

# Ultracold bosons in a synthetic periodic magnetic field: Mott phases and re-entrant superfluid-insulator transitions

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We study Mott phases and superfluid-insulator (SI) transitions of ultracold bosonic atoms in a two-dimensional square optical lattice at commensurate filling and in the presence of a synthetic periodic vector potential characterized by a strength  $p$  and a period  $l = qa$ , where  $q$  is an integer and  $a$  is the lattice spacing. We show that the Schrödinger equation for the non-interacting bosons in the presence of such a periodic vector potential can be reduced to an one-dimensional Harper-like equation which yields  $q$  energy bands. The lowest of these bands have either single or double minima whose position within the magnetic Brillouin zone can be tuned by varying  $p$  for a given  $q$ . Using these energies and a strong-coupling expansion technique, we compute the phase diagram of these bosons in the presence of a deep optical lattice. We chart out the  $p$  and  $q$  dependence of the momentum distribution of the bosons in the Mott phases near the SI transitions and demonstrate that the bosons exhibit several re-entrant field-induced SI transitions for any fixed period  $q$ . We also predict that the superfluid density of the resultant superfluid state near such a SI transition has a periodicity  $q$  ( $q/2$ ) in real space for odd (even)  $q$  and suggest experiments to test our theory.

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## I. INTRODUCTION

Several experiments on ultracold trapped atomic gases have opened a new window onto the phases of quantum matter<sup>1</sup>. A gas of bosonic atoms in an optical or magnetic trap has been reversibly tuned between superfluid and insulating ground states by varying the strength of a periodic potential produced by standing optical waves<sup>1,2</sup>. This transition has been explained on the basis of the Bose-Hubbard model with on-site repulsive interactions and hopping between nearest neighboring sites of the lattice<sup>3-7</sup>. In fact, experiments on the superfluid-insulator (SI) transitions of such bosonic atoms in two-dimensional (2D) optical lattices<sup>8</sup> is found to agree with predictions of theoretical studies of the Bose-Hubbard model quite accurately<sup>3,7,9</sup>.

More recently, several experiments have successfully generated time- or space- dependent effective vector potentials for neutral bosons. Such synthetic vector potentials are created by generating temporally or spatially dependent optical coupling between the internal states of these bosonic atoms<sup>10-12</sup>. We note that this experimental technique involves production of a specific effective vector potential for the atoms and hence corresponds to a fixed gauge. In the simplest experimental setup, these vector potentials are typically chosen to represent a constant magnetic field in the asymmetric gauge. However, a few experiments have also generated vector potentials which correspond to spatially varying synthetic magnetic fields<sup>12</sup>. Several theoretical studies have been carried on the properties of the bosons in deep optical lattice in the presence of a constant synthetic magnetic field<sup>13</sup>. In particular, the SI phase boundary has been computed both using mean-field theory<sup>14</sup> and excitation energy calculation which relies on a perturbative expansion in the hopping parameter<sup>15</sup>. More recently, experimentally relevant

issues, such as the momentum distribution of the bosons in the Mott phase, the critical theory of the SI transition, and the nature of the superfluid ground states and collective modes near criticality have also been addressed<sup>16,17</sup>. However, in spite of the possibility of direct experimental realization<sup>12</sup>, the phase diagram of these bosons in the presence of a spatially dependent magnetic field has not been theoretically investigated.

