent anomaly current

$$J^m = -\sqrt{2\alpha'}\sqrt{\gamma}\gamma^{mn}\left(\partial_n Y - \sqrt{2\alpha'}A_n\right) - \sqrt{2\alpha'}\epsilon^{mn}\partial_n Y. \tag{2.44}$$

Note that the total action is gauge invariant provided Y transforms along with A_μ and $B_{\mu\nu}$, so that

$$\begin{aligned}\delta A_\mu &= \nabla_\mu \Lambda \\ \delta B_{\mu\nu} &= 2\alpha' \Lambda F_{\mu\nu} \\ \delta Y &= \sqrt{2\alpha'}\Lambda.\end{aligned} \tag{2.45}$$

It is worth pointing out that the source terms (2.42) are consistent with dimensional reduction. If we reduce the spacetime and source action with no gauge fields on a circle of constant radius, then we obtain a spacetime action with left-moving and right-moving gauge fields. Using the formula presented in the next section one can show that the source terms precisely give (2.42) after setting the right-moving gauge fields to zero. From this perspective, however, the constraint (2.43) must be imposed by hand.

The equations of motion for the string coordinates X can be straightforwardly derived and for our ansatz we are led to the first two equations in (2.33) with T replaced with W, while the i component is modified to give

$$\partial_i\left(e^{2\phi}W\right)\partial_+ V\partial_- V - \partial_i\left(e^{2\phi}\right)\partial_+ U\partial_- V + 2\sqrt{2\alpha'}\partial_i A_v \partial_+ Y\partial_- V = 0. \tag{2.46}$$

The equation of motion for Y is implied by the constraint (2.43). The Virasoro constraints are given by

$$\begin{aligned} T_{++} &= -e^{2\phi}\left(\partial_+ V \partial_+ U - W \partial_+ V \partial_+ V\right) + (\partial_+ Y - \sqrt{2\alpha'} A_+)^2 = 0 \\ T_{--} &= -e^{2\phi}\left(\partial_- V \partial_- U - T \partial_- V \partial_- V\right) = 0. \end{aligned} \tag{2.47}$$

The equations of motion for the spacetime fields reduce to the first three equations in (2.35) and in addition:

$$\begin{aligned} W\partial^2 e^{-2\phi} + & e^{-2\phi}\partial^2 W + 4\alpha' e^{-4\phi}\partial_i A_v \partial^i A_v \\ &= -\frac{\kappa^2}{\pi\alpha'}\int d\sigma^+ d\sigma^- \partial_+ U \partial_- U \delta^D(\vec{x} - \vec{X}) \\ A_v \partial^2 e^{-2\phi} - & \partial^2\left(A_v e^{-2\phi}\right) \\ &= \frac{\kappa^2}{4\pi\alpha'\sqrt{2\alpha'}}\int d\sigma^+ d\sigma^- \left(\partial_+ Y - \sqrt{2\alpha'} A_v \partial_+ V\right)\partial_- U \delta^D(\vec{x} - \vec{X}) \end{aligned} \tag{2.48}$$

coming from the uu component of the Einstein equation and the u component of the A_μ equation, respectively.

We can now solve these equations for the general charged, traveling-wave solution given in (2.19) except that we take $p = 0$. The first three equations in (2.35) imply that $\partial_- V = 0$ and that the constant Q is given by (2.37). Whereas T was zero at $r = 0$ for the pure traveling wave solution, for the general charged solution, W at $r = 0$ is $2\alpha' q^2/Q^2$. This means that, while the second two terms on the right-hand side of the first equation in (2.48) are zero distributions, the first term is no longer zero. To solve this equation we must take

$$\partial_+ U = \frac{2\alpha' q^2}{Q^2}\partial_+ V. \tag{2.49}$$

Finally, we must solve the second equation in (2.48). Noting that $A_v = q/Q$ at $r = 0$, we find that the two terms on the left-hand side of the equation, each a delta function in the transverse space, cancel against each other. Solving for the right-hand side equal to zero, we get the further condition

$$\partial_+ Y = \frac{\sqrt{2\alpha'} q}{Q}\partial_+ V. \tag{2.50}$$

Of course the value of the gauge field at $r = 0$ can always be changed by a gauge transformation. In general we find the condition (2.50) is $\partial_+ Y = \sqrt{2\alpha'}\, A_v|_{r=0}\, \partial_+ V$, which is a gauge covariant expression. We note that if the string is not wound around the internal Y direction then we can always find a gauge transformation to put $Y = 0$. This is the analog of choosing coordinates where the traveling wave profile is flat.

With the conditions (2.49) and (2.50) we find that the string equations of motion are also satisfied, except for the v component of the X^μ equation which reads

$$e^{2\phi}\partial_+\partial_- U = 0. \tag{2.51}$$

implying that $U = U_+(\sigma_+) + U_-(\sigma_-)$. In particular we note that (2.50) implies that there is no contribution from Y to the Virasoro constraints. From the left-moving constraint (2.43) on Y we find that $\partial_- Y = 0$.

As before, using the residual conformal invariance, we can rewrite the solution for the string coordinates in the convenient form

$$\begin{aligned} U &= (2Rn + a_F + a_q)\sigma^- + \frac{2\alpha'}{Q^2}\int^V q^2 \\ V &= 2Rn\sigma^+ \\ Y &= \frac{\sqrt{2\alpha'}}{Q}\int^V q, \end{aligned} \tag{2.52}$$

where now

$$a_F \equiv \frac{1}{\pi}\int_0^{2\pi Rn} \dot{F}^2 \qquad a_q \equiv \frac{1}{\pi}\int_0^{2\pi Rn} \frac{2\alpha'}{Q^2}q^2. \tag{2.53}$$

Transforming to manifestly asymptotically flat coordinates this becomes

$$\begin{aligned} U' &= (2Rn + a_F + a_q)\sigma^- + \int^{V'} \left(\dot{F}^2 + \frac{2\alpha'}{Q^2}q^2\right) \\ V' &= 2Rn\sigma^+ \\ \vec{X}' &= \vec{F}(V') \\ Y &= \frac{\sqrt{2\alpha'}}{Q}\int^{V'} q \end{aligned} \tag{2.54}$$

which appears to describe an oscillating string with a profile $\vec{F}(V')$ in the transverse space and a profile $\int^{V'} q$ in the internal space, which is manifest as a charged wave in the external space. To pin down the state of the source string, we again go to the weak coupling limit by letting $\kappa^2 \to 0$. One can then directly verify that the configuration (2.54) satisfies the corresponding flat Virasoro constraints, and so we may interpret the source as a oscillating string state carrying a charge wave. Calculating the momentum, p^μ, and winding, n^μ, vectors in $({X'}^0, \vec{X}', {X'}^{D-1})$ coordinates gives $n^\mu = (0, \vec{0}, n)$ and $p^\mu = (2\alpha')^{-1}(2nR + a_F + a_q, \vec{0}, a_F + a_q)$. The momentum in the internal Y direction is a_q. Thus we can identify the momentum in the ${X'}^{D-1}$ direction, the classical oscillator number N_L and internal charges q_L^2 as

$$m = -\frac{(a_F + a_q)R}{2\alpha'} \qquad N_L = \frac{(a_F + a_q)nR}{2\alpha'} \qquad q_L^2 = -\frac{a_q nR}{\alpha'}. \tag{2.55}$$

Again the ADM momentum of the full solution (2.20) is consistent with the momentum of the string source (2.54) in flat space.

