CALT-68-2051
DOE RESEARCH AND
DEVELOPMENT REPORT

hep-th/9604178

STRINGS ON ORIENTIFOLDS

Atish Dabholkar[1] and Jaemo Park[2]

Lauritsen Laboratory of High Energy Physics

California Institute of Technology

Pasadena, CA 91125, USA

ABSTRACT

We construct several examples of compactification of Type IIB theory on orientifolds and discuss their duals. In six dimensions we obtain models with $N = 1$ supersymmetry, multiple tensor multiplets, and different gauge groups. In nine dimensions we obtain a model that is dual to M-theory compactified on a Klein bottle.

April, 1996

[1] e-mail: atish@theory.caltech.edu

[2] e-mail: jpk@theory.caltech.edu

1. Introduction

In this paper we discuss string compactifications on orientifolds to six and higher dimensions. Orientifolds are a generalization of orbifolds [1,2,3,4] in which the orbifold symmetry includes orientation reversal on the worldsheet (for a review see [5] and references therein). Orientifolding allows one to construct new perturbative vacua that cannot be obtained by usual Calabi-Yau compactification of string theory. One can thus explore different regions in the moduli space of string vacua that were previously not accessible.

In six dimensions we focus on orientifolds of Type IIB theory compactified on a $K3$ orbifold to obtain six dimensional theories with $N = 1$ spacetime supersymmetry. It has recently become clear that the dynamics of $D = 6$, $N = 1$ string theories is quite rich and offers many surprises. There are points in the moduli spaces of these theories where tensionless strings appear which makes it possible to have non-trivial dynamics in the infra-red [6,7]. In particular, there can be phase transitions in which the number of tensor multiplets can change. It is therefore quite interesting to analyze different branches of the tensor-multiplet moduli space. Usual Calabi-Yau compactifications can give only one tensor multiplet. In [8] an orientifold was constructed that has nine tensor multiplets. In this paper we discuss some generalizations that give models with five, seven, nine, or ten tensor multiplets with different gauge groups. Models with multiple tensor-multiplets can also be obtained by compactifications of M-theory [6,9,10], or of F-theory [11,12]. The orientifolds that we construct allow one to study the duals of some of these compactifications as perturbative string theories.

In nine dimensions we consider an orientifold of Type IIB theory compactified on a circle to obtain the dual of M-theory compactified on a Klein bottle. It is interesting to note that the compactification of M-theory on a circle gives the Type IIA theory, on an interval the $E_8 \times E_8$ heterotic string [13], on a Möbius strip a CHL string[14,15], and on a torus the Type II string [16]. Thus, compactification on a Klein bottle completes this list of Ricci-flat compactifications to nine and ten dimensions. We also discuss some issues regarding the compactification of Type I theory on a torus.

This paper is organized as follows. In section two we first discuss some generalities about orientifolds. In section three we discuss orientifolds of toroidal compactifications. In section four we discuss orientifolds of Type IIB theory compactified on K3 orbifolds. The calculation of tadpoles and the relevant partition sums are summarized in the Appendix.

2. Some Generalities about Orientifolds

In general our starting point will be some $\mathbf{Z}_N$ orbifold of toroidally compactified Type IIB theory. We can then take the orientifold projection $(1+\Omega\beta)/2$, where Ω is the orientation reversal on the worldsheet and β is some $\mathbf{Z}_2$ involution of the orbifold. If the orbifold group $\mathbf{Z}_N$ is generated by the element α, then the total projection we would like to perform is given by $(\frac{1+\alpha+\ldots+\alpha^{N-1}}{N})(\frac{1+\Omega\beta}{2})$ in both the twisted and the untwisted sectors of the orbifold. The orientifold group G can be written as $G = G_1 + \Omega G_2$ such that $\Omega h \Omega h' \in G_1$ for $h, h' \in G_2$.

The closed string sector of the orientifold is obtained by projecting the spectrum of the original orbifold onto states that are invariant under the orientifold symmetry. The open-string sector of the orientifold arises as follows. Orientifolding introduces unoriented surfaces in the closed-string perturbation theory. The unoriented surfaces such as the Klein bottle can have tadpoles of R-R fields in the closed string tree channel. The tadpoles correspond to the fact that the equations of motion for some R-R fields are not satisfied because the orientifold plane acts as the source of the R-R fields [1]. By including the right number of D-branes which are also sources for the R-R fields with opposite charge, one can cancel these tadpoles. This introduces the open-string sector with appropriate boundary conditions and Chan-Paton factors. As we shall see, sometimes the Klein bottle amplitude turns out to have no tadpoles; in these cases there is no need to introduce the open-string sector, and the closed-string sector by itself describes a consistent theory.

An open string can begin on a D-brane labeled by i and end on one labeled by j. The label of the D-brane is the Chan-Paton factor at each end. Let us denote a general state in the open string sector by $|\psi, ij\rangle$. An element of G_1 then acts on this state as

$$g: \qquad |\psi, ij\rangle \ \to (\gamma_g)_{ii'} |g\cdot\psi, i'j'\rangle (\gamma_g^{-1})_{j'j}, \tag{2.1}$$

for some unitary matrix γ_g corresponding to g. Similarly, an element of ΩG_2 acts as

$$\Omega h: \qquad |\psi, ij\rangle \ \to (\gamma_{\Omega h})_{ii'} |\Omega h\cdot\psi, j'i'\rangle (\gamma_{\Omega h}^{-1})_{j'j}. \tag{2.2}$$

The relevant partition sums for the Klein bottle, the Möbius strip, and the cylinder are respectively $\int_0^\infty dt/2t$ times

$$
\begin{aligned}
\mathrm{KB}: &\quad \mathrm{Tr}^{\mathrm{U+T}}_{\mathrm{NSNS+RR}}\left\{\frac{\Omega\beta}{2}\,\frac{1+\alpha+...+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi t(L_0+\tilde{L}_0)}\right\}\\
\mathrm{MS}: &\quad \mathrm{Tr}^{\lambda\lambda}_{\mathrm{NS-R}}\left\{\frac{\Omega\beta}{2}\,\frac{1+\alpha+...+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi t L_0}\right\}\\
\mathrm{C}: &\quad \mathrm{Tr}^{\lambda\lambda'}_{\mathrm{NS-R}}\left\{\frac{1}{2}\,\frac{1+\alpha+...+\alpha^{N-1}}{N}\,\frac{1+(-1)^F}{2}e^{-2\pi t L_0}\right\}.
\end{aligned}
\tag{2.3}
$$