In this work, we present a theory of the SI transition for ultracold bosons in a 2D square optical lattice with commensurate filling  $n_0$  and in the presence of a periodic synthetic vector potential given by  $\vec{A}^* = (0, A_y^*)$  with  $A_y^* = A_0^* \sin(2\pi x/l)$ , where  $l = qa$  is the period of the vector potential,  $q$  is an integer,  $a$  is the lattice spacing, and  $A_0^*$  is the maximum value of the vector potential on any lattice site. At the outset, we introduce a dimensionless number  $p = 2\pi q^* A_0^* a / hc$ , (where  $q^*$  is the effective charge of the bosons<sup>11</sup>,  $c$  is the speed of light, and  $h = 2\pi\hbar$  is the Planck's constant) which will be used in the rest of this work to characterize the strength of the vector potential. We first consider the problem of non-interacting bosons in a lattice in the presence of such a periodic vector potential and show that the corresponding single particle Schrödinger equation can be reduced to a one-dimensional Harper-like equation<sup>18,19</sup>. The solution of this equation yields an energy spectra with  $q$  bands (with energies  $\epsilon_\alpha^q(\mathbf{k}; p)$  for  $\alpha = 0..q-1$ ) all of which have a periodicity of  $2\pi/q$  along  $k_x$ . The lowest of these bands  $\epsilon_0^q(\mathbf{k}; p)$  has, depending on  $p$ , either a single minimum at  $\mathbf{k} \equiv (k_x, k_y) = (0, 0)$  or  $(0, \pi)$  or doubly degenerate minima either at  $(0, 0)$  and  $(0, \pi)$  or at  $(0, \pm k_y^{\min})$  where  $k_y^{\min}$  can vary continuously as a function of  $p$  for a given  $q$ . The minimum energy of the lowest band,  $\epsilon_{\min}$ , turns out to be a non-monotonic function of  $p$  for a fixed  $q$ . Using these properties of the single particle energy bands and a strong coupling expansion<sup>7,16</sup>, we

analyze the Mott phase and SI phase transition of these bosons in the presence of a deep optical lattice. We show that, depending on  $p$  and  $q$ , the momentum distribution of these bosons in the Mott phase near the SI transition will exhibit single (double) precursor peak(s) at the position of the minimum (minima) of  $\epsilon_0^q(\mathbf{k}; p)$ . We determine the SI phase boundary and demonstrate that the bosons exhibit a series of re-entrant field-induced SI transitions as a function of the vector potential strength  $p$  for any period  $q$ . We also construct an effective Landau-Ginzburg action for the SI transition and show, by analyzing this action at a mean-field level, that the resultant superfluid state has a  $q$  ( $q/2$ ) periodic structure in real space for any odd (even)  $q$ . We show that the reason for such a period-halving of the superfluid density for even  $q$  can be traced back to the properties of the Harper-like equation obeyed by the non-interacting bosons. We discuss several experiments that can probe our theory.

The rest of the paper is organized as follows. In Sec. II, we introduce the relevant tight-binding Hamiltonian of the bosons in an optical lattice in the presence of the periodic vector potential and obtain the energy spectrum when the interaction between these bosons is set to zero. This is followed by Sec. III, where we introduce the strong coupling expansion for the bosons and use it to compute the boson momentum distribution in the Mott phase and the SI phase boundary. In Sec. IV, we show that the superfluid state into which the transition takes place exhibits a  $q$ -periodic superfluid density. We conclude with a discussion of possible experiments to test our theory in Sec. V.

## II. NON-INTERACTING BOSON SPECTRUM

The Hamiltonian of a system of bosons in the presence of an optical lattice and a synthetic periodic vector field is given by<sup>1,3,8,14,15</sup>

$$\mathcal{H} = \sum_{\mathbf{r}, \mathbf{r}'} t'_{\mathbf{r}\mathbf{r}'} b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} + \sum_{\mathbf{r}} [-\mu \hat{n}_{\mathbf{r}} + \frac{U}{2} \hat{n}_{\mathbf{r}}(\hat{n}_{\mathbf{r}} - 1)] \quad (1)$$

where  $\mu$  is the chemical potential,  $U$  is the on-site Hubbard interaction,  $b_{\mathbf{r}}$  ( $\hat{n}_{\mathbf{r}} = b_{\mathbf{r}}^\dagger b_{\mathbf{r}}$ ) is the boson annihilation (density) operator, the hopping matrix  $t'_{\mathbf{r}\mathbf{r}'}$  is given by

$$t'_{\mathbf{r}\mathbf{r}'} = -t' e^{-iq^* \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}^* \cdot d\mathbf{l}} / \hbar c, \quad (2)$$

if  $\mathbf{r} \equiv (x, y) = (m, n)a$  and  $\mathbf{r}'$  are nearest neighboring sites and is zero otherwise, and  $t'$  is the hopping amplitude of the bosons between the nearest neighboring sites. In the rest of this work, we set the lattice spacing  $a$ ,  $\hbar$ , and  $c$  to unity. Our aim in this work is to analyze the phases of  $\mathcal{H}$ .