So far we have discussed the solution only to the lowest order in α', but it may be possible to extend these results to higher orders. Away from the string source, all solutions that we have described, essentially belong to a general class of solutions called 'chiral null models' by Horowitz and Tseytlin [43]. These authors have shown that there exists a particular renormalization scheme in the worldsheet sigma-model in which the lowest-order solution of a chiral null model satisfies the string equations of motion to all orders in α'. Moreover, Tseytlin has suggested [44] that, even near the core, the only effect of α' corrections will be to smooth out the source on the scale of α'. Tseytlin has also proposed [45] an attractive and plausible picture of how the sources might arise by resumming 'thin handles' of the full string partition function. These results suggest that an exact conformal field theory should exist corresponding to a macroscopic string. Such a 'conformal field theory' with sources would be an interesting generalization of the usual notion of an exact string solution.

2.7. Dimensional Reduction

Under toroidal compactification, the infinite tower of macroscopic strings reduces to various pointlike and string-like BPS-saturated states in lower dimensions. One can generate the solutions corresponding to these states by dimensional reduction of the macroscopic string solutions described earlier.

A macroscopic string solution can be reduced in two ways: either in a direction transverse to the string to obtain a string-like solution, or in the longitudinal direction to obtain a point-like solution. We discuss the former in this subsection and the latter in section three. Dimensional reduction in a transverse direction can be achieved by considering an infinite periodic array of strings. This is similar to the way magnetic monopoles or H-monopoles can be obtained from periodic arrays of instantons [46,47,48].

For simplicity let us consider a macroscopic string in six dimensions that wraps around the x^5 direction on a large circle of radius R. We would like to reduce this solution along one of the transverse directions, say x^4, on a circle of radius R_c, $x^4 \equiv x^4 + 2\pi R_c$. We can take R_c to be much smaller than R but still much larger than the string scale. To reduce the solution we consider an infinite periodic array of oscillating strings localized at $\vec{x} = \vec{x}_0, x^4 = x^4_0 + 2\pi R_c k$ where $\vec{x} = (x^1, x^2, x^3)$ is a three-dimensional vector and k is an integer, with the same profile of oscillation $\vec{F}(v)$. This multi-string solution is determined by the dilaton (2.18)

$$e^{-2\phi} = 1 + \sum_{k=-\infty}^{\infty} \frac{Q}{r^2 + (x^4 - x^4_0 - 2\pi R_c k)^2} \tag{2.56}$$

with $r^2 = |\vec{x} - \vec{x}_0 - \vec{F}(x)|^2$. The infinite sum can be readily performed [46] to obtain

$$e^{-2\phi} = 1 + \frac{Q}{2R_c r} \sinh\frac{r}{R_c} (\cosh\frac{r}{R_c} - \cos\frac{x^4 - x^4_0}{R_c})^{-1}. \tag{2.57}$$

For fixed Q, in the asymptotic limit $r \gg R_c$ or, equivalently, averaging over x^4, (2.57) reduces to

$$e^{-2\phi} = 1 + \frac{Q}{2R_c r} + O(e^{-r/R_c}). \tag{2.58}$$

We recognize this solution as the five-dimensional dilaton given by (2.18) with $D = 5$ if we identify the five-dimensional dilaton charge with $Q/2R_c$. We have thus constructed a solution that at distances much larger than the scale of compactification looks exactly like a five-dimensional oscillating string. On smaller scales, however, it differs substantially. In particular, near the core it dynamically decompactifies and matches onto a six-dimensional source.

It is easy to generalize this construction to an arbitrary toroidal compactification of an oscillating string. We simply consider an array of strings on an arbitrary lattice using the general multi-string solution (2.18). In general, it is not possible to perform the infinite discrete sum analytically but its asymptotic form can easily be extracted.

These considerations clarify the nature of sources in the previous subsection. When we solved the constraint equations for a string in D large dimensions we ignored the contributions to the world-sheet stress tensor from the internal conformal field theory. If the the conformal field theory is a torus, then this is justified when the size of the torus R_c is much smaller than the radius R of the string. The solution then matches onto a D-dimensional source. However, one can also consider intermediate scales when R and R_c are both comparable and much larger than the string scale. In this case, we effectively have a ten-dimensional theory and the solution matches onto a ten-dimensional source. Similar considerations should apply to a more general compactification.

We can also use dimensional reduction to understand the relation between different types of string solutions. For example, consider a traveling wave solution with the profile pointing in the compact direction. Then using the solution in asymptotically flat coordinates and the techniques in the following section we get precisely the charged string solution after taking the limit $r >> R$. In particular the charge wave is given by $q(v) = Q\dot{F}/\sqrt{2}$.

3. Black Holes as Strings

We now turn to dimensional reduction of macroscopic strings in the longitudinal direction to obtain point-like "black hole" solutions in one lower dimension.

In toroidal compactification of string theories, there exists an infinite tower of BPS-saturated, electrically-charged, point-like states in the perturbative spectrum, which is closely related to the tower of macroscopic strings discussed earlier. For example, in heterotic string theory compactified to four dimensions on a six-torus, there is a tower of states satisfying the mass-shell conditions

$$\begin{aligned} N_L &= 1 + \alpha' \left(q_R^2 - q_L^2 \right), \qquad N_R = 0, \\ M^2 &= 4 q_R^2. \end{aligned} \tag{3.1}$$

Here q_R and q_L are the right-moving and the left-moving charge-vectors respectively; together they define a point in a 28-dimensional, self-dual, Lorentzian lattice with signature $(22, 6)$. A typical such state has nonzero angular momentum J which satisfies a Regge bound [35]. The right-moving ground state corresponds to a $N = 4$ vector-multiplet in four dimensions so can have maximum spin one. The maximum left-moving spin is $|N_L|$, so the total angular momentum satisfies the Regge bound:

$$J \leq |2 + \alpha' \left(q_R^2 - q_L^2 \right) |. \tag{3.2}$$

Starting from these states, S-duality predicts [23] an infinite tower of magnetically charged states, which must exist as solitons. We shall discuss these solitons in section four.

Corresponding to the tower of first-quantized states we expect to find BPS-saturated electric black-holes. The most general, rotating, charged, stationary black hole solution of the low-energy heterotic string has been constructed by Sen [22]. We can try to identify this solution with the states discussed above by choosing appropriate asymptotic quantum numbers, so that in particular, the solution is supersymmetric. However, we immediately run into a difficulty because to obtain the supersymmetric limit one has to extend well beyond extremality when the angular momentum is non-zero and the solution has a severe ring-like naked singularity [49]. Moreover, the angular momentum of these solutions can be arbitrary and is not Regge-bounded.

The new solutions that we discuss in this section are obtained by dimensionally reducing an oscillating string in higher dimensions along the length of the string. These solutions are asymptotically identical to the supersymmetric, stationary, rotating, electrically-charged, black-hole solutions but near the core they have a much milder singularity of a higher-dimensional string source. Their angular momentum is naturally Regge-bounded because the source solves the string equations of motion and constraints.

In this section we restrict our attention to those states in the spectrum whose entire left-moving and right-moving charge arises from the winding and momentum along a single internal circle. In this case, the charges q_R and q_L in (3.1) are identical to the momenta p_R and p_L respectively in (2.2). For simplicity, we shall also restrict to the dimensional reduction of oscillating strings in five dimensions to four. These considerations can be easily extended to more general cases.