Here F is the worldsheet fermion number, and as usual $\frac{1+(-1)^F}{2}$ performs the GSO projection. The Klein bottle includes contributions both from the untwisted sector(U) and the twisted sectors(T) of the original orbifold. Orientation reversal Ω takes NS-R sector to R-NS sector, so these sectors do not contribute to the trace. The labels λ and λ' refer to the type of D-brane an open string ends on. For example, in a theory with both 5-branes and 9-branes, λ and λ' are either 5 or 9; one has to include the sectors 55 and 99 for the Möbius strip, and the sectors 55, 99, 59, and 95 for the cylinder[17]. The tadpoles can be extracted by factorizing the loop-amplitude in the tree channel. Tadpole cancellation then determines the number of D-branes as well as the form of the γ matrices introduced earlier, which in turn determines the open string sector completely. In fact in many examples that we consider, spacetime supersymmetry and anomaly cancellation usually place powerful constraints which determine the spectrum even without knowing the full form of the γ matrices.

Many of the details of the tadpole calculation are similar to those discussed in [5,8,17] and will not be repeated here. We give a collection of relevant partition sums and their factorized forms in the tree channel in the Appendix.

3. Orientifolds of Toroidally Compactified Type IIB theory.

3.1. An Example in Nine Dimensions

Consider Type IIB theory compactified say in the X^9 direction on a circle S_9 of radius r_9. We can take an orientifold with the group $\{1, S\Omega\}$ where S is a half-shift along the circle, $X^9 \to X^9 + \pi r_9$. The closed-string sector of this theory is obtained by projecting onto states that are invariant under $S\Omega$. The massless bosonic spectrum of Type IIB theory in ten dimensions consists of the metric g_{MN}, the dilaton ϕ^1, and a two-form B^2_{MN} from the NS-NS sector; a two-form B^1_{MN}, a scalar ϕ^2, and a four-form A_{MNPQ} with self-dual field strength from the R-R sector. The fields g_{MN}, ϕ^1, and B^1_{MN} are all even under Ω, whereas the fields A_{MNPQ}, B^2_{MN}, and ϕ^2 are odd. If we were projecting only under Ω, we would obtain the spectrum of Type I strings; the superscript 1 above refers to the fields that survive this projection.

Now, if we expand a given field Ψ in terms of the Kaluza-Klein momentum modes Ψ_m carrying quantized momentum m/R then the modes with even m are even under S, whereas the modes with odd m are odd. Thus, the combined projection under ΩS eliminates all odd momentum modes of the fields g_{MN}, ϕ^1, and B^1_{MN}, but all even momentum modes of A_{MNPQ}, B^2_{MN}, and ϕ^2. In particular, once we restrict ourselves to zero momentum modes to obtain the massless spectrum in nine dimensions, we obtain the closed string sector of the Type I string reduced to nine dimensions.

Let us now look at the open-string sector. As explained in the previous section, open-string sector arises from the addition of D-branes to cancel tadpoles in the Klein bottle amplitude. Now, because of the half-shift that accompanies Ω, only states with odd winding appear in the crosscap state and are thus massive. Another way to see this is to first compute the amplitude in the loop channel and then factorize in the tree channel. The loop channel momentum sum gives a term proportional to $\sum_m (-1)^m e^{\frac{-t\alpha' m^2}{r_9^2}}$ where t is the loop-channel parameter. To see the tadpoles in the tree channel we use Poisson resummation formula and take the limit $t \to 0$ corresponding to long, thin tubes; it is easy to see that in this limit the amplitude vanishes, and there is no tadpole. Therefore, to obtain a consistent orientifold there is no need to add any branes.

To see what this theory is dual to, we compactify further on a circle S_8 of radius r_8 in the direction X_8. The Type IIB theory is T-dual to Type IIA under $r_8 \to 1/r_8$, and moreover the operation Ω in IIB is dual to $R_8\Omega$ in IIA where R_8 is the reflection $X_8 \to -X_8$ [5]. Now Type IIA theory is M-theory compactified on a circle S_{10} in the X^{10} direction. The operation $R_8\Omega$ corresponds, in M-theory, to taking $X^8 \to -X^8$, at the same time flipping the sign of the three-form potential C_{MNP} of the eleven dimensional supergravity. In M-theory we can interchange the two circles S^8 and S^{10}. Therefore, the combined operation $S\Omega$ in Type IIB theory correponds, in M-theory, to $X^{10} \to -X^{10}$, $X^9 \to X^9 + \pi r_9$ which is nothing but the Z_2 transformation that turns the torus $T_{9,10}$ into a Klein bottle. Notice that this is not a purely geometric operation in M-theory but is accompanied by a simultaneous change of sign of the three-form potential. Under the interchange of the two circles S_{10} and S_8, the symmetry $R_8\Omega$ in Type IIA theory is conjugate to the symmetry $(-1)^{F_L}$, where F_L is the spacetime fermion number coming from the left-movers [18]. All R-R fields are odd under this symmetry and all NS-NS fields are even. Thus, the strong coupling limit of the orbifold of Type-IIA theory under the combined operation $(-1)^{F_L}$ and $X_9 \to X_9 + \pi r_9$ is given by M-theory compactified on a Klein bottle.

It is amusing that we have an example of a compactification on a non-orientable surface. Another example is M-theory on a Möbius strip which is dual to a CHL compactification [14,15]. Recall that the $E_8 \times E_8$ string is dual to M-theory on an interval in the tenth direction: the two E_8 factors live at the two endpoints of the interval [13]. Compactifying further on a circle, we obtain M-theory on a cylinder. The CHL string is obtained as a Z_2 orbifold of the heterotic string in nine dimensions. The orbifold symmetry corresponds to an interchange of the two E_8 factors accompanied by a half shift on the circle. The combined operation is again $X^{10} \to -X^{10}$, $X^9 \to X^9 + \pi r_9$ which turns the cylinder into a Möbius strip [19].

