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tt^* Geometry and closed string tachyon potential

Atish Dabholkar* **and Cumrun Vafa**

Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, USA
E-mail: atish@theory.tifr.res.in, vafa@physics.harvard.edu

Abstract: We propose a closed string tachyon action including kinetic and potential terms for non-supersymmetric orbifolds. The action is given in terms of solutions to tt^* equations which captures the geometry of vacua of the corresponding $N = 2$ worldsheet theory. In certain cases the solutions are well studied. In case of tachyons of $\mathbb{C}/\mathbb{Z}_n$, solutions to affine toda equations determine the action. We study the particular case of $\mathbb{C}/\mathbb{Z}_3 \to \mathbb{C}$ in detail and find that the Tachyon action is determined in terms of a solution to Painleve III equation.

Keywords: Superstrings and Heterotic Strings, Black Holes in String Theory.

*Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai 400005, India.

JHEP02(2002)008

Contents

1. Introduction

Recently the dynamics of closed string tachyons has been addressed in a number of situations involving non-supersymmetric orbifolds [1]–[7]. The aim of this note is to propose a tachyon action for these cases. Our proposal utilizes the $N = 2$ supersymmetry that the worldsheet continues to enjoy upon tachyon condensation [6]. In particular we use the ground state metric and a notation of an algebraic "c-function" defined in [8] for $N = 2$ theories in 2 dimensions, to construct the kinetic and potential terms for the Tachyon action. The geometry underlying these structures is known as the tt^* geometry. The "c-function" can also be interpreted as a kind of supersymmetric index $\mathrm{Tr}(-1)^F F e^{-\beta H}$ [9].

The plan of this note is as follows: in section 2 we propose the tachyon action for general $N = 2$ worldsheet theories, focusing for simplicity on the $\mathbb{C}/\mathbb{Z}_n$ case. We use the results in [4, 5, 6, 8] to motivate our proposal. In section 3 we consider some examples. The case of $\mathbb{C}/\mathbb{Z}_3 \to \mathbb{C}$ is the simplest example and is worked out in great detail.

2. Tachyon action and $N = 2$ worldsheet supersymmetry

In this section we make a proposal for tachyon action involving both the kinetic and potential terms. The action can be stated in full generality in terms of purely $N = 2$ worldsheet data. However for the sake of definiteness we will concentrate on the case of $\mathbb{C}/\mathbb{Z}_n$. We will see that the natural candidate for the tachyon potential is given in terms of the highest axial charge of the Ramond ground states. The natural candidate for the tachyon kinetic term is given in terms of the ground state metric of the Ramond states. Both of these are captured by tt^* equations which we will briefly review.

Tachyon condensation in the context of $\mathbb{C}/\mathbb{Z}_n$ was studied in [4]. In order to avoid tachyon in the bulk, one has to consider odd n and take the $\mathbb{Z}_n$ action to be given by

$$g = \exp\left(\frac{4\pi i J}{n}\right),$$

where J is the generator of rotation on the complex plane x:

$$\exp(i\theta J)\cdot x = e^{i\theta}x\,.$$

It was argued in [4] that giving vev to the tachyon field will give a flow to $\mathbb{C}/\mathbb{Z}_k$ theories, where k is an odd integer satisfying $k < n$. In [6] it was shown that the worldsheet theory is mirror to the $N = 2$ Landau-Ginzburg theory with a superpotential given by

$$W = u^n + \sum_{i=1,3,\ldots,n-2} t_i u^i\,,$$

where t_i corresponds to giving vev to the tachyon in the i-th twisted sector. The fundamental field here is the chiral field Y where $u = e^{-Y}$. Let us set $t_i = 0$. The chiral ring in this case is generated by u modulo the equation $dW/dY = -udW/du = 0$, which gives the ring elements

$$u, u^2, \ldots, u^{n-1}$$

in one to one correspondence with the twist fields of the $\mathbb{C}/\mathbb{Z}_n$ orbifold. The fact that we are restricted to odd powers of u is related to the requirement of GSO projection which requires a $\mathbb{Z}_2$ symmetry under which $W \to -W$, which is realized here by $u \to -u$. The charge of the field u^k is given by k/n, where we assign charge 1 to W. The charge in question here is the U(1) charge of the $N = 2$ algebra. More precisely the (left, right)-moving U(1) charge of u^k is given by $(k/n, k/n)$. In particular this state has total axial charge $Q^5 = F_L + F_R = 2k/n$. Note that the relevant ring of the LG model when u is the natural variable instead of Y, is given by $1, \ldots, u^{n-2}$. This would correspond to $N = 2$ minimal model [10, 11].