3.1. Dimensional Reduction to Four Dimensions

The five-dimensional action is obtained by specializing (2.5) to $D = 5$ and setting the gauge fields to zero:

$$S = \frac{1}{2\tilde{\kappa}^2} \int d^5x \sqrt{-\tilde{g}} e^{-2\tilde{\phi}} \left(R(\tilde{g}) + 4(\nabla\tilde{\phi})^2 - \frac{1}{12}\tilde{H}^2 \right), \tag{3.3}$$

where the tilde $\tilde{}$ is used to denote five-dimensional quantities. For dimensional reduction, it is convenient to introduce [50] the 'four-dimensional' fields $g_{\mu\nu}, B_{\mu\nu}, \phi, A^a_\mu$ and σ $(0 \leq \mu \leq 3, a = 1, 2)$ through the relations

$$\begin{aligned} e^{-4\sigma} &= \tilde{g}_{44}, & \phi &= \tilde{\phi} + \sigma, \\ A^1_\mu &= \frac{1}{2}\tilde{g}_{4\mu} e^{4\sigma}, & A^2_\mu &= \frac{1}{2}\tilde{B}_{4\mu}, \\ g_{\mu\nu} &= \tilde{g}_{\mu\nu} - \tilde{g}_{4\mu}\tilde{g}_{4\nu} e^{4\sigma}, \\ B_{\mu\nu} &= \tilde{B}_{\mu\nu} - 2(A^1_\mu A^2_\nu - A^1_\nu A^2_\mu). \end{aligned} \tag{3.4}$$

The low-energy lagrangian of a string compactified on a circle is expected to have $SO(1,1)$ symmetry. This is broken to the perturbative $SO(1,1,Z)$ T-duality symmetry by world-sheet instantons. To make this symmetry manifest, we define

$$M = \begin{pmatrix} e^{4\sigma} & 0 \\ 0 & e^{-4\sigma} \end{pmatrix} \qquad\qquad L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.5}$$

which satisfy

$$MLM^T = L, \qquad M^T = M \tag{3.6}$$

To obtain the effective action in four dimensions, we express the five-dimensional fields in (3.3) in terms of the four-dimensional fields and take all fields to be independent of x^4. The resulting action is

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2 - F^a(LML)_{ab}F^b\right) \\ + \frac{1}{8}\mathrm{Tr}(\nabla ML\nabla ML), \end{aligned} \tag{3.7}$$

where all spacetime indices are contracted with the four-dimensional metric and $\kappa^2 \equiv \frac{\tilde{\kappa}^2}{2\pi R}$ is the four-dimensional gravitational coupling. The field strengths are defined by

$$F^a = dA^a, \qquad H = dB + 2A^a \wedge F^b L_{ab}. \tag{3.8}$$

Note that a Chern-Simons term appears in the definition of H in four dimensions even though it was absent in five dimensions. For later use, it is more convenient to use 'left-moving' and 'right-moving' gauge fields defined by

$$A^L = \frac{-A^1 + A^2}{\sqrt{2}}, \qquad\qquad A^R = \frac{A^1 + A^2}{\sqrt{2}}, \tag{3.9}$$

so that the anomaly term for the two gauge fields is diagonal.

By applying these reduction formulae to the neutral oscillating string solution (2.15) with $p = 0$ and reducing along the direction x^4 (recall $u = x^0 - x^4$, $v = x^0 + x^4$) one obtains the four-dimensional quantities:

$$
\begin{aligned}
e^{-4\sigma} &= \left(e^{2\tilde{\phi}} - \dot{F}^2(e^{2\tilde{\phi}} - 1)\right), \\
e^{4\phi} &= -\, g_{00} = e^{4\tilde{\phi}+4\sigma}, \\
g_{ij} &= \delta_{ij} - \dot{F}_i \dot{F}_j e^{4\sigma}(e^{2\tilde{\phi}} - 1)^2, \\
g_{0i} &= \dot{F}_i \dot{F}^2 e^{4\sigma}(e^{2\tilde{\phi}} - 1)^2 + \dot{F}_i(e^{2\tilde{\phi}} - 1), \\
A^1_0 &= -\frac{1}{2}\dot{F}^2 e^{4\sigma}(e^{2\tilde{\phi}} - 1), \qquad A^1_i = \frac{1}{2}\dot{F}_i e^{4\sigma}(e^{2\tilde{\phi}} - 1), \\
A^2_0 &= -\frac{1}{2}(e^{2\tilde{\phi}} - 1), \qquad A^2_i = \frac{1}{2}\dot{F}_i(e^{2\tilde{\phi}} - 1), \\
B_{0i} &= \dot{F}_i(e^{2\tilde{\phi}} - 1) + \frac{1}{2}\dot{F}_i(\dot{F}^2 - 1)e^{4\sigma}(e^{2\tilde{\phi}} - 1)^2,
\end{aligned}
\tag{3.10}
$$

where

$$
e^{-2\tilde{\phi}} = 1 + \frac{Q}{|\vec{x} - \vec{F}(v)|}. \tag{3.11}
$$

Note that this solution still has dependence on the internal coordinate. To obtain a true four-dimensional solution one must do the usual Kaluza-Klein truncation as we explain in the following. Before continuing to analyze this solution we first review the stationary extremal black hole solutions of [22]. We shall see that the solution (3.10) asymptotically approaches these solutions.

3.2. Supersymmetric Stationary Black holes

The most general stationary solution of (3.7) that describes a rotating, electrically-charged black hole was constructed by Sen [22]. For our purpose we shall be interested in only the supersymmetric solutions, which in the notation of [22], correspond to the limit $\beta \to \infty, m \to 0$ keeping $m_0 = m\cosh(\beta)$ and α fixed. The resulting solution depends upon three parameters, a, m_0, and α. Introducing

$$
\Delta = (r^2 + a^2\cos^2\theta)^2 + 2m_0\cosh\alpha(r^2 + a^2\cos^2\theta) + m_0^2 r^2 \tag{3.12}
$$

the solution can be written as

$$
\begin{aligned}
e^{4\phi} &= \Delta^{-1}(r^2+a^2\cos^2\theta)^2, \\
ds^2 &\equiv g_{\mu\nu}dx^\mu dx^\nu \\
&= -e^{4\phi}dt^2+(r^2+a^2\cos^2\theta)(r^2+a^2)^{-1}dr^2+(r^2+a^2\cos^2\theta)\big\{d\theta^2 \\
&\quad +\Delta^{-1}\sin^2\theta\big[\Delta+a^2\sin^2\theta(r^2+a^2\cos^2\theta+2m_0r\cosh\alpha)\big]\,d\varphi^2 \\
&\quad -2m_0ar\sin^2\theta\Delta dtd\varphi\big\}, \\
A_0^L &= -\frac{m_0n_L}{\sqrt{2}}\Delta^{-1}r\sinh\alpha(r^2+a^2\cos^2\theta), \\
A_\varphi^L &= -\frac{m_0^2n_L}{\sqrt{2}}\Delta^{-1}a\sinh\alpha r^2\sin^2\theta(r^2+a^2\cos^2\theta), \\
A_0^R &= -\frac{m_0n_R}{\sqrt{2}}\Delta^{-1}r\cosh\alpha(r^2+a^2\cos^2\theta), \\
A_\varphi^R &= +\frac{m_0n_R}{\sqrt{2}}\Delta^{-1}ar\sin^2\theta(r^2+a^2\cos^2\theta+m_0r\cosh\alpha), \\
B_{0\varphi} &= -m_0a\sin^2\theta r\Delta^{-1}(r^2+a^2\cos^2\theta+m_0r\cosh\alpha),
\end{aligned}
\tag{3.13}
$$

where n_L and n_R determine the sign of the charges and can be either ±1. For discussing the asymptotic behavior of these fields at large r, it is simpler to use cartesian coordinates t, z, x^1, x^2. The axis of rotation of the black hole is in the z direction, $x^i (i = 1, 2)$ are the coordinates in the transverse plane, and $r^2 = z^2 + (x^i)^2$. The asymptotic fields are then given by