3.2. Type I Theory in Eight Dimensions

Type I theory compactified in the 8 and 9 directions to eight dimensions can be viewed as an orientifold of the Type IIB theory on the torus T_{89}. It is straightforward to find the massless spectrum, but there is one subtlety in taking the T-dual of this theory which is worth mentioning.

Let us T-dualize first in the X^9 direction. T-duality is a one sided parity transform [5] which means that in the RNS formulation of the superstring, only the left-moving coordinate $\tilde{X}^9$ and its fermionic partner $\tilde{\Psi}^9$ change sign. Thus, T-duality takes Type IIB theory to Type IIA theory, and takes Ω to $R_9\Omega$, where R_9 is the reflection in the X^9

direction. If we dualize again in the X^8 direction, we would get Type IIB theory back; Ω goes to $R_{89}\Omega$, where R_{89} reflects both X^8 and X^9. This identification leads to the following puzzle for the orientifold with the group $\{1, R_{89}\Omega\}$. Under Ω the four-form field A_{MNPQ} is odd, therefore the modes like A_{MNP9} and A_{MNP8} which are 3-forms in eight dimensions would be even under the combined operation $R_{89}\Omega$ and would survive the projection. But $N = 1$ supersymmetry in $D = 8$ uniquely determines the massless field content and does not allow a three-form potential. Therefore, supersymmetry is broken by this projection. On the other hand, the orientifold with the group $\{1, R_{89}\Omega\}$ is T-dual to the one with the group $\{1, \Omega\}$, and we cannot break supersymmetry by a T-duality transformation. We should really have obtained the T-dual of Type I strings in eight dimensions. The reason for this discrepancy is that Type IIB theory has an additional symmetry $(-1)^{F_L}$ under which all R-R fields are odd. The correct projection that gives the T-dual of Type I theory involves the combined operation $R_{89}(-1)^{F_L}$ instead of just the geometric reflection.

It is easy to see this ambiguity on the worldsheet. In the Ramond sector, the zero modes $\tilde{\Psi}^M$ correspond to the Γ^M matrices of the spacetime Clifford algebra. Under the T-duality transformation $\tilde{\Psi}^9 \to -\tilde{\Psi}^9$, the spinors transform as

$$
\begin{aligned}
&S \to S \\
&\tilde{S} \to \Gamma^9 \Gamma \tilde{S},
\end{aligned}
\tag{3.1}
$$

where S and $\tilde{S}$ are the right-moving and left-moving spacetime spinors respectively, and Γ as usual is the matrix that anticommutes with all Γ^M matrices and squares to one. If we T-dualize further in the X^8 direction then S goes to itself, and $\tilde{S}$ goes to $\Gamma^8\Gamma\Gamma^9\Gamma\tilde{S} = \Gamma^9\Gamma^8\tilde{S}$. Let us now see how the massless fields from the Ramond-Ramond sector transform. The vertex operator for an n-form field strength $H_{M_1 \dots M_n}$ is proportional to $\bar{S}\Gamma_{M_1\dots M_n}\tilde{S}$ where $\Gamma^{M_1 \dots M_n} = \frac{1}{n!}(\Gamma^{M_1} \dots \Gamma^{M_n} \pm \text{permutations})$. It is easy to see that the effect of T-duality on the R-R field strengths $H_{M_1 \dots M_n}$ and the corresponding potentials is to remove the $8, 9$ indices if they are present and add them if they are not. For example, the vertex operator for H^1_{M89} is proportional to $\bar{S}\Gamma_M\Gamma_8\Gamma_9\tilde{S}$. Under T-duality, it would map onto $\bar{S}\Gamma_M\tilde{S}$ which is the vertex operator for the field strength of a scalar. Thus, B^1_{89} maps onto the scalar ϕ^2. However, because $\Gamma^8\Gamma$ and $\Gamma^9\Gamma$ anticommute with each other, there is a choice of sign for the action on the R-R fields, which corresponds precisely to the choice between R_{89} and $R_{89}(-1)^{F_L}$. This ambiguity is, of course, fixed by the correct choice of the orientifold symmetry.

4. Orientifolds of Type IIB theory on $K3$

4.1. General Remarks

Let us review some relevant facts about the $K3$ surfaces which can be represented as $\mathbf{Z}_N$ orbifolds of the 4-torus T^4 [20]. Let (z_1, z_2) be the complex co-ordinates on the torus, and consider the $\mathbf{Z}_N$ transformation generated by

$$g: \quad (z_1, z_2) \rightarrow (e^{2\pi i/N} z_1, e^{-2\pi i/N} z_2). \tag{4.1}$$

The $\mathbf{Z}_N$ group must be a subgroup of $SU(2)$ to obtain unbroken supersymmetry in six dimensions. The torus T^4 is obtained by identifying a lattice Λ of points in R^4, so the orbifold group must leave the lattice invariant to have a sensible action on the torus. This crystallographic condition allows only four possibilities: the groups $\mathbf{Z}_2$ and $\mathbf{Z}_4$ when Λ is the square ($SU(2)^4$) lattice given by the identifications $z_i \sim z_k + 1, \sim z_k + i, k = 1, 2$; or $\mathbf{Z}_3$ and $\mathbf{Z}_6$ when Λ is the hexagonal ($SU(3)^2$) lattice given by the identifications $z_i \sim z_k + 1, \sim z_k + e^{2\pi i/3}, k = 1, 2$. At a fixed point of a $\mathbf{Z}_k$ symmetry there is a curvature singularity. A smooth $K3$ can be obtained by blowing up the singularity by replacing a ball around the fixed point by an appropriate smooth non-compact Ricci-flat surfaces E_k whose boundary at infinity is $S^3/\mathbf{Z}_k$.

In this section we consider two classes of orientifold projections $(1 + \Omega\beta)/2$ of Type IIB theory on these orbifolds. In the first class of models we take β to be identity, whereas in the second class we take β to be a specific $\mathbf{Z}_2$ involution S of $K3$ that has 8 fixed points. We shall give an explicit description of this involution in the following subsections.