To this end, we first analyze the boson spectrum in the non-interacting limit  $\mu = U = 0$ . In this case, non-

interacting boson Hamiltonian becomes

$$\mathcal{H}_0 = -t' \sum_{m,n} \left[ b_{mn}^\dagger \left( b_{m+1,n} + b_{m,n+1} e^{-ip \sin(2\pi m/q)} \right) + \text{h.c.} \right], \quad (3)$$

where we have used  $p = 2\pi q^* A_0^* a / \hbar c$ . To obtain the spectrum for  $\mathcal{H}_0$ , we use the identity

$$e^{iz \sin x} = \sum_{r=-\infty}^{\infty} J_r(z) e^{ixr}, \quad (4)$$

where  $J_r(z)$  denotes Bessel functions with integer  $r$ , and write  $\mathcal{H}_0$  in momentum-space representation as

$$\begin{aligned} \mathcal{H}_0 &= -t' \sum_{\mathbf{k}} \left[ 2 \cos(k_x) b^\dagger(k_x, k_y) b(k_x, k_y) \right. \\ &\quad \left. + \left( \sum_{r=-\infty}^{\infty} J_r(p) e^{-ik_y r} b^\dagger(k_x, k_y) b(k_x + 2\pi r/q, k_y) \right. \right. \\ &\quad \left. \left. + \text{h.c.} \right) \right] \\ &= -t' \sum_{\mathbf{k}} \left[ 2(\cos(k_x) + S_0(p) \cos(k_y)) \right. \\ &\quad \times b^\dagger(k_x, k_y) b(k_x, k_y) + \sum_{r=1}^{q-1} S_r(p) e^{-ik_y} \\ &\quad \left. \times b^\dagger(k_x, k_y) b(k_x + 2\pi r/q, k_y) + \text{h.c.} \right], \quad (5) \end{aligned}$$

where  $b(k_x, k_y) = \sum_{\mathbf{k}} \exp(i(k_x m + k_y n)) b_{mn}$ . In Eq. 5,  $S_r(p)$  is given by

$$S_r(p) = \sum_{n=-\infty}^{\infty} J_{qn+r}(p), \quad (6)$$

where  $n$  takes integer values,  $S_r(p) = S_{r+q}(p)$ , and we have used the  $2\pi$  periodicity of  $b(k_x, k_y)$ :  $b(k_x + 2\pi, k_y) = b(k_x, k_y)$ . Note that for even  $q$ ,  $S_r = 0$  for all odd integer  $r$  which follows from the well-known property of the Bessel functions  $J_n(p) = (-1)^n J_{-n}(p)$  for any integer  $n$ .

The Schrödinger equation obtained from Eq. 1 can be written by expressing the eigenfunctions as<sup>19</sup>

$$|\psi\rangle = \sum_{\alpha=0}^{q-1} \psi_\alpha b^\dagger(k_x + 2\pi\alpha/q, k_y) |0\rangle, \quad (7)$$

where  $\psi_\alpha = \psi_{\alpha+q}$ , and obtaining the equations of  $\psi_\alpha$  from  $\mathcal{H}_0|\psi\rangle = E|\psi\rangle$ . This yields a one-dimensional Harper-like equation for  $\psi_\alpha$

$$\begin{aligned} \epsilon \psi_\alpha &= -t' \left[ 2(\cos(k_x) + S_0(p) \cos(k_y)) \psi_\alpha \right. \\ &\quad \left. + \sum_{r=1}^{q-1} S_r(p) (e^{-ik_y} \psi_{\alpha+r} + e^{ik_y} \psi_{\alpha-r}) \right]. \quad (8) \end{aligned}$$

Eq. 8 can easily be cast in the form of  $q \times q$  dimensional Hermitian matrix equation  $\Lambda^q(\mathbf{k}; p)\psi = \epsilon\psi$ . The