Aspects of chrial rings in the conformal case have been studied [12]. There is a $1-1$ correspondence between chiral fields and ground states of Ramond-Ramond sector: for each chiral field of charge Q, there is a Ramond ground state with left/right charge $q = Q - \hat{c}/2$, where $\hat{c}$ is the complex dimension of the conformal theory. Note that the spectrum in the Ramond ground state is symmetric under $q \to -q$ as follows from CTP. In the case of $\mathbb{C}/\mathbb{Z}_n$ theory the (left or right) charges of the Ramond states are given by

$$\left(-\frac{1}{2}+\frac{1}{n}, -\frac{1}{2}+\frac{2}{n}, \ldots, \frac{1}{2}-\frac{1}{n}\right),$$

where we have used $\hat{c} = 1$ for this case. Note that the charges of the ground states for $N = 2$ minimal models also vary exactly over this range, as follows from noting that $\hat{c} = 1 - \frac{2}{n}$ in this case. As discussed in detail in [12] for $N = 2$ SCFT with discrete spectrum, the range of axial charges Q^5 of the Ramond ground state, which is left- plus right-moving fermion number varies from $-\hat{c}, \ldots, +\hat{c}$. In particular the absolute value of the highest axial charge in the Ramond ground state is given by $\hat{c}$. In the case of $\mathbb{C}/\mathbb{Z}_n$ the highest charge state is not $\hat{c} = 1$, as that is not a compact SCFT. However its highest charge state is given by $1 - \frac{2}{n}$, suggesting that this is the *effective* $\hat{c}$ for the tachyon fields.

Now consider deforming the theory by turning on tachyon fields. The worldsheet theory will flow in such a case. For example suppose we consider the deformation given by the superpotential

$$W = u^n + tu^k\,.$$

with $k < n$ and k odd. In this case the IR theory will be given by $W = u^k$ which corresponds to the $\mathbb{C}/\mathbb{Z}_k$ theory. It is natural to ask how the chiral fields and the corrsponding Ramond vacua of the $\mathbb{C}/\mathbb{Z}_n$ theory and those of $\mathbb{C}/\mathbb{Z}_k$ theory are related. Note that by considering isolated solutions of $udW/du = 0$ we have $n - k$ massive vacua which decouple at the IR. In addition we have $k - 1$ vacua which in the IR continue to exist corresponding to the Ramond vacua of the $\mathbb{C}/\mathbb{Z}_k$ theory. Note that the charges have now changed. It was shown in [8] how to follow the corresponding charges of the vacua of the Ramond sector: one studies the space of all the Ramond vacua as a function of the deformation parameters. This forms a holomorphic vector bundle over the complex moduli space t_i, which inherits a natural connection as considered in the context of Quantum mechanics by Berry. Let

$$g_{i\bar{j}} = \langle i|\bar{j}\rangle$$

denote the overlap metric of the corresponding Ramond ground states in a holomorphic parameterization of the vacua (as defined in [8]). Then the metric satisfies a rich set of equations discovered in [8] known as tt^* equations (topological /anti-topological equations). The main equation being that the curvature of this bundle is given by

$$\overline{\partial}_i(g\partial_j g^{-1}) = -[\overline{C}_i, C_j]\,,$$

where C_i denotes the matrix for the multiplication of the Ramond ground states with the $i-$th chiral field. Moreover, it was shown in [8, 9] that the axial charge acting on the Ramond ground states is related to the tt^* metric by

$$\frac{1}{2}[Q^5 + m]|a\rangle = -(g\partial_\tau g^{-1})_a^b|b\rangle\,,$$

where Q^5 denotes the axial charge ($Q^5 = F_L + F_R$) and m denotes the total number of fields (which in the $\mathbb{C}/\mathbb{Z}_n$ case is just 1), and τ denotes the RG paramater and corresponds to $W \to e^\tau W$. Note that Q^5 is not conserved during the flow and is conserved only at the two ends of flow, where the theory becomes conformal. Moreover the Q^5 charges flow down in absolute value. It was shown in [9] that Q^5 is also an "index" given by

$$Q^5_{ab} = \frac{i\tau}{L}\,\mathrm{Tr}_{ab}(-1)^F F e^{-\tau H}\,,$$

where this is on a dual channel where space is of length L (in the limit as $L \to \infty$) and a, b denotes the left/right vacua on the space. F here is the vectorial fermion number $F_L - F_R$ and is related to the above axial charge by the change of channel which converts axial charge to vector charge (i.e. $Q^5 L \leftrightarrow iF\tau$).