$$
\begin{aligned}
e^{4\phi} &\sim 1-\frac{2m_0\cosh\alpha}{r}, \\
ds^2 &\sim -(1-\frac{2m_0\cosh\alpha}{r})dt^2+dz^2+dx^idx^i-\frac{2m_0a}{r}\epsilon_{ij}x^idx^j\,dt, \\
A_0^L &\sim -\frac{m_0n_L\sinh\alpha}{\sqrt{2}r}, \qquad A_i^L \sim 0, \\
A_0^R &\sim -\frac{m_0n_R\cosh\alpha}{\sqrt{2}r}, \qquad A_i^R \sim \frac{m_0n_Ra\epsilon_{ij}x^j}{\sqrt{2}r^3}, \\
B_{0i} &\sim -\frac{m_0a\epsilon_{ij}x^j}{r^3},
\end{aligned}
\tag{3.14}
$$

where ϵ_{ij} is an antisymmetric tensor with $\epsilon_{12} = 1$. From these asymptotic formulae, one can easily read off the mass M, the angular momentum J, the left-moving and right-moving charges Q_L and Q_R, and the magnetic moments μ_L and μ_R to obtain

$$\begin{aligned} M &= \frac{1}{2} m_0 \cosh\alpha, & J &= \frac{1}{2} m_0 a, \\ Q_L &= \frac{m_0 n_L}{\sqrt{2}} \sinh\alpha, & Q_R &= \frac{m_0 n_R}{\sqrt{2}} \cosh\alpha, \\ \mu_L &= 0, & \mu_R &= \frac{m_0 n_R a}{\sqrt{2}}. \end{aligned} \tag{3.15}$$

Recall that for a black hole of mass M and angular momentum J, the canonical Einstein metric $\hat{g}_{\mu\nu}$ takes the asymptotic form [51]

$$d\hat{s}^2 \equiv \hat{g}_{\mu\nu} dx^\mu dx^\nu \sim -(1 - \frac{2M}{r})dt^2 + dz^2 + dx^i dx^i - \frac{4J}{r^3}\epsilon_{ij} x^i dx^j dt. \tag{3.16}$$

In four dimensions, the string metric is related to the Einstein metric by the relation $g_{\mu\nu} = \hat{g}_{\mu\nu} e^{2\phi}$.

3.3. Supersymmetric Black Holes as Strings

To obtain a rotating black hole in four dimensions from our dimensionally reduced string solutions, we choose the profile of oscillation $\vec{F}$ in (3.10) to be of the form

$$\vec{F} = A(\hat{e}_1 \cos\omega t + \hat{e}_2 \sin\omega t), \tag{3.17}$$

where $\hat{e}_i$ are unit vectors in the ith direction. This corresponds to a string configuration that is rotating in the $x^1 - x^2$ plane with amplitude A and frequency ω. The period of oscillation T can be taken to be very small, and on time scales much larger than T, only time-averaged quantities will be observable. These time-averaged quantities can be compared with the parameters of the stationary solution. There is an ambiguity in directly time-averaging the fields themselves. For example, the average of a product of two fields will not be the same as the product of averages if there are correlations between the two fields. Of course, all physical quantities such as geodesic distance, forces, and in particular all asymptotic charges will have a well defined time-average. Therefore, we consider time-averages of only the asymptotic fields.

The time-averaging is equivalent to the usual dimensional reduction. In order to reduce a solution that depends on the internal coordinate x^4, one has to first expand every field Ψ in terms of the momentum modes: $\Psi(x^4) = \sum_n \Psi_n e^{inx^4/R}$. From the lower-dimensional point of view, the modes Ψ_n are massive with masses of order n/R. Unless we probe the structure of the solution on scales much shorter than R, we see only the zero momentum mode Ψ_0 which is nothing but the average of Ψ. Because our solution depends on x^4 only through the combination $x^4 + t$, averaging over x^4 is the same as time-averaging.

The two basic time-averages are

$$\begin{aligned} \langle F_i \dot{F}_j \rangle &= \frac{A^2\omega}{2}\epsilon_{ij}, \\ \langle \dot{F}_i \dot{F}_j \rangle &= \frac{A^2\omega^2}{2}\delta_{ij}, \end{aligned} \tag{3.18}$$

where we use $\langle A \rangle$ to denote time-average of the quantity A over a period of oscillation. Using these relations we obtain

$$\begin{aligned} \langle e^{4\phi} \rangle &\sim -\langle g_{00} \rangle \sim 1 - \frac{Q}{r}(1 + A^2\omega^2), \\ \langle g_{0i} \rangle &\sim -\frac{QA^2\omega}{2r^3}\epsilon_{ij}x^j, \\ \langle B_{0i} \rangle &\sim -\frac{QA^2\omega}{2r^3}\epsilon_{ij}x^j, \\ \langle A_0^1 \rangle &\sim \frac{QA^2\omega^2}{2r} \qquad \langle A_i^1 \rangle \sim -\frac{QA^2\omega}{4r^3}\epsilon_{ij}x^j, \\ \langle A_0^2 \rangle &\sim \frac{Q}{2r} \qquad \langle A_i^2 \rangle \sim -\frac{QA^2\omega}{4r^3}\epsilon_{ij}x^j. \end{aligned} \tag{3.19}$$

One can easily read off various asymptotic quantities:

$$\begin{aligned} M &= \frac{Q(1 + A^2\omega^2)}{4}, & J &= \frac{QA^2\omega}{4}, \\ Q_L &= -\frac{Q}{2\sqrt{2}}(A^2\omega^2 - 1), & Q_R &= -\frac{Q}{2\sqrt{2}}(1 + A^2\omega^2), \\ \mu_L &= 0, & \mu_R &= -\sqrt{2}J. \end{aligned} \tag{3.20}$$

These are exactly the same parameters as in (3.15) with $n_L = n_R = -1$. The three parameters m_0, α, and a can be easily computed in terms of Q, A, and ω. The left-moving and right-moving gyromagnetic ratios are defined by the relations $\mu_L = \frac{g_L Q_L J}{2M}$ and $\mu_R = \frac{g_R Q_R J}{2M}$. One easily obtains

$$g_L = 0, \qquad g_R = 2 \tag{3.21}$$

These gyromagnetic ratios are related to the left-moving and right-moving angular momenta by the relations [15,22]

$$g_L = 2\frac{J_R}{J_R + J_L}, \qquad g_R = 2\frac{J_L}{J_R + J_L}. \tag{3.22}$$

Thus, from these asymptotic quantities we find that the angular momentum of our solution is carried entirely by left-movers consistent with the fact that the underlying string solution has only left-moving oscillations. It is also satisfying that the right-moving gyromagnetic ratio is two as expected for a fundamental object.

The discussion in the preceding paragraphs can be easily extended to nonrotating charged black holes by considering a slightly different profile of oscillation $\vec{F}$:

$$\vec{F}(t) = A\hat{e}_1 \cos\omega t. \tag{3.23}$$

The time-averages are somewhat different from (3.18)

$$\begin{aligned} \langle F_i \dot{F}_j \rangle &= 0, \\ \langle \dot{F}_i \dot{F}_j \rangle &= \frac{A^2\omega^2}{2}\delta_{i1}\delta_{j1}, \end{aligned} \tag{3.24}$$

The asymptotic parameters are then given by

$$\begin{aligned} M &= \frac{Q(2 + A^2\omega^2)}{8}, & J &= 0, \\ Q_L &= -\frac{Q}{4\sqrt{2}}(A^2\omega^2 - 2), & Q_R &= -\frac{Q}{4\sqrt{2}}(A^2\omega^2 + 2), \end{aligned} \tag{3.25}$$

which again agrees with (3.14) if we set $a = 0$. More generally, one can consider an arbitrary Fourier expansion for $\vec{F}$.