One immediate question is whether the projection leaves any supersymmetries unbroken. In the case of Ω the combination $Q_\alpha + \Omega\tilde{Q}_\alpha$ of the left-moving and right-moving supercharges will be invariant; supersymmetry will be broken by half, giving us $N = 1$ supersymmetry starting from $N = 2$. When we combine Ω with S, we do not want to break the supersymmetry further, so S should leave all $N = 2$ supersymmetries invariant. This is possible if the rotational part of the symmetry S is a subgroup of $SU(2)$, or equivalently if it leaves the holomorphic 2-form invariant. It is useful to consider the example of $\mathbf{Z}_2$ orbifold. In this case we have $\alpha : (z_1, z_2) \rightarrow (-z_1, -z_2)$ which generates a discrete subgroup of the $SU(2)$ holonomy group of a smooth $K3$, and therefore leaves two supercharges invariant giving us $N = 2$ supersymmetry. The symmetry S is given by $S : (z_1, z_2) \to (-z_1 + \frac{1}{2}, -z_2 + \frac{1}{2})$ which is a combination of a shift and a rotation [8]. The shift has no effect on the supercharges; the rotation is again a subgroup of the holonomy group $SU(2)$ and therefore does not break any supersymmetries by itself. Thus the combined operation $S\Omega$ gives $N = 1$ supersymmetry as required. Now, the $\mathbf{Z}_2$ orbifold admits other involutions; for example, the Enriques involution $E : (z_1, z_2) \to (-z_1 + \frac{1}{2}, z_2 + \frac{1}{2})$ which does not leave the holomorphic 2-form invariant, and cannot be used for orientifolding if we want unbroken supersymmetry.

The closed-string sector of an orientifold can be determined by index theory and by appropriate projection. Recall that the massless representations in $D = 6$ are labeled by the representations of the little group which is $Spin(4) \sim SU(2) \times SU(2)$. The massless $N = 1$ supermultiplets are

1. the gravity multiplet: $(3, 3) + (1, 3) + 2(2, 3)$,
2. the vector multiplet: $(2, 2) + 2(1, 2)$
3. the tensor multiplet: $(3, 1) + (1, 1) + 2(2, 1)$
4. the hyper multiplet: $4(1, 1) + 2(2, 1)$.

To determine the massless modes we need to know the Dolbeault cohomology [21], and how the symmetry $\Omega\beta$ acts on the cohomology. For a smooth $K3$, the nonzero Hodge numbers are $h^{00} = h^{22} = h^{02} = h^{20} = 1$, and $h^{11} = 20$. Among the 2-forms the $(0, 2)$, $(2, 0)$, and the Kähler $(1, 1)$ form are self-dual, and the remaining 19 $(1, 1)$ forms are anti-self-dual. The manifolds E_k have $(k-1)$ anti-self-dual $(1, 1)$ harmonic forms, and one $(0, 0)$ form. In the orbifold limit, each fixed point that is repaired by E_k contributes $(k-1)$ anti-self-dual $(1, 1)$ forms which together with the $(1, 1)$ forms of the original torus that are invariant under the orbifold group give the 20 $(1, 1)$ forms of $K3$.

It is useful to think in terms of Type I theory compactified on a smooth $K3$. In this case, the orientation reversal symmetry in ten dimensions, which we shall call Ω_0 has the effect of flipping the sign of A_{MNPQ}, ϕ^2, and B^2_{MN}, leaving other massless fields invariant. The resulting theory has $h^{11}(= 20)$ hypermultiplets which come from the zero modes of B^1_{MN} and g_{MN}. There is only one tensor multiplet from contracting B^1_{MN} with the $(0, 0)$

form. Now imagine performing a projection not with Ω_0 but with $\Omega_0 T$ where T is some geometric symmetry under which n_T $(1,1)$ forms are odd and all others are even. In this case, by contracting A_{MNPQ} with these $(1,1)$ forms, one can obtain n_T additional tensor multiplets that are invariant under the combined operation $\Omega_0 T$. At the same time, n_T hyper-multiplets are now projected out changing their total number to $(20 - n_T)$. This reasoning gives the simple equation

$$n_T + n_H^c = 20, \tag{4.2}$$

where n_H^c refers to the number of hypermultiplets arising from the closed string sector, and $n_T + 1$ is the total number of tensor multiplets. Moreover, no vector multiplets arise from the closed string sector because there are no harmonic odd forms on $K3$, so starting with even forms and the metric in ten dimensions, one cannot obtain a one-form vector potential. We can thus read off the closed string spectrum immediately from the geometric data of the orientifold.

In the orbifold limit, the orientifold symmetry Ω, for the purposes of counting of states, is really a combination of $\Omega_0 T$ where T is some geometric symmetry that has nontrivial action on the cohomology. This is because at each fixed point, Ω takes the sector twisted by ρ to the one twisted by ρ^{-1}. If we repair the singularity at the fixed point of a $\mathbf{Z}_k$ symmetry by the smooth surfaces E_k then the $(k-1)$ $(1,1)$-forms coming from E_k correspond to the $(k-1)$ twisted sectors. If we think of the orbifold as a limit of a smooth K_3, then except in the case when α is a $\mathbf{Z}_2$ twist, we get a nontrivial action on the cohomology denoted by T. This information is sufficient to work out the spectrum of the orientifold in the closed-string sector.

Let us now discuss the massless bosonic spectrum coming from the NS open-string sector. The states

$$\psi^\mu_{-1/2}|0, ij\rangle\lambda_{ji}, \quad \mu = 1, 2, 3, 4, \tag{4.3}$$

belong to the vector multiplets whereas the states

$$\psi^m_{-1/2}|0, ij\rangle\lambda_{ji}, \quad m = 6, 7, 8, 9, \tag{4.4}$$

belong to the hypermultiplets. We have to keep only the states that are invariant under α and $\Omega\beta$. For this purpose we need to know the form of the γ matrices defined in (2.1) and (2.2) which are determined by the requirement of tadpole cancelation. The Chan-Paton wave functions λ_{ij} allowed by these projections determine the gauge group and the matter representations.