2.1 Proposal for the potential term

We are now ready to present our proposal for the tachyon potential. As noted in [8] the axial charge matrix Q^5 is a natural matrix generalization of Zamolodchikov's c-function to the case at hand. In particular the absolute value of the highest eigenvalue of Q^5 was

JHEP02(2002)008

proposed as a candidate for a c-function defined for arbitrary massive $N = 2$ theories. Our proposal for the tachyon potential as a function of $V(t_i, \bar{t}_i)$ is

$$V(t_i, \bar{t}_i) = \max |Q^5| \,. \tag{2.1}$$

where the action involves $S \sim \frac{1}{g_s^2} \int d^D x V$. The most natural motivation for this conjecture is that the closed string theory has a cosmological constant given by the central charge of the theory, minus a constant. In particular $V \propto \hat{c} + \text{const.}$, is very natural. The only difference here is that the potential lives on a submanifold of the spacetime, where the tachyon field is localized. Thus we have to find the effective $\hat{c}$ corresponding to the tachyon degrees of freedom on the worldsheet. Since $\max |Q^5|$ effectively plays the role of the relevant piece of $\hat{c}$ for tachyonic fields, therefore the above identification is rather natural. For the $\mathbb{C}/\mathbb{Z}_n$ vacuum the height of the tachyon potential is given by

$$Q^5_{\text{max}} = \left(1 - \frac{2}{n}\right).$$

This is in line with a recent proposal for the height of the tachyon potential [5]. The case studied there corresponds to modding out the space by the rotation given by

$$\exp\left(\frac{2\pi i J}{N}\right)$$

which to compare it to the case at hand we set $N = n/2$ to obtain

$$g = \exp\left(\frac{4\pi i J}{n}\right).$$

It was argued that the tachyon potential in this case should be $(1 - \frac{1}{N})$ which by the substitution $N \to n/2$ we get the above result.[1] In fact one can essentially map the logic of the derivation of [5] to the above proposal: the proposal of [5] was based on identification of the tachyon potential with the holonomy of the spin connection at infinity. Consider a string state in the twisted sector, and compute its vectorial fermion number. This is the same as the pull back of the holonomy of the spin connection along this path. In the limit the theory is the orbifold we can take this path at any distance from the orbifold fixed point, and in particular we can take it to correspond to the ground state of the twisted sector. Upon mirror symmetry, getting the u field, vectorial fermion number gets converted to the axial charge of the lowest twisted state of the LG theory, which is the above proposal for the tachyon potential.

Note that when we flow to the supersymmetric theory, such as by considering

$$W = u^n + tu$$

the Q^5_{max} coming from the discrete states flow to zero, as expected for the vanishing height of the potential at the supersymmetric point. In particular the height of the potential is $1 - \frac{2}{n}$ only for $n > 1$.

In our proposal for the tachyon potential (2.1) we have to search for the maximum charge Q^5 among the states that can flow to zero upon deformations. Otherwise we would be also including supersymmetric states that have zero potential.

[1] This is presumably related to the fact that one has to take spin connection rather than SO(2) connection.

2.2 Proposal for the kinetic term

In case the target theory is supersymmetric, the fields t_i correspond to massless fields whose kinetic term is given as [8]

$$\int d^D x G_{i\bar{j}}(t^k, \bar{t}^k)\partial_\mu t^i \partial_\mu \bar{t}^j ,$$

where $G_{i\bar{j}}$ is defined as follows. Let $|\rho\rangle$ denote the ground state of the Ramond sector with lowest axial charge. Suppose the superpotential involves $W = \int t_i \Phi_i$. Define

$$|i\rangle = \Phi_i |\rho\rangle .$$

Then

$$G_{i\bar{j}} = \frac{\langle \bar{j}|i\rangle}{\langle \bar{\rho}|\rho\rangle} . \tag{2.2}$$

This definition of $G_{i\bar{j}}$ is simply the natural normalized metric of the corresponding chiral fields. In fact in the case the target is supersymmetric more is true: G is a Kahler metric, i.e. $G = \partial\bar{\partial}K$ where $K = \log\langle\bar{\rho}|\rho\rangle$ [8, 13].