As a consistency check we note that, both for the rotating and the nonrotating solution, the asymptotic form of the antisymmetric tensor is also in agreement with the stationary solution. In four dimensions, the antisymmetric tensor is equivalent to an axion field b through the relation $db = *dH$. Now, because of the anomaly term $dH \sim \vec{E}\cdot\vec{B}$, there is a source term for the axion whenever there is a non-vanishing $\vec{E}\cdot\vec{B}$. A rotating charged object has both charge and magnetic moment and accordingly a nonzero $\vec{E}\cdot\vec{B}$ that falls off as $1/r^5$. One then expects a nonzero value of the axion that falls off as $1/r^3$. In the non-rotating case, on the other hand, one expects a faster fall-off. This is precisely the behavior we find for the time-averaged fields in the two cases.

We now compare these results with the perturbative spectrum (3.1). For this purpose we need to properly take into account different normalizations that we have used. The normalization of charges Q_L and Q_R depends on the normalization of the gauge fields, and is different from the normalization q_L and q_R. Furthermore, in the black-hole formulae above, we have chosen units in which the four-dimensional Newton's constant $G \equiv \kappa^2/(8\pi)$ is one. From the normalization of the sigma model action and the dimensional reduction formulae, we readily see that $Q_{L,R} = 2\sqrt{2}q_{L,R}$, and from (2.37) that $Q = 4nR/\alpha'$. If we consider a string that wraps n times around the internal circle of radius R then the periodicity of string coordinates implies that $\omega = \ell/nR$ for some integer ℓ.

Using these formulae it is easy to see that various asymptotic parameters of our black hole are in complete agreement with the quantum numbers of a perturbative state in the spectrum (3.1). In particular, $M = 2q_R$ and the angular momentum is given by

$$J = \frac{A^2\ell}{\alpha'}. \tag{3.26}$$

The maximum angular momentum on the other hand is given by (3.2) which from (3.20) equals $A^2\ell^2/\alpha'$. We thus see that the angular momentum is Regge-bounded and the leading Regge trajectory corresponds to taking $l = 1$ for arbitrary A. This is in precise agreement with the behavior expected from a string source at the core.

We would now like to comment upon the singularities of the reduced string solution vis-a-vis those of the stationary Sen solution. Recall, that the stationary Sen solution is obtained by twisting the Kerr solution in Einstein gravity. One of the peculiarities of the Kerr solution is the existence of the Kerr singular ring [49] which is a branch-cut connecting spacetime on 'negative' and 'positive' sheets. Across the branch-cut various fields change their signs and directions. A similar naked singularity exists also for the Sen solution; across the branch-cut the dilaton becomes imaginary. By contrast, although the reduced string solution asymptotically approaches the Sen solution after time-averaging, the solution has only a mild singularity of a string source from the five-dimensional point of view. In four dimensions, the string source with a rotating profile is squashed to a disk and it may appear to have a disk-like naked singularity.

We note that this is reminiscent of the proposal of [52] in which the negative sheet of the Kerr solutions is replaced by a disk-like source bounded by the Kerr ring. It was argued in [52] that the disk must be super-conducting and in rigid rotation, and in [49] that the boundary of the disk could be identified with some string-like source.

It is also worth comparing our solutions with those in [53] where rotating charged black holes were constructed in dimensions $D > 4$ with one non-zero component of the angular momentum. It was found that for these dimensions the BPS limit coincides with the extremal limit. By taking periodic arrays of these solutions in six dimensions, a four-dimensional rotating black hole was constructed by dimensional reduction that asymptotically approached the four-dimensional stationary charged rotating black hole solution. For these solutions the ring singularity of the four- dimensional black hole is replaced by a six-dimensional BPS charged rotating extremal black hole singularity. One difficulty in relating these solutions to the four-dimensional BPS states of string theory is that, unlike our solutions, the angular momentum is not Regge bounded.

We conclude this section with some comments concerning the dimensional reduction of solutions with $F = 0, p \neq 0$. Starting with the five-dimensional solution with $F = 0, p \neq 0$ we obtain four-dimensional static supersymmetric black hole solutions with mass and charge given by

$$M = \frac{Q+p}{4}, \qquad Q_L = \frac{1}{2\sqrt{2}}(Q-p), \qquad Q_R = \frac{1}{2\sqrt{2}}(Q+p). \tag{3.27}$$

For $p > 0$ this static black hole solution is precisely the same as the asymptotic behavior of the black hole solution that arises from the profile (3.24). This is to be expected because even before dimensional reduction, the solution with $p > 0$ is asymptotically the same as the time-averaged, non-rotating oscillating string solution. This can be seen, for example, from the ADM tensor (2.16).

It is also illuminating to consider the behavior of these static black hole solutions under the T-duality transformations of Buscher [54]. In our notation these transformations take the form

$$\begin{aligned} \sigma' &= -\sigma, \qquad \phi' = \phi + \sigma, \\ A^{1'} &= A^2, \qquad A^{2'} = A^1, \\ g'_{\mu\nu} &= g_{\mu\nu}, \qquad B'_{\mu\nu} = B_{\mu\nu}. \end{aligned} \tag{3.28}$$

One finds that this has the effect of simply interchanging the parameters Q and p in the solution. Now let us consider a neutral (m, n) state. We have seen in the last section that this corresponds to the neutral, oscillating-string solution with $p = 0$, Q given by (2.37), and a profile F satisfying (2.40). Comparing how F and p contribute to the ADM momentum we deduce that asymptotically the state (m, n) corresponds to the static black hole with parameters given by

$$\begin{aligned} Q &= \frac{n}{2\pi\alpha'}\frac{\tilde{\kappa}^2}{2\pi} \\ p &= -\frac{m}{2\pi R^2}\frac{\tilde{\kappa}^2}{2\pi} \end{aligned} \tag{3.29}$$

Replacing the five-dimensional gravitational coupling $\tilde{\kappa}^2$ with $2\pi R\kappa^2$ and recalling that the four-dimensional coupling κ^2 is held fixed under T-duality, we see that the interchange of p and Q coming from spacetime duality corresponds to the interchange of n and $-m$ along with R and α'/R which are the worldsheet T-duality transformations.

It is also interesting to consider the solution with $p < 0$. From (3.27) we see that we have constructed massless black holes when we choose $p = -Q$. In fact this construction is essentially the same as that in [55]. Note that for the value of p corresponding to the heterotic string state $(1, 1)$, $p = -QR^2/\alpha'$, the black holes are massless precisely at the self dual radius $R^2 = \alpha'$.

4. Strings as Solitons

At present there are essentially three independent exact string-string dualities which we list below.

I. The Type-IIA string compactified on a $K3$ surface is conjectured to be dual to the heterotic string compactified on a torus T^4.

II. The Type-IIB string in ten dimensions is conjectured to be dual to itself.

III. In ten dimensions, the Type-I string and the heterotic string with gauge group $SO(32)$ are conjectured to be dual to each other.

For our present purpose we shall be interested in only a few general aspects of these dualities. A more detailed description can be found, for example, in [56,32].

If two string theories are exactly dual to each other at a microscopic level, the spectrum of BPS-saturated states in the two theories must match. Under duality, perturbative states in one theory typically match onto nonperturbative states in the other. In this section we discuss the nonperturbative states that are dual to the infinite tower of states analyzed in previous sections. We then describe a variety of string-like and point-like solitons in the low energy theory that correspond to these states.