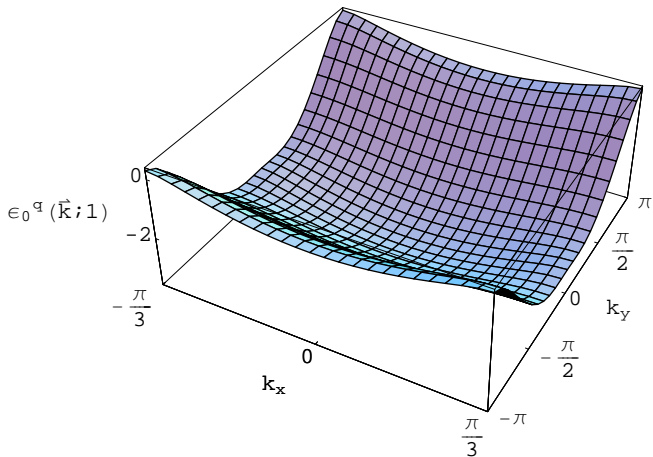


FIG. 1: (Color online) Plot of  $\epsilon_0^q(\vec{k}; 1)$  showing a single minima at  $(k_x, k_y) = (0, 0)$ .

diagonal elements of  $\Lambda^q(\mathbf{k}; p)$  are given by  $\Lambda_{nn}^q(\mathbf{k}; p) = -2(\cos(k_x + 2\pi(n-1)/q) + S_0(p)\cos(k_y))$  and the off-diagonal elements by  $\Lambda_{n, n+r}^q(\mathbf{k}; p) = \Lambda_{n+r, n}^{q*}(\mathbf{k}; p) = -S_r(p)e^{-ik_y}$ . The difference of  $\Lambda^q(\mathbf{k}; p)$  with its counterpart in the constant magnetic field<sup>19</sup> is two-fold. First,  $\Lambda^q(\mathbf{k}; p)$  no longer remains a tri-diagonal matrix. However, the  $2\pi/q$  periodicity of its eigenvalues, which is a consequence of the periodicity of the magnetic field, is still retained. This property is most easily seen by noting that a shift of  $k_x \rightarrow k_x + 2\pi/q$  in Eq. 7 amounts to a shift of  $\psi_\alpha \rightarrow \psi_{\alpha+1}$ . Second, for even  $q$ , where  $\Lambda_{n, n+r}^q(\mathbf{k}; p) = \Lambda_{n+r, n}^{q*}(\mathbf{k}; p) = 0$  for all odd  $r$ ,  $\Lambda^q(\mathbf{k}; p)$  separates into two block-diagonal matrices of dimension  $q/2$  leading to  $q/2$  non-zero elements of the eigenvector  $\psi$  for any eigenvalue  $\epsilon$ . Note that for  $q = 2$ , which correspond to  $A_y^* = 0$  on all sites, we have  $S_0(p) = 1$  and  $S_1(p) = 0$  so that Eq. 8 reduces to the standard tight-binding Hamiltonian in zero magnetic field.

For  $q \geq 3$ , a straightforward numerical diagonalization of  $\Lambda$  leads to  $q$  energy bands with energy dispersions  $\epsilon_\alpha^q(\mathbf{k}; p)$ , where  $\alpha = 0..q-1$ , which have a period of  $2\pi/q$  along  $k_x$ . This periodicity is a manifestation of the  $q$ -fold folding of the Brillouin zone due to the presence of the periodic vector potential. The lowest energy band  $\epsilon_0^q(\mathbf{k}; p)$ , shown in Fig. 1 for  $p = 1$  and  $q = 3$ , displays a single minima at  $(k_x, k_y) = (0, 0)$  within the magnetic Brillouin zone  $(-\pi/q \leq k_x \leq \pi/q$  and  $-\pi \leq k_y \leq \pi)$ . This minima structure changes with increasing  $p$  as shown in Fig. 2 for  $q = 3, 4, 5$  and  $6$ . For  $q = 3, 5$  and  $6$ , we find that beyond a critical strength of the vector potential  $p_1(q)$ ,  $\epsilon_0^q(\mathbf{k}; p)$  has two minima at the  $(0, \pm k_y^{\min}(p))$ . As  $p$  is increased,  $k_y^{\min}$  increases monotonically from 0 to  $\pi$  until it reaches  $\pi$  at  $p = p_2(q)$ , where we recover the single minima structure of  $\epsilon_0^q(\mathbf{k}; p)$  with the minima at  $(0, \pi)$ . As  $p$  is further increased, till a value  $p_3(q)$ ,  $k_y^{\min}$  remains at  $\pi$ . Beyond