We propose the same kind of kinetic term also for tachyonic deformations. Namely again let $|\rho\rangle$ denote the ground state of the Ramond sector with the lowest axial charge. By this we mean a normalizable such state. For example for $\mathbb{C}/\mathbb{Z}_n$ orbifold, with mirror LG given by $W = u^n$, the state $|\rho\rangle = |u\rangle$. Of course if we deform W, the state $|\rho\rangle$ may no longer be given by $|u\rangle$ as the state with minimal Q^5 charge may be a combination of various states. However for deformations of the form $W = u^n + tu$ it is easy to see by symmetry arguments (as discussed in [8]) that $|u\rangle$ has the lowest axial charge for all t. At any rate our proposal for the tachyon kinetic term is

$$K = \int G_{i\bar{j}} \partial_\mu t^i \partial_\mu \bar{t}^j ,$$

with $G_{i\bar{j}}$ defined by (2.2). For example consider the case with $W = u^n + tu$. In this case the kinetic part of the action involving t can be written as

$$\int \frac{\langle \bar{u}^2|u^2\rangle}{\langle \bar{u}|u\rangle} \partial_\mu t \partial_\mu \bar{t} . \tag{2.3}$$

To see this, note that since t multiplies the field u in the superpotential, and since $|\rho\rangle = |u\rangle$ we have

$$u|\rho\rangle = u|u\rangle = |u^2\rangle$$

which yields from (2.2)

$$G_{t\bar{t}} = \frac{\langle \bar{u}^2|u^2\rangle}{\langle \bar{u}|u\rangle} .$$

In the non-supersymmetric case G is no longer a Kahler metric. We are now ready to consider some examples.

JHEP02(2002)008

3. Examples

In this section we consider some examples. Consider again $\mathbb{C}/\mathbb{Z}_n$. The tt^* geometry in this case is exactly the same as the corresponding $N=2$ minimal model with a shift $u^l \to u^{l+1}$, as discussed in the previous section. For example, let us consider the $\mathbb{C}/\mathbb{Z}_3$ case. In this case we have

$$W = \frac{1}{3}u^3 - tu\,.$$

There are two ground states in the R sector with charges $(-1/6,-1/6)$ and $(1/6,1/6)$ at $t=0$. As $t\to\infty$ the corresponding charges, while opposite in sign, go to 0. As shown in [8] the maximal charge Q^5, which we identify with the Tachyon potential, can be computed using tt^* equations and is given by

$$V = Q^5 = \frac{-1}{2} z \frac{\partial v}{\partial z}\,, \tag{3.1}$$

where

$$z = \frac{4}{3}|t|^{3/2}\,, \qquad |t|e^v = \frac{\langle \overline{u}^2|u^2\rangle}{\langle \overline{u}|u\rangle}$$

and v is a special solution to affine A_1 toda equation[2]

$$v_{zz} + \frac{v_z}{z} = 4\sinh(v) \tag{3.2}$$

with boundary conditions dictated by the regularity of the solution and given by

$$v(z) \sim \frac{-2}{3}\log z + s + 2\left(\frac{3}{4}\right)^2 e^s z^{4/3} + O(z^{8/3})\,, \qquad z \longrightarrow 0\,,$$
$$v(z) \sim \frac{1}{\sqrt{\pi z}}\exp(-2z)\,, \qquad z \gg 0\,,$$

where

$$e^{s/2} = 2^{2/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}\,.$$

Similarly the Kinetic term can be computed using (2.3). Putting these together we obtain the tachyon action

$$S \sim \frac{1}{g_s^2}\int |t|e^v \partial_\mu t \partial_\mu \overline{t} + \frac{-z}{2}\frac{\partial v}{\partial z}$$

(with $z = \frac{4}{3}|t|^{3/2}$ and $v(z)$ defined above).