4.1. Infinite Tower of Solitonic Strings

Note that all string theories in the above list, with the exception of the Type-I theory, are supersymmetric theories of closed strings only. Consequently, they contain an infinite tower of macroscopic strings in their perturbative spectrum. Which states do these map onto under duality? We address this question below.

A basic requirement of exact duality is that the low-energy action for the massless fields in the two theories must agree after appropriate field-redefinitions. One feature common to all dualities is that strong coupling must match onto weak coupling, so the dilaton ϕ maps onto $-\phi$. Furthermore, the Einstein-Hilbert action should be left invariant, so $\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu}$ where the canonical metric $\hat{g}$ is related to the string metric g by $\hat{g}_{\mu\nu} = e^{-4\phi/(D-2)} g_{\mu\nu}$ in

D 'large' dimensions. Transformations of the remaining fields such as the gauge fields and the antisymmetric fields are specific to each case, and are often nonlocal, but they can all be deduced uniquely from the low-energy actions.

Given the duality map for the massless modes, and their coupling to a fundamental macroscopic string, one can immediately see that the infinite tower of macroscopic strings in each case must map onto a nonperturbative state in the corresponding dual theory. A macroscopic fundamental string couples universally and locally to the second-rank, antisymmetric, Kalb-Ramond field $B_{\mu\nu}$ through the action (2.7). So the duality transformation of this field is particularly important. For example, in the case of duality I above, the $B_{\mu\nu}$ fields of the two theories are related by a nonlocal transformation involving the field-strengths H, $H \to *H$. Thus, the state that is dual to a macroscopic string couples non-locally to the dual antisymmetric tensor field. Such a state must arise nonperturbatively because all perturbative states have local couplings with this field. Similarly, in the case of dualities II and III, the $B_{\mu\nu}$ field maps onto a field coming from the Ramond-Ramond sector. Once again there are no perturbative states that couple to the Ramond-Ramond field through the action (2.7), so these must also be nonperturbative.

It is trivial to find the solitonic solutions of the low-energy equations of motion that correspond to this infinite tower of states once we know the duality map for the massless modes. A solitonic solution is obtained simply by rewriting a solution in the previous sections in terms of the new dual variables.

Even though the two solutions are related by a field redefinition, their interpretation and properties are completely different. First, the mass per unit length of a solitonic string in a given theory is inversely proportional to the coupling constant of that theory. So these states are infinitely massive at weak coupling, as they should be. Second, a fundamental string solution discussed in previous sections has a source at the core; it therefore describes the fields around a perturbative macroscopic string state. One cannot really regard them as solitons any more than one can regard the fields around a classical electron as a soliton in quantum electrodynamics. A solitonic solution, on the other hand, has no sources. Third, for a fundamental string, the coupling constant vanishes near the core whereas for a solitonic string it diverges. Consequently, the structure of singularities of the string metric is quite different in the two cases. Finally, the nature of higher order α' corrections to the solutions also differs substantially in the two cases.

To illustrate these points let us consider the infinite tower of solitonic strings required to exist by duality in Type-IIA theory compactified on a $K3$ surface. The mass of these states goes as $1/\lambda^2$ where λ^2 is the loop counting parameter. These are the solitons that are dual to the infinite tower of perturbative states in the heterotic string theory compactified on T^4. The solution corresponding to an infinitely long straight-string soliton was analyzed in [6,7]. At a generic point in the moduli space of $K3$, where the gauge symmetry is $U(1)^{24}$, it was shown in [6,7] that the charged zero modes in the background of such a string live on an even, self-dual, Lorentzian lattice with signature $(20,4)$. This is precisely the structure expected for a charged heterotic soliton with $N_L = 0$. We would now like consider a more general soliton corresponding to the state (m,n) at finite radius with nonzero oscillations.

We start with the general uncharged solution (2.11) corresponding to an oscillating, heterotic macroscopic string in six dimensions and rewrite it in terms of Type-IIA variables. The resulting solution is given by,

$$\begin{aligned} ds^2 =& -\left(dudv - \vec{f}(v)\cdot\vec{x}dv^2\right) + e^{2\phi}d\vec{x}\cdot d\vec{x} \\ H_{ijk} =& -\epsilon_{ijk}{}^{l}\nabla_l\phi \\ e^{2\phi} =& 1 + \frac{Q}{r^{D-4}}, \end{aligned} \tag{4.1}$$

where i,j,k,l are the indices in the space transverse to the string, ϵ is the completely antisymmetric tensor, and f is an arbitrary function as before. We have used the coordinate frame that is not asymptotically flat because we wish to concentrate on the properties of the solution near the core. Notice that near the core, $\vec{x}\to 0$, the solution reduces to the one with no oscillations *i.e.* the straight-string solution discussed in [6,7]. This has two important consequences. First, the entire infinite tower of solitons is smooth near the core and has an infinite throat just like the straight-string soliton. Second, near the core they will be described by an exact superconformal field theory [57,58] that consists of a product of a WZW model and a free field with a linear dilaton, with an additional perturbation corresponding to the dv^2 term in the metric that becomes smaller and smaller down the throat. Of course, because of the growing dilaton, the theory eventually gets into strong coupling and the semi-classical approximation breaks down.

If we consider the duality III, then we expect to find heterotic winding states as solitons in Type-I theory. Because the mass of these solitons goes as $1/\lambda$ they are very different from the usual solitons in field theory. These solitons are also a source of the Kalb-Ramond field in the R-R sector of the Type-I theory, and correspond to Dirichlet one-branes [59,60]. The straight-string soliton was constructed in [8,9]. For weak coupling and low winding number, interactions are small, and this soliton has a description in terms of an exact conformal field theory of a one-brane [60]. It would be interesting to generalize these results to oscillating one-branes.

4.2. Infinite tower of Magnetically charged States

The three basic string-string dualities, I-III, discussed above imply other dualities in lower dimensions after toroidal compactification. Under this reduction, a solitonic string gives rise to different point-like and string-like solitons, which is what we describe now. Our main interest will be the magnetically charged states predicted by S-duality in four dimensions.

Let us consider the heterotic string compactified on $T^4 \times T^2$ which is dual to Type-IIA on $K3 \times T^2$. It has been shown that the six-dimensional string-string duality implies that each of the four-dimensional theories are S-dual [32,61]. In both theories there is an infinite tower of electrically charged point-like states given by a formula analogous to (3.1). Four-dimensional S-duality then predicts an infinite tower of magnetically charged states.

We would now like to show that the solutions corresponding to many of these states can be obtained directly from solitonic strings in six dimensions.

For example, consider the Type-IIA string which has a heterotic string soliton in six dimensions. We now consider a two step reduction of the soliton to four dimensions. We first reduce in the x^5 direction, as in §2.7, by taking a periodic array of solitons, to obtain a five-dimensional string soliton. We then reduce in the x^4 direction, as in §3.3, to obtain a point-like solution in four dimensions. The original string soliton has a 'magnetic' charge $\int_\Sigma H$ where H is the field strength of $B_{\mu\nu}$ and Σ is an asymptotic three-surface on the spatial slice transverse to the string. As a result, the dimensionally reduced soliton is magnetically charged under the gauge field coming from $B_{4\mu}$. This is precisely the state that is S-dual to a perturbative state that is electrically charged under the gauge field coming from $g_{4\mu}$. Starting with an oscillating string soliton in six dimensions, one can obtain an infinite tower of magnetically charged states. The degeneracy of these states is naturally related to the degeneracy of the oscillating string soliton. Moreover, because these solitons are obtained from the smooth string soliton, they are smooth near the core and have an infinite throat much like the stationary solution [62]. The low lying states in this tower are particularly interesting. For example, the lowest level of this tower corresponds to magnetic monopoles [63], the first level to H-monopoles [64,47,48] and so on. Most of the analysis in this paper is applicable to states with large oscillation number. It would be interesting to extend it to these low-lying states.