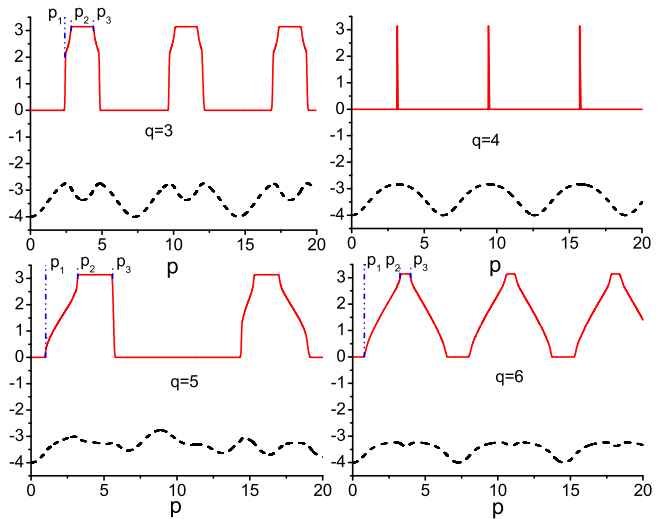


FIG. 2: (Color online) Plot of  $k_y^{\min}$  (red solid line) and  $\epsilon_{\min}/t'$  (black dashed line) as a function of the vector potential strength  $p$  for  $q = 3$  (left top panel), 4 (right top panel), 5 (left bottom panel), and 6 (right bottom panel), showing non-monotonic periodic behavior. See text for details.

$p_3(q)$ , for  $q = 3$  and  $6$ , we find that  $\epsilon_0^q(\mathbf{k}; p)$  again has two minima at  $(0, \pm k_y^{\min}(p))$  and  $k_y^{\min}(p)$  monotonically decreases from  $\pi$  to 0 as  $p$  is increased. For  $q = 5$ , beyond  $p_3(q)$ , we find a discontinuous change in  $k_y^{\min}$  from  $\pi$  to 0, and  $\epsilon_0^q(\mathbf{k}; p)$  retains its single minima structure. For  $q = 4$ , we always have a single minima of  $\epsilon_0^q(\mathbf{k}; p)$  at  $(k_x, k_y) = (0, 0)$ , except at  $p = n\pi$  where there are two degenerate minima at  $(0, 0)$  and  $(0, \pi)$ . We also note from Fig. 2, that the minimum value of the energy,  $\epsilon_{\min}$ , is a non-monotonic function of  $p$  for all  $q \leq 6$ . We have checked that these features remain qualitatively similar for  $q > 6$  and we shall not discuss those cases further here. In the next section, we shall utilize these properties of  $\epsilon_0^q(\mathbf{k}; p)$  to understand the phase diagram of these bosons in the presence of a deep optical lattice.

Before ending this section, we note that there is an alternative method of finding the energy eigenvalues of the Hamiltonian Eq. 3 by constructing the Schrödinger equation in real space and using the  $q$  periodicity of the eigenfunctions along  $x$ . This has been carried out in Ref. 20 and yields identical results to the method elaborated here. We also point out that, although we have, keeping in mind the simplicity of experimental realization, considered a relatively simple sinusoidal form of the vector potential, our method can be easily generalized to treat more complicated periodic vector potentials. Also, we note that since the vector potential  $A_y^*$  is not a *gauge field*, there is no gauge freedom in the choice of the eigenfunctions (Eq. 7). Thus the flux of the vector potential appears only in the coefficients of the Harper equation and not in the choice of the eigenfunctions which is in contrast to the case of periodic magnetic fields with gauge freedom treated in Ref. 21.

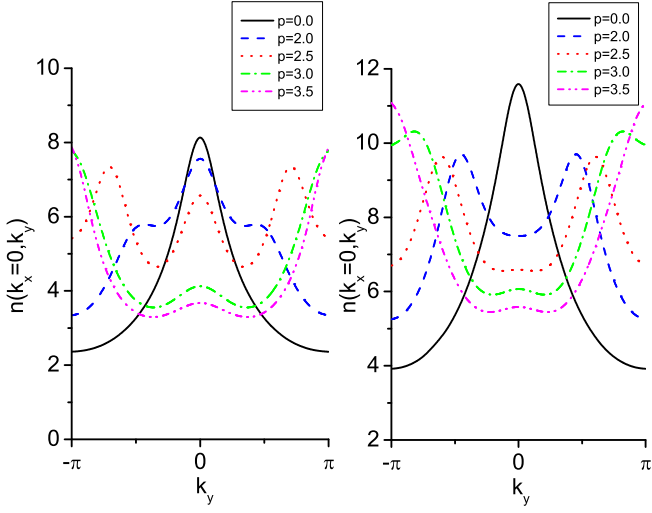


FIG. 3: (Color online) Plot of  $n(k_x = 0, k_y)$  as a function of  $k_y$  for different representative value of  $p$ , and for  $q = 3$  (panel 1) and  $q = 5$  (panel 2) at  $\mu/U = 0.414$  and  $t'(p)/t'_c(p) = 0.95$ .