Let us see if this action has some of the expected properties: let us study the action near $t=0$. At $t=0$ we have

$$\langle \overline{u}|u\rangle|_{t=0} = \left(\frac{3}{4}\right)^{-1/3} e^{-s/2} = \left.\frac{1}{\langle \overline{u}^2|u^2\rangle}\right|_{t=0}\,.$$

Using this, if we identify the normalized tachyon field T by $T = t\sqrt{\langle \overline{u}^2|u^2\rangle/\langle \overline{u}|u\rangle}|_{t=0}$ we find that S reduces near $T\sim 0$ to

$$S \sim \frac{1}{g_s^2}\int \partial_\mu T \partial_\mu \overline{T} + V(T,\overline{T})\,,$$

[2]This is a special case of Painleve III equation $Y'' = (Y')^2/Y - Y'/z + Y^3 - 1/Y$ where $Y^2 = e^v$.

where

$$V \sim \frac{1}{3} - \frac{4}{3}|T|^2 + O(|T|^4)\,, \qquad |T| \sim 0\,.$$

Note that with this normalization the tachyon potential has the right height as well as the correct negative mass squared for the tachyon (we are using the convention $\alpha' = 1$). This is a rather non-trivial check of our proposal.

We would also like to study the solution for large t. In this limit we have

$$V \sim a|t|^{3/4} \exp\left(-\frac{8}{3}|t|^{3/2}\right), \qquad |t| \gg 0\,.$$

where a is a positive constant. This potential approaches 0 exponentially fast for large $|t|$.

We would also like to prove that V has no extra critical points. Note that $t = 0$ and $t = \infty$ are two critical points and we do not expect any other critical points. In particular $dV/dt = 0$ should have no solution for finite $t > 0$. V starts from a positive value 1/3 at $t = 0$ and approaches 0 at infinity, and we will now show that it is monotonic. If $dV/dt = 0$ for some finite non-zero t, it follows that $dV/dz = 0$ for some $z > 0$. We know that

$$\frac{dV}{dz} = \frac{-1}{2}\frac{d}{dz}\left[\frac{zdv}{dz}\right] = -2z\sinh(v)\,,$$

where we used the definition of V in (3.1) and the equation satisfied by v (3.2). For this to be zero, for non-zero z we must have $v = 0$ for some finite positive z. $v \gg 0$ at $t = 0$ and $v = 0$ at $t = \infty$. So if $v = 0$ for some finite z it must become negative for finite values of z. Given the fact that v approaches zero as $z \to \infty$ it implies that there must be at least some critical points $v_z = 0$ where $v < 0$ and $v_{zz} > 0$, which is incompatible with eq. (3.2). This proves that V has no other critical points, as expected.

One can also ask how the tachyon field disappears in the $|t| \gg 0$ regime. Note that in this limit using $e^v \sim 1$ the kinetic term of the action becomes

$$\int |t|\partial_\mu t \partial_\mu \bar{t}\,.$$

As t rolls off to infinity, the above kinetic term has a huge prefactor $|t|$, and this implies that for finite action the field t cannot vary appreciably over spacetime. In other words the field t gets frozen out. However this is not very convincing, because by going to a field redefinition namely z, which is the radial part of the tachyon field, we see that the kinetic term will become $\partial z \partial z$ and the potential $a\sqrt{z}e^{-2z}$. It would be interesting to study the meaning of the tachyon field at the other end.

We can also consider other $\mathbb{C}/\mathbb{Z}_n$ cases. In all such cases the tt^* equations yields the tachyon potential V. However an exact solution to tt^* equations is not possible for general deformations. For some special cases of deformations the tt^* equations are better studied. For example for

$$W = u^n + tu$$

the tt^* equations boils down to affine A_{n-2} toda equations [8]. This and some other special cases have been studied in [8, 9] to which we refer the interested reader.

We can also consider higher-dimensional orbifolds such as $\mathbb{C}^2/\mathbb{Z}_n$ studied in [4, 6]. Also in these case our proposal for the tachyon potential V has the right features to correspond to tachyon potential. For example for the non-supersymmetric orbifold $(\exp(2\pi i k_1/n), \exp(2\pi i k_2/n))$ we obtain the height of the potential

$$V = 2\left(\max_{l=1}^{n-1} \frac{[lk_1]+[lk_2]}{n} - 1\right),$$

where $[lk_i] = lk_i \mod n$ and $0 \leq [lk_i] < n$.

For a generic deformation all the charges go to zero and so $V \to 0$, flowing to a supersymmetric vacuum. Note that $V = 0$ in the supersymmetric case, which corresponds to $k_1 = n - k_2$. A priori this did not have to be the case, because our proposal for (2.1) is the maximum charge among all the states that can flow to zero, and this is not the case for supersymmetric states. Similarly one can consider the kinetic term and the variation of potential term by studying solutions to tt^* equations.

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