This discussion makes it clear that the problem of finding H-monopoles in four-dimensional heterotic string theory is closely related to the problem of finding the Type-IIA solitonic string moving on $K3$ in six-dimensional heterotic string theory. It is difficult to see how one can obtain translational zero modes for a soliton that parametrize a complicated surface like $K3$. The main difficulty is that such a motion is not associated with any charges at infinity as is the case with the motion on a torus. In particular, the charged zero modes which are present for the heterotic string soliton in Type IIA theory can be inferred using an anomaly inflow argument directly from the low-energy theory [65]. This is not true for the IIA string as a soliton in the heterotic theory because the zero mode structure after reduction to six dimensions is completely non-chiral. These arguments seem to suggest that the detailed zero mode structure of the Type-IIA soliton cannot be understood entirely within the massless theory. It would follow that obtaining the precise degeneracy of H-monopoles also requires a treatment beyond field theory, something that was also apparent in [66] for somewhat different reasons.

We do not have much to say about this important problem except to point out the following possibility. Consider a Type-IIA theory compactified on a K3 that is geometrically an orbifold obtained from flat space by modding out with a discrete group G. One can consider a ten-dimensional macroscopic string that is located at a point other than the fixed points of G. Its image under G generates a 'periodic' array of strings. On distances larger than the characteristic size of the $K3$, this periodic array will describe a macroscopic Type-IIA string moving in six dimensions. The zero modes that move it along the $K3$ will be visible, however, only on scales smaller than the size of the $K3$. By taking the duality transform of this solution, one can obtain, at least at large distances and away from the orbifold singularities, the soliton in heterotic string theory that corresponds to the Type-IIA string.

5. Conclusions

In this paper we have shown that there is a very satisfying agreement between the structure of BPS states in toroidal compactifications of supersymmetric strings and supersymmetric solutions to the low-energy effective theory when source terms are included. This leaves little doubt that the solutions we have discussed are just the fields outside of elementary string states.

This point of view has in the past led to a puzzle when considering BPS fundamental string states carrying angular momentum. In string theory the states have Regge-bounded angular momentum and one expects source terms for the external fields. On the other hand to reach the supersymmetric limit for the known rotating black hole solutions one must go beyond the extremal limit to very singular configurations, and there is no reason for the angular momentum to be bounded. We think we have resolved this puzzle by showing that the rotating BPS string configurations are stable but depend on both time and the internal coordinate. This leads to a substantial modification of the short distance structure of the solution which is consistent with an underlying string source. Upon time-averaging the solution, which is equivalent to the usual Kaluza-Klein reduction, one obtains the asymptotic form of the previously studied solutions. This point of view may have broader applicability to black hole physics in that it shows that information about black hole structure may be carried by configurations which vary in time on the string scale and which are averaged out in a low-energy description of black hole physics.

There are a number of directions related to this work which would be interesting to pursue. It would be very interesting to extend this analysis to non-extremal black holes. Both the uncharged and the charged fundamental-string solutions are the extremal limit of a larger class of black-string solutions representing string-like singularities surrounded by event horizons [67]. Because the charged black-string solutions do not possess a null Killing vector, it is not possible to use the generating technique to construct traveling-wave solutions on these backgrounds. Such a construction remains an open problem.

It would also be interesting to see whether some of the problems of black hole information loss can be resolved using a picture in terms of microscopic string states as has been advocated by Susskind and others [13,14,15]. A related question is the statistical interpretation of black hole entropy. The entropy of supersymmetric black holes as considered in [17] now appears to have a nice state-counting interpretation in terms of oscillations of an underlying string. This entropy has so far been computed only for non-rotating black holes [17]. To make this picture complete, one would like to generalize this computation to rotating black holes. We expect that the solutions discussed in this paper will be useful for this purpose. One puzzle is that in Sen's calculation the relevant scale for determining the location of the stretched horizon is the string scale and not the scale of compactification. On the other hand, we have seen in this paper that the four-dimensional black hole solution matches not onto a four-dimensional point source but rather onto a five-dimensional string source. Thus, beyond the scale of compactification one should really use a five-dimensional description. We hope to return to these questions in the near future.

We have also provided more evidence for the idea that fundamental strings are solitons in the dual theory. We have shown that the entire spectrum of macroscopic BPS states exists as solitons. This makes it more plausible that even small, oscillating loops of fundamental strings, which in general are unstable, have a dual description. There are many open questions in string duality related to BPS states and macroscopic strings. We have sketched a few possible applications in section four; it would be nice to make these ideas more concrete. We have so far explored the detailed structure of these states only in theories with maximal supersymmetry. Many new dynamical features emerge in theories with reduced supersymmetry, so it would be useful to explore the structure of BPS string states and their duals in these theories as well.

It has also become clear recently that extended objects, p-branes, play a fundamental role in duality. We have seen that for strings, which are one-branes, BPS states arise as purely left-moving excitations of the underlying string. This suggests, at least for odd p, that there may be a connection between excited BPS states of these p-branes and "chiral brane waves." This would be particularly interesting for D-branes [59,60,31,68].

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Note Added

After this work was completed we became aware of a paper with some overlap with ours [69].

References

[1] E. Witten and D. Olive, *Phys. Lett.* **78B** (1978) 97.

[2] C. Montonen and D. Olive, *Phys. Lett.* **72B** (1977) 117;
P. Goddard, J. Nuyts and D. Olive, *Nucl. Phys.* **B125** (1977) 1;
H. Osborn, *Phys. Lett.* **83B** (1979) 321.

[3] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994) 19, hep-th/9407087.

[4] N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484, hep-th/9408099.

[5] J. A. Harvey and G. Moore, "Algebras, BPS states, and strings," hep-th/9510182.

[6] A. Sen, *Nucl. Phys.* **B450** (1995) 103, hep-th/9504027.

[7] J. A. Harvey and A. Strominger, *Nucl. Phys.* **B449** (1995) 535, hep-th/9504047.

[8] A. Dabholkar, *Phys. Lett.* **357B** (1995) 307, hep-th/9506160.

[9] C. M. Hull, *Phys. Lett.* **357B** (1995) 545, hep-th/9506194.

[10] E. Witten, *Phys. Lett.* **B153** (1985) 243.

[11] A. Dabholkar and J. A. Harvey, *Phys. Rev. Lett.* **63** (1989) 478.

[12] A. Dabholkar, G. Gibbons, J. A. Harvey and F. R. Ruiz-Ruiz, *Nucl. Phys.* **B340** (1990) 33.

[13] L. Susskind, "Some Speculations about Black Hole Entropy in String Theory," RU-93-44, hep-th/9309145.

[14] L. Susskind and J. Uglum, *Phys. Rev.* **D50** (1994) 2700, hep-th/9401070.

[15] J. Russo and L. Susskind, *Nucl. Phys.* **B437** (1995) 611, hep-th/9405117.

[16] C. Vafa, as referenced in [13].

[17] A. Sen, *Mod. Phys. Lett.* **A10** (1995) 2081, hep-th/9504147.