There are some features of the tadpole calculation that are common to all orbifolds. First, by the arguments given in [17], only 5-branes and 9-branes appear. Let v_6 and v_4 be the regularized volumes of the noncompact and the compact spaces in string units. If we look at the the Klein bottle amplitude in the tree channel then non-zero tadpoles proportional to $v_6 v_4$ correspond to 10-form exchange requiring addition of 9-branes. Similarly a term proportional to v_6/v_4 corresponds to the exchange of 6-forms from the untwisted sector, requiring addition of 5-branes, and the terms proportional to v_6 correspond to the exchange of 6-forms from the twisted sector and must cancel without the addition of any branes. Now with the orientifold group $G = G_1 + \Omega G_2$, 9-branes can arise only if G_2 contains the identity, and 5-branes arise only if G_2 contains the element R that reflects all four internal co-ordinates. In these cases the determination of the 10-form and the untwisted 6-form tadpoles is identical to the calculation in [17] which requires 32 9-branes with $\gamma_{\Omega,9}^T = \gamma_{\Omega,9}$, and/or 32 5-branes with $\gamma_{\Omega,5}^T = -\gamma_{\Omega,5}$.

4.2. $\mathbf{Z}_2$ *Orbifold*

For the $\mathbf{Z}_2$ orbifold, the model in the first class with the projection $(1+\Omega)/2$ has been discussed in [17], and the model in the second class with the projection $(1+S\Omega)/2$ in [8]. We would now like to consider a model that is closely related to the one in [8]. Let us recall that in [8] the symmetry S was chosen to be such that $S^2 = 1$. However, if we are on a $\mathbf{Z}_2$ orbifold, then the symmetry can square to the element α that generates the orbifold group. We choose

$$S: \quad (z_1, z_2) \to (iz_1, -iz_2). \tag{4.5}$$

Now S has 4 fixed points and not 8. However, they are also the fixed points of α which is a $\mathbf{Z}_2$ symmetry. So on the orbifold, the fixed point of S should be regarded as having

Euler character 2 giving us the total Euler character of 8 in agreement with the Lefschetz number [22].

Obviously, the spectrum consists of the closed string sector found in [8] giving us $n_T = 8$, $n_H = 12$ and the gravity multiplet. However, because now neither R nor the identity are elements of G_2, there is no need to add any branes, and there is no open-string sector. One nontrivial check is that the tadpoles of the R-R fields from the twisted sector now have to cancel by themselves for the Klein bottle without any contribution from the open-string sector. It is easy to see using the formulae in the Appendix that the tadpoles from the untwisted sector cancel against those from the sector twisted by $\frac{1}{2}$ giving us a consistent theory. Gravitational anomalies cancel completely as expected.

4.3. $\mathbf{Z}_3$ *Orbifold*

The orbifold symmetry in this case has nine fixed points of order 3 which contribute two anti-self-dual (1, 1) forms each giving 18 in all. Out of the six 2-forms on the torus one anti-self-dual (1, 1) form and the remaining three self-dual 2-forms are invariant under α giving us 22 2-forms of the $K3$.

Let us first consider the projection under Ω. As explained in §4.1, at each fixed point of the orbifold Ω interchanges the sector twisted by α to that twisted by α^{-1} besides flipping the sign of all R-R fields. This means that of the two tensor multiplets coming from each fixed point, only one will be invariant, giving us $n_T = 9$ from the nine fixed points, and $n_H^c = 11$ from (4.2).

To determine the open-string sector we note that, by the general arguments mentioned in §4.1, there will be 32 9-branes, and we can choose $\gamma_\Omega = \mathbf{1}$ by a unitary change of basis [17]. The requirement that $(\Omega\alpha)^2 = \alpha^2$ implies

$$\gamma_{\alpha^2} = \gamma_\alpha^2 = \gamma_{\Omega\alpha}(\gamma_{\Omega\alpha}^{-1})^T. \tag{4.6}$$

Using the fact the the γ matrices are unitary, and $\gamma_{\Omega\alpha} = \gamma_\Omega\gamma_\alpha$, we conclude that γ_α is real. Furthermore, because $\gamma_\alpha^3 = 1$, the only eigenvalues are cube-roots of unity. If n eigenvalues are $e^{2\pi i/3}$, then n will be $e^{-2\pi i/3}$, and $32 - 2n$ will be 1. We can then write γ in a block-diagonal form where in a $2n$ dimensional subspace it acts as a $2\pi/3$ rotation and in $32 - 2n$ dimensional subspace it equals the identity matrix. This information and anomaly cancellation is enough to determine that $n = 8$. We can also verify this by a detailed calculation of tadpoles as discussed in the Appendix. The gauge group will then be given by $SO(16) \times U(8)$ with hypermultiplets in $(1, 28) + (16, 8)$. It is easy to see that the anomaly terms proportional to $\mathrm{tr}(F^4)$ and $\mathrm{tr}(R^4)$ vanish. It is not necessary for the remaining anomaly to factorize because we have more than one tensor multiplet available, and the anomalies can be canceled by the generalized Green-Schwarz mechanism as in [23,24,8].

Let us now describe the action of S on the $\mathbf{Z}_3$ orbifold. It is given by

$$S: \quad (z_1, z_2) \to (-z_1, -z_2). \tag{4.7}$$

S has 16 fixed points on the torus but on the orbifold they split into one singlet and five triplets of $\mathbf{Z}_3$. The Euler character of the fixed point at the origin which is a singlet under the $\mathbf{Z}_3$ is 3 and that of the 5 triplets is 1 each giving 8 altogether. Now, because S is just a reflection of all co-ordinates, the orientifold with the projection $(1 + S\Omega)/2$ is T-dual to the one described in the previous paragraphs with the projection$(1 + \Omega)/2$. T-duality turns 9-branes into 5-branes, but the spectrum remains unchanged.