### III. STRONG COUPLING EXPANSION

In this section, we analyze the phases of  $\mathcal{H}$  in the limit of  $t'/U \ll 1$ , where the bosons are in a Mott insulating state. We note that the effect of the magnetic field manifests itself in the first term of Eq. 1 and thus vanishes in the local limit ( $t' = 0$ ). In this limit the boson Green function can be exactly computed<sup>7,9,16</sup> and is given, at  $T = 0$ , by

$$G_0(i\omega_n) = \frac{(n_0 + 1)}{i\omega_n - E_p} - \frac{n_0}{i\omega_n + E_h}. \quad (9)$$

Here  $\omega_n$  denote bosonic Matsubara frequencies and  $E_h = \mu - U(n_0 - 1)$  ( $E_p = -\mu + Un_0$ ) are the energy cost of adding a hole (particle) to the Mott state. To address the effects of the hopping term, we resort to the coherent state path integral description of these bosons. The partition function of the system can then be written as

$$\begin{aligned} Z &= \int D\tilde{\psi} D\tilde{\psi}^* e^{-(S_0 + S_1)}, \\ S_0 &= \int_0^\beta d\tau \sum_{\mathbf{r}} \left[ \tilde{\psi}_{\mathbf{r}}^*(\tau) \partial_\tau \tilde{\psi}_{\mathbf{r}}(\tau) - \mu n_{\mathbf{r}}(\tau) \right. \\ &\quad \left. + \frac{U}{2} n_{\mathbf{r}}(\tau) (n_{\mathbf{r}}(\tau) - 1) \right], \\ S_1 &= \int_0^\beta d\tau \sum_{\mathbf{r}, \mathbf{r}'} t'_{\mathbf{r}\mathbf{r}'} \tilde{\psi}_{\mathbf{r}}^*(\tau) \tilde{\psi}_{\mathbf{r}'}(\tau). \end{aligned} \quad (10)$$

Here  $\tau$  is the imaginary time,  $\tilde{\psi}$  denote boson fields in the path integral representation,  $n_{\mathbf{r}}(\tau) = \tilde{\psi}_{\mathbf{r}}^*(\tau) \tilde{\psi}_{\mathbf{r}}(\tau)$ ,  $\beta = 1/k_B T$  is the inverse temperature ( $T$ ), and  $k_B$  is the Boltzman constant. Following Ref. 7, we then decouple the hopping term introducing a Hubbard-Stratonovitch

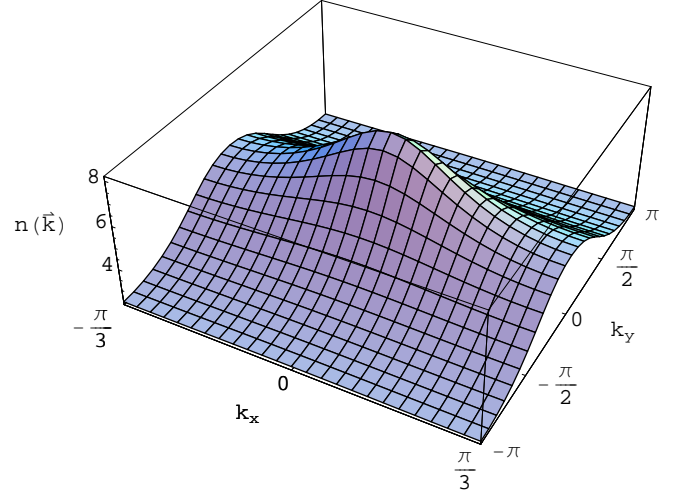


FIG. 4: (Color online) Plot of  $n(\mathbf{k})$  for  $q = 3$  and  $p = 1$  at  $\mu/U = 0.414$  and  $t'/t'_c = .95$  showing peaks at minima  $(0, 0)$ .