[18] D. Garfinkle, *Phys. Rev.* **D46** (1992) 4286

[19] T. Vachaspati, *Nucl. Phys.* **B277** (1986) 593;
Vachaspati and T. Vachaspati, *Phys. Lett.* **B238** (1990) 41;
D. Garfinkle, *Phys. Rev.* **D41** (1990) 1112;
D. Garfinkle and T. Vachaspati, *Phys. Rev.* **D42** (1990) 1960.

[20] A. Sen. *Phys. Lett.* **271B** (1991) 295;
A. Sen. *Phys. Lett.* **274B** (1991) 34;
S. Hassan and A. Sen, *Nucl. Phys.* **B375** (1992) 103.

[21] A. Sen, *Nucl. Phys.* **B388** (1992) 457, hep-th/9206016.

[22] A. Sen, *Nucl. Phys.* **B440** (1995) 421

[23] A. Sen, *Int. J. Mod. Phys.* **A9** (1994) 3707, hep-th/9402002.

[24] D. Waldram, *Phys. Rev.* **D47** (1993) 2528.

[25] J. A. de Azcarraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Phys. Rev. Lett.* **63** (1989) 2443.

[26] M. Dine, P. Huet and N. Seiberg, *Nucl. Phys.* **B322** (1989) 301;
J. Dai, R. G. Leigh, J. Polchinski, *Mod. Phys. Lett.* **A4** (1989) 2073.

[27] J. Gauntlett and J. Harvey, "S-Duality and the Dyon Spectrum in $N = 2$ SuperYang-Mills Theory", CALT-68-2017/EFI-95-56, hep-th/9508156.

[28] S. Sethi, M. Stern, E. Zaslow, "Monopole and Dyon Bound States in $N = 2$ Supersymmetric Yang-Mills Theories", HUTP-95-A031, hep-th/9508117.

[29] A. Strominger, *Nucl. Phys.* **B451** (1995) 96, hep-th/9504090.

[30] K. Becker, M. Becker, and A. Strominger, "Fivebranes, Membranes and Non-Perturbative String Theory", NSF-ITP-95-62, hep-th/9507158.

[31] E. Witten, "Bound States of Strings and P-Branes", IASSNS-HEP-95-78, hep-th/9510335.

[32] E. Witten, *Nucl. Phys.* **B443** (1995) 85, hep-th/9503124.

[33] J. Schwarz and A. Sen, *Phys. Lett.* **312B** (1993) 105, hep-th/9305185.

[34] J. Schwarz, "An SL(2,Z) Multiplet of Type IIB Superstrings,"
CALT-68-2013, hep-th/9508143.

[35] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vols. I and II, Cambridge University Press (1987).

[36] G. Veneziano, *Phys. Lett.* **265B** (1991) 287;
K. Meissner and G. Veneziano, *Phys. Lett.* **267B** (1991) 33;
M. Gasperini, J. Maharana and G. Veneziano, *Phys. Lett.* **272B** (1991) 277.

[37] C. G. Callan, J. A. Harvey and A. Strominger, *Nucl. Phys.* **B367** (1991) 60.

[38] C. Hull and E. Witten, *Phys. Lett.* **160B** (1985) 398.

[39] R. Geroch and J. Traschen, *Phys. Rev* **D36** (1987) 1017.

[40] J. Gauntlett, J. Harvey, M. Robinson, D. Waldram, *Nucl. Phys.* **B411** (1994) 461.

[41] C. G. Callan and J. A. Harvey, *Nucl. Phys.* **B250** (1985) 427.

[42] S. Naculich, *Nucl. Phys.* **B296** (1988) 837.

[43] G. Horowitz and A. Tseytlin, *Phys. Rev.* **D50** (94) 5204, hep-th/9408040; *Phys. Rev.* **D51** (95) 2896, hep-th/9409021.

[44] A. A. Tseytlin, "On Singularities of Spherically Symmetric Backgrounds in String Theory," Imperial/TP/94-95/54, hep-th/9509050.

[45] A. A. Tseytlin, *Phys. Lett.* **251B** (1990) 530.

[46] B. Harrington and H. Shepard, *Phys. Rev.* **D17** (1978) 2122.

[47] R. Khuri, *Nucl. Phys.* **B387** (1992) 315.

[48] J. P. Gauntlett, J. A. Harvey, and J. T. Liu, *Nucl. Phys.* **B409** (1993) 363.

[49] A. Ya. Burinskii, "Complex String as a Source of Kerr Geometry," hep-th/9503094; "Some Properties of the Kerr Solution to Low Energy String Theory," IBRAE-95-08, hep-th/9504139.

[50] J. Schwarz, "Dilaton - Axion Symmetry", CALT-68-1815 (1992), hep-th/9209125;
J. Maharana and J. Schwarz, *Nucl. Phys.* **B390** (1993) 3;
A. Sen, *Nucl. Phys.* **B404** (11993) 109.

[51] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freedman and Company (1973).

[52] W. Israel, *Phys. Rev.* **D2** (1970) 641;
C. A. Lo' pez, *Phys. Rev.* **D30** (1984) 313;
A. Burinskii, *Phys. Lett.* **216B** (1989) 123.

[53] G. Horowitz and A. Sen, "Rotating Black Holes which Saturate a Bogomol'nyi Bound," UCSBTH-95-27, hep-th/9509108.

[54] T. H. Buscher, *Phys. Lett.* **194B** (1987) 59; *Phys. Lett.* **201B** (1988) 466.

[55] K.Behrndt, "About a Class of Exact String Backgrounds", HUB-EP-9516, hep-th/9506106;
R. Kallosh and A. Linde, "Exactly Supersymmetric Massive and Massless Black Holes", SU-ITP-95-14, hep-th/9507022.

[56] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B438** (1995) 109, hep-th/9410167.

[57] C. G. Callan, J. A. Harvey and A. Strominger, *Nucl. Phys.* **B359** (1991) 611.

[58] C. G. Callan, J. A. Harvey and A. Strominger, "Supersymmetric String Solitons" in String Theory and Quantum Gravity 1991: Proceedings of the Trieste Spring School, World Scientific, Singapore, 1991, hep-th/9112030.

[59] J. Polchinski, "Dirichlet Branes and Ramond-Ramond Charges", NSF-ITP-95-122, hep-th/9510017.

[60] J. Polchinski and E. Witten, "Evidence for Heterotic-Type I String Duality", IASSNS-HEP-95-81/NSF-ITP-95-135, hep-th/9510169.

[61] M. Duff, *Nucl. Phys.* **B442** (1995) 47, hep-th/9501030.

[62] D. Garfinkle, G. Horowitz and A. Strominger, *Phys. Rev.* **D43** (1991) 3140.

[63] J. A. Harvey and J. Liu, *Phys. Lett.* **B268** (1991) 40.

[64] R. Rohm and E. Witten, *Ann. Phys.*(NY) **170** (1986) 454.

[65] J. Blum and J. A. Harvey, unpublished; M. J. Duff, J. T. Liu, and R. Minasian, *Nucl. Phys.* **B452** (1995) 261, hep-th/9506126.

[66] J. Gauntlett and J. Harvey, "S-Duality and the Spectrum of Magnetic Monopoles in Heterotic String Theory", EFI-94-36, hep-th/9407111.

[67] G. T. Horowitz and A. Strominger, *Nucl. Phys.* **B360** (1991) 197.

[68] P. Horava and E. Witten, "Heterotic and Type-I String Dynamics from Eleven Dimensions," IASSNS-HEP-95-86/PUPT-1571, hep-th/9510209.

[69] C. G. Callan, J. M. Maldacena and A. W. Peet, "Extremal Black Holes as Fundamental Strings", PUPT-1565, hep-th/9510134.