4.4. $\mathbf{Z}_4$ *Orbifold*

The $\mathbf{Z}_4$ orbifold has four fixed points of order 4. Each contributes three tensor multiplets out of which only one is invariant under the action Ω. No additional tensors arise from the six doublets of fixed points of order 2. Altogether $n_T = 4$, and $n_H^c = 16$. In this case both 5-branes and 9-branes will be present, and we can choose

$$\gamma_{\Omega,9} = \mathbf{1}, \quad \gamma_{\Omega,5} = \mathbf{J} \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{4.8}$$

The remaining algebra is determined in terms of the matrices $\gamma_{\alpha,9}$ and $\gamma_{\alpha,5}$. Tadpoles are canceled if $\mathrm{Tr}\left(\gamma_{\alpha,9}\right) = \mathrm{Tr}\left(\gamma_{\alpha,9}\right)^2 = \mathrm{Tr}\left(\gamma_{\alpha,9}\right)^3 = 0$ and similarly for the matrices with subscript 5. This determines the γ matrices completely. Moreover $\gamma_{\alpha,9} = \gamma_{\alpha,5}$, and their eigenvalues are such that each forth root of unity appears eight times. The gauge group is $U(8) \times U(8) \times U(8) \times U(8)$ with hypermultiplets in $(28, 1, 1, 1) + (1, 28, 1, 1) + (1, 1, 28, 1) +$

$(1,1,1,28)+(8,8,1,1)+(1,1,8,8)+(8,1,8,1)+(1,8,1,8)$. Once again the anomaly terms proportional to $\mathrm{tr}\left(F^4\right)$ for each factor, and the coefficient of $\mathrm{tr}\left(R^4\right)$ vanish.

Let us now consider the action of the symmetry S which is given by

$$S: \quad (z_1, z_2) \rightarrow (-z_1 + \frac{1+i}{2}, -z_2 + \frac{1+i}{2}). \tag{4.9}$$

This form is determined by the requirement that S has to preserve the orbifold symmetries; in particular, it should map a fixed point of a given order to a fixed point of the same order. It is easy to check that eight $(1,1)$ forms are odd under S. The 16 fixed points form four quartets under $\mathbf{Z}_4$. In addition, S leaves two doublets under α invariant which should be regarded as fixed points on K_3 with Euler character 2. The total Euler character of the fixed point set adds up to 8.

If we consider the orientifold with the projection $(1+\Omega S)$, then only 32 5-branes are required. As in [8] we find $n_T = 8$, $n_H^c = 12$ from the closed-string sector. We can place 16 branes at a fixed point of α^2 which is in a doublet of α that is left invariant by S, and 16 at its image under α. For example, we can place 16 branes at the $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the remaining 16 at $\left(\frac{i}{2}, \frac{i}{2}\right)$. In this case the gauge group is $U(8) \times U(8)$, with charged hyper-multiplets in $2\,(8,8)$. This is exactly the spectrum of the model considered in [8] for the $\mathbf{Z}_2$ orbifold. If we place 16 branes at the fixed point of α, and 16 at its image under S, then the gauge group is $U(4) \times U(4) \times U(4) \times U(4)$ with hypermultiplets in $(4,4,1,1)+(4,1,4,1)+(1,4,1,4)+(1,1,4,4)$.

4.5. $\mathbf{Z}_6$ *Orbifold*

In this case, we get two tensors from the fixed points of order 6 and one each from the four fixed points of order 3 giving us $n_T = 6$ and $n_H^c = 14$. The open-string sector has both 5-branes and 9-branes. The eigenvalues of the matrix $\gamma_{\alpha,5} = \gamma_{\alpha,9}$ are as follows: 1 and -1 appear eight times each and the other sixth roots of unity appear four times each. The resulting gauge-group is $U(4) \times U(4) \times U(8)$ with hypermultiplets in $(6,1,1)+(1,6,1)+(4,1,8)+(1,4,8)$ from the 55 sector, and identical spectrum from the 99 sector. The 59 sector contributes hypermultiplets in $(4,1,1,4,1,1)+(1,4,1,1,4,1)+(1,1,8,1,1,8)$.

Appendix A. Tadpole Calculation

For evaluating the traces in the loop-channel we need the determinants of chiral bosons and fermions with twisted boundary conditions. Let us denote by $D_F\left[{a \atop b}\right]$ the fermion determinant of a chiral Dirac operator $(\nabla^z_{-\frac{1}{2}})$ which corresponds to the path integral of a complex chiral fermion with boundary condition $\psi(\sigma_1 + 2\pi\,,\,\sigma_2) = -e^{2\pi i a}\psi(\sigma_1\,,\,\sigma_2)$, and $\psi(\sigma_1\,,\,\sigma_2 + 2\pi) = -e^{2\pi i b}\psi(\sigma_1\,,\,\sigma_2)$. It is straightforward to evaluate this determinant in the operator formalism[25]. Writing $q = e^{2\pi i\tau}$, and using the standard relation between the path integral and the operator formalism, it is equal to the trace $\operatorname*{Tr}_{\mathcal{H}}\,(h_b\,q^{H_a})$. H_a is the Hamiltonian of a chiral, twisted fermion:

$$H_a = \sum_{n=1}^{\infty}(n - \frac{1}{2} + a)d_n^\dagger d_n + (n - \frac{1}{2} - a)\bar{d}_n^\dagger \bar{d}_n + \frac{a^2}{2} - \frac{1}{24} \tag{A.1}$$

The fermionic oscillators satisfy canonical anticommutation relations $\{d_n^\dagger, d_m\} = \delta_{mn}$ and $\{\bar{d}_n^\dagger, \bar{d}_m\} = \delta_{mn}$, and $\mathcal{H}$ is the usual Fock space representation of these commutations. The group Z_N acts on this Fock space through $hdh^{-1} = -e^{-2\pi ib}d$, $h\bar{d}h^{-1} = -e^{2\pi ib}\bar{d}$. The trace equals (up to an arbitrary phase)

$$e^{2\pi iab}q^{\frac{a^2}{2}-\frac{1}{24}}\prod_{n=1}^{\infty}(1 + q^{n-\frac{1}{2}+a}e^{2\pi ib})\,(1 + q^{n-\frac{1}{2}-a}e^{-2\pi ib})\ . \tag{A.2}$$