field  $\phi_{\mathbf{r}}(\tau)$ . The partition function can then be written as

$$\begin{aligned} Z &= \int D\tilde{\psi} D\tilde{\psi}^* D\phi D\phi^* e^{-(S_0 + S'_1 + S'_2)}, \\ S'_1 &= \int_0^\beta d\tau \sum_{\mathbf{r}} (\tilde{\psi}_{\mathbf{r}}^*(\tau) \phi_{\mathbf{r}}(\tau) + \text{h.c.}), \\ S'_2 &= - \int_0^\beta d\tau \sum_{\mathbf{r}, \mathbf{r}'} t'_{\mathbf{r}\mathbf{r}'}^{-1} \phi_{\mathbf{r}}^*(\tau) \phi_{\mathbf{r}'}(\tau). \end{aligned} \quad (11)$$

Finally, we introduce a second Hubbard-Stratonovitch field  $\psi_{\mathbf{r}}(\tau)$  and decouple  $S'_2$  to obtain

$$\begin{aligned} Z &= \int D\tilde{\psi} D\tilde{\psi}^* D\phi D\phi^* D\psi D\psi^* e^{-(S_0 + S'_3 + S'_4)}, \\ S'_3 &= \int_0^\beta d\tau \sum_{\mathbf{r}} \left[ (\tilde{\psi}_{\mathbf{r}}^*(\tau) - \psi_{\mathbf{r}}^*(\tau)) \phi_{\mathbf{r}}(\tau) + \text{h.c.} \right], \\ S'_4 &= \int_0^\beta d\tau \sum_{\mathbf{r}, \mathbf{r}'} t'_{\mathbf{r}\mathbf{r}'} \psi_{\mathbf{r}}^*(\tau) \psi_{\mathbf{r}'}(\tau). \end{aligned} \quad (12)$$

Note that integrating out  $\phi_{\mathbf{r}}(\tau)$  in Eq. 12 would lead to the constraint  $\psi_{\mathbf{r}} = \tilde{\psi}_{\mathbf{r}}$  on  $Z$ . It can also be shown that  $\psi$  and  $\tilde{\psi}$  fields have identical correlation functions<sup>7</sup>. Next, we follow Refs. 7,16 to integrate out the  $\tilde{\psi}$  and  $\phi$  fields and obtain an effective action in terms of  $\psi$ . The details of this procedure has been elaborated in Ref. 7. The effective action so obtained is given by<sup>7,16</sup>

$$\begin{aligned} S_{\text{eff}} &= S_0 + S_1 \\ S_0 &= \int_{\mathbf{k}} \psi_q^*(i\omega_n, \mathbf{k}) [-G_0^{-1}(i\omega_n) I + \Lambda^q(\mathbf{k})] \psi_q(i\omega_n, \mathbf{k}), \\ S_1 &= g/2 \int_0^\beta d\tau \int d^2 r |\psi_q^*(\mathbf{r}, \tau) \psi_q(\mathbf{r}, \tau)|^2, \end{aligned} \quad (13)$$

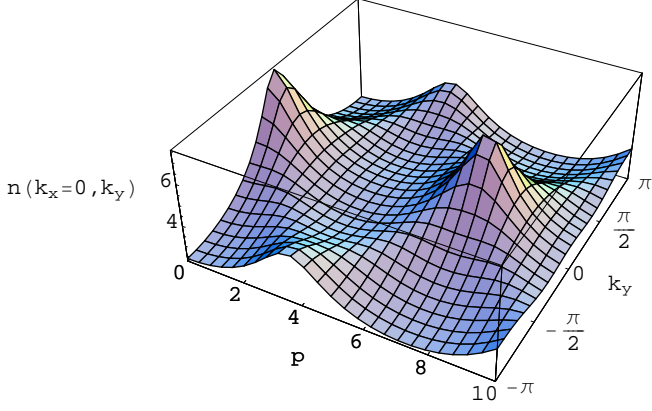


FIG. 5: (Color online) Plot of  $n(k_x = 0, k_y)$  as a function of  $k_y$  and  $p$  for  $q = 3$  at  $\mu/U = 0.414$  and  $t'/U = 0.04$ .