Using the product representation of the theta function $\vartheta\left[{a \atop b}\right](\tau)$ with characteristics [26] , we see that

$$D_F\Big[{a \atop b}\Big] = \operatorname*{Tr}_{\mathcal{H}}\,(h_b\,q^{H_a}) = \frac{\vartheta\left[{a \atop b}\right](0|\tau)}{\eta(\tau)}\ , \tag{A.3}$$

where $\eta(\tau)$ is the Dedekind η function. The chiral boson determinant is the inverse of the chiral fermion determinant, except for $a = \frac{1}{2}$ when one needs to be careful about the zero modes. Note that untwisted NS fermions with half-integer modings and antiperiodic boundary conditions for the trace corresponds to $a = 0, b = 0$; an untwisted boson with periodic boundary condition along the σ_2 direction corresponds to $a = \frac{1}{2}, b = \frac{1}{2}$. Using these formulae one can write down the traces by inspection. The tadpole calculation corresponding to the 10-form and the untwisted 6-form exchange are identical to the one in [17], and will not be repeated here. We shall be interested in the tadpole of only the 6-form from the twisted sector which corresponds to the boundary conditions for the determinant for internal bosons that have only oscillator sums but no momentum or winding sums.

Let us first evaluate the traces in (2.3) for the Klein bottle. The total trace can be written as

$$\frac{(1-1)v_6}{64N}\int_0^\infty \frac{dt}{t^4} 8\sum_{a,b} Z\big[{a \atop b}\big], \tag{A.4}$$

where the $(1-1)$ refers to NSNS - RR exchange in the tree channel, v_6 is $V_6/(4\pi\alpha')^3$; $b = k/N, k = 1, ..., (N-1)$ corresponding to the terms with α^k in the trace. Only the untwisted sector and the sector twisted by $\frac{1}{2}$ contribute because for other twisted sectors Ω is off-diagonal; a is therefore either 0, or $\frac{1}{2}$. From the untwisted sector we get

$$Z\big[{0 \atop b}\big] = 4\sin^2(2\pi b)\frac{\vartheta\big[{0 \atop \frac{1}{2}}\big]^2 \vartheta\big[{0 \atop 2b+\frac{1}{2}}\big]\vartheta\big[{0 \atop -2b-\frac{1}{2}}\big]}{\eta^6 \vartheta\big[{\frac{1}{2} \atop 2b+\frac{1}{2}}\big]\vartheta\big[{\frac{1}{2} \atop -2b-\frac{1}{2}}\big]}; \tag{A.5}$$

and from the sector twisted by $\frac{1}{2}$ at each fixed point that is left invariant by α^k, we get

$$Z\big[{\frac{1}{2} \atop b}\big] = -\frac{\vartheta\big[{0 \atop \frac{1}{2}}\big]^2 \vartheta\big[{\frac{1}{2} \atop 2b+\frac{1}{2}}\big]\vartheta\big[{\frac{1}{2} \atop -2b-\frac{1}{2}}\big]}{\eta^6 \vartheta\big[{0 \atop 2b+\frac{1}{2}}\big]\vartheta\big[{0 \atop -2b-\frac{1}{2}}\big]}, \tag{A.6}$$

where $\tau = 2it$ and $b = k/N$. Let us now turn to the traces for the cylinder. In this case, in general we can have $55, 99, 59$, or 95 sectors. The partition sum is given by

$$\frac{(1-1)v_6}{64N}\int_0^\infty \frac{dt}{t^4} \sum_{\lambda,\lambda',b} Z\big[{\lambda\lambda' \atop b}\big]\mathrm{Tr}\,(\gamma_{b,\lambda})\mathrm{Tr}\,(\gamma^{-1}_{b,\lambda'}), \tag{A.7}$$

where λ and λ' take values either 5 or 9, and $\gamma_{\lambda,b}$ refers to the matrix $\gamma_{\lambda,\alpha^k}$ for $b = k/N$. We obtain

$$\begin{aligned} Z\big[{99 \atop b}\big] =& Z\big[{55 \atop b}\big] = 4\sin^2(\pi b)\frac{\vartheta\big[{0 \atop \frac{1}{2}}\big]^2 \vartheta\big[{0 \atop b+\frac{1}{2}}\big]\vartheta\big[{0 \atop -b-\frac{1}{2}}\big]}{\eta^6 \vartheta\big[{\frac{1}{2} \atop b+\frac{1}{2}}\big]\vartheta\big[{\frac{1}{2} \atop -b-\frac{1}{2}}\big]}, \\ Z\big[{59 \atop b}\big] =& Z\big[{95 \atop b}\big] = -\frac{\vartheta\big[{0 \atop \frac{1}{2}}\big]^2 \vartheta\big[{\frac{1}{2} \atop b+\frac{1}{2}}\big]\vartheta\big[{\frac{1}{2} \atop -b-\frac{1}{2}}\big]}{\eta^6 \vartheta\big[{0 \atop b+\frac{1}{2}}\big]\vartheta\big[{0 \atop -b-\frac{1}{2}}\big]}, \end{aligned} \tag{A.8}$$

with $\tau = it$. The Möbius strip amplitude is given by

$$\frac{(1-1)v_6}{64N}\int_0^\infty \frac{dt}{t^4} \sum_{\lambda,b} Z\big[{\lambda\lambda \atop b}\big]\mathrm{Tr}\,(\gamma^T_{b\Omega,\lambda}\gamma^{-1}_{b\Omega,\lambda}), \tag{A.9}$$

where only 55 and 99 sector contribute. We obtain

$$Z\big[{}^{99}_{\ b}\big] = \tan^2(\pi b) Z\big[{}^{55}_{\ b}\big] = -4\sin^2(\pi b)\frac{\vartheta\big[{}^{0}_{\frac12}\big]^2\vartheta\big[{}^{\frac12}_{0}\big]^2\vartheta\big[{}^{0}_{b+\frac12}\big]\vartheta\big[{}^{0}_{-b-\frac12}\big]\vartheta\big[{}^{\frac12}_{b}\big]\vartheta\big[{}^{\frac12}_{-b}\big]}{\eta^6\vartheta\big[{}^{0}_{0}\big]^2\vartheta\big[{}^{\frac12}_{b+\frac12}\big]\vartheta\big[{}^{\frac12}_{-b-\frac12}\big]\vartheta\big[{}^{0}_{b}\big]\vartheta\big[{}^{0}_{-b}\big]}, \quad \text{(A.10)}$$

with $\tau = 2it$

To factorize in the tree channel we use the modular transformations under $\tau \to -1/\tau$:

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) &= (-i\tau)^{-\frac12} e^{-2\pi iab}\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(-1/\tau) \\ \eta(\tau) &= (-i\tau)^{-\frac12}\,\eta\,(-1/\tau)\,, \end{aligned} \quad \text{(A.11)}$$

and take the limit $t \to 0$. While writing the tadpoles we also have to take into account that the tree channel length l is equal to $1/4t$, $1/2t$, and $1/8t$ for the Klein bottle, the cylinder, and the Möbius strip respectively. The twisted-sector tadpole is then proportional to $\frac{(1-1)v_6}{8N}\int dl$. In this common normalization, we get,

$$\begin{aligned} \text{KB}: \quad & (16)^2\sin^2(2\pi b), && a=0, b\neq 0, \\ & -64, && a=\frac12, b\neq 0, \frac12; \\ \text{C}: \quad & 4\sin^2(\pi b)\mathrm{Tr}\,(\gamma_{b,\lambda})\mathrm{Tr}\,(\gamma^{-1}_{b,\lambda}), && b\neq 0, \lambda = 5 \,\text{or}\, 9, \\ & -\mathrm{Tr}\,(\gamma_{b,5})\mathrm{Tr}\,(\gamma^{-1}_{b,9}) - (9\leftrightarrow 5)\,, && b\neq 0; \\ \text{MS}: \quad & -64\sin^2(\pi b)\mathrm{Tr}\,(\gamma^{T}_{b\Omega,9})\mathrm{Tr}\,(\gamma^{-1}_{b\Omega,9}), && b\neq 0, \frac12 \\ & -64\cos^2(\pi b)\mathrm{Tr}\,(\gamma^{T}_{b\Omega,5})\mathrm{Tr}\,(\gamma^{-1}_{b\Omega,5}), && b\neq 0, \frac12. \end{aligned} \quad \text{(A.12)}$$

The Klein bottle contributes -64 from each sector twisted by $\frac12$ for each fixed point that is left invariant by α^k.

Acknowledgements

This work was supported in part by the U. S. Department of Energy under Grant No. DE-FG03-92-ER40701.

Note Added:

In a paper [27] that appeared after this work was completed many of the orientifolds of $K3$ discussed in §4 have been found independently.

References

[1] J. Polchinski, *Phys. Rev. Lett.* **75** (1995) 4724, hep-th/9510017.

[2] J. Dai, R. G. Leigh, and J. Polchinski, *Mod. Phys. Lett.* **A4** (1989) 2073;
R. G. Leigh, *Mod. Phys. Lett.* **A4** (1989) 2767.

[3] A. Sagnotti, in Cargese '87, "Non-perturbative Quantum Field Theory," ed. G. Mack et. al. (Pergamon Press, 1988) p. 521; *Some Properties of Open-String Theories,* preprint ROM2F-95/18, hep-th/9509080;
M. Bianchi and A. Sagnotti, *Phys. Lett.* **B247** (1990) 517; *Nucl. Phys.* **B361** (1991) 519

[4] P. Horava, *Nucl. Phys.* **B327** (1989) 461; *Phys. Lett.* **B231** (1989) 251; *Phys. Lett.* **B289** (1992) 293; *Nucl. Phys.* **B418** (1994) 571.

[5] J. Polchinski, S. Chaudhuri, C. V. Johnson, "Notes on D-Branes," NSF-ITP-96-003, hep-th/9602052.

[6] N. Seiberg and E. Witten, "Comments on String Dynamics in Six Dimensions," RU-96-12, IASSNS-HEP-96/19, hep-th/9603003.

[7] E. Witten, "Five-branes and M-Theory on an Orbifold," hep-th/9512219.

[8] A. Dabholkar and J. Park, "An Orientifold of Type IIB theory on K3," CALT-68-2038, hep-th/9602030.

[9] A. Sen, "M-Theory on $(K3 \times S^1)/Z_2$," hep-th/9602010 ; "Orbifolds of M-Theory and String Theory," hep-th/9603113.

[10] A. Kumar and K. Ray, "M-Theory on Orientifolds of $K_3 \times S^1$," hep-th/9602144.

[11] C. Vafa, "Evidence for F-theory," HUPT-96/A004, hep-th/9602022.

[12] C. Vafa and D. Morrison, "Compactifications of F-Theory on Calabi-Yau Threefolds-I, II," hep-th/9602114, hep-th/9603161.

[13] P. Horava and E. Witten, *Nucl. Phys.* **B460** (1996) 506.

[14] S. Chaudhuri, G. Hockney, J. Lykken, *Phys. Rev. Lett.* **75** (1995) 2264.

[15] S. Chaudhuri and J. Polchinski, *Phys. Rev.* **D52** (1995) 7168.

[16] J. H. Schwarz, *Phys. Lett.* **B371** (1996) 223, hep-th/9512053.

[17] E. G. Gimon and J. Polchinski, "Consistency Conditions for Orientifolds and D-manifolds," hep-th/9601038.

[18] C. Vafa and E. Witten, "Dual String Pairs with $N = 1$ and $N = 2$ Supersymmetry in Four Dimensions," HUTP-95-A23, hep-th/9507050.

[19] This possibility has been considered in informal discussions but we are not certain of its origin.

[20] M. A. Walton, *Phys. Rev.* **D37** (1988) 377.

[21] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Vol. II , Cambridge University Press (1987).

[22] T. Eguchi, P. B. Gilkey, and A. J. Hanson, *Phys. Rept.* **66** (1980) 213.

[23] M. B. Green and J. H. Schwarz, *Phys. Lett.* **B149** (1984) 117; *Phys. Lett.* **B151** (1985) 21.

[24] A. Sagnotti, *Phys. Lett.* **B294** (1992) 196.

[25] L. Alvarez-Gaumé, G. Moore and C. Vafa, *Comm. Math. Phys.* **106** (1986) 40.

[26] D. Mumford, *Tata Lectures on Theta* I, Birkhäuser (1983).

[27] E. Gimon and C. Johnson, "K3 Orientifolds," hep-th/9604129.