where  $\psi_q = (\psi_0(k_x, k_y) \dots \psi_{q-1}(k_x, k_y))^T$  with  $\psi_\alpha(k_x, k_y) = \psi(k_x + 2\pi\alpha/q, k_y)$  denoting the  $q$ -component of the auxiliary field  $\psi$  in momentum space,  $\int_{\mathbf{k}} \equiv (1/\beta) \sum_{\omega_n} \int d^2k / (2\pi)^2$ ,  $I$  denotes the unit matrix, and  $g > 0$  is the static limit of the exact two-particle vertex function of the bosons in the local limit which has been computed in Ref. 7. Note that  $S_0$  reproduces exact bosons propagator both in the local ( $t' = 0$ ) and the non-interacting ( $U = 0$ ) limits and therefore provides a suitable starting point for the strong coupling approximation. In the next subsection, we shall compute the momentum distribution function of the bosons from  $S_0$ .

### A. Momentum distribution of the bosons in the Mott phase

The momentum distribution of the bosons in the Mott phase can be computed from  $S_0$ <sup>7,16</sup>

$$n(\mathbf{k}) = -\lim_{T \rightarrow 0} (1/\beta) \sum_{\omega_n} \text{Tr} G(i\omega_n, \mathbf{k}),$$

$$G(i\omega_n, \mathbf{k}) = [-G_0^{-1}(i\omega_n)I + \Lambda^q(\mathbf{k}; p)]^{-1}. \quad (14)$$

To compute  $n(\mathbf{k})$ , we note that  $G_0^{-1}$  is independent of momenta. Hence finding  $G(i\omega_n, \mathbf{k})$  amounts to inverting  $\Lambda^q(\mathbf{k}; p)$ . To this end we introduce a unitary transformation where the transformation matrix  $U_q(\mathbf{k})$  diagonalizes  $\Lambda^q(\mathbf{k}; p)$  to obtain a diagonal Green function  $G^d(i\omega_n, \mathbf{k}) = U_q^{-1}(\mathbf{k})G(i\omega_n, \mathbf{k})U_q(\mathbf{k})$  whose diagonal elements are given by

$$\begin{aligned} G_{\alpha\alpha}^d(i\omega_n, \mathbf{k}) &= [-G_0^{-1}(i\omega_n) + \epsilon_\alpha^q(\mathbf{k}; p)]^{-1} \\ &= \frac{i\omega_n + \mu + U}{(i\omega_n - E_q^{\alpha+}(\mathbf{k}; p))(i\omega_n - E_q^{\alpha-}(\mathbf{k}; p))}, \end{aligned} \quad (15)$$

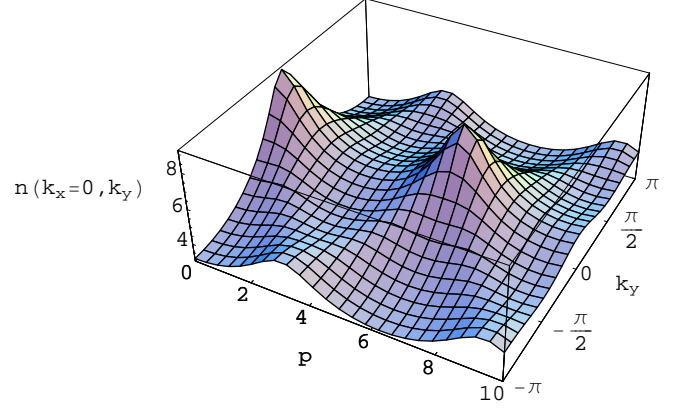


FIG. 6: (Color online) Same as Fig. 5 for  $q = 4$ .

where we have used the expression of  $G_0$  from Eq. 9 and  $E_q^{\alpha\pm}(\mathbf{k}; p)$  denote the location of the poles of the interacting boson Green function and are given by

$$\begin{aligned} E_q^{\alpha\pm}(\mathbf{k}; p) &= -\mu + U(n_0 - 1/2) + \epsilon_\alpha^q(\mathbf{k}; p)/2 \pm \frac{1}{2} \\ &\quad \times \sqrt{\epsilon_\alpha^q(\mathbf{k}; p)^2 + 4\epsilon_\alpha^q(\mathbf{k}; p)U(n_0 + 1/2) + U^2}. \end{aligned} \quad (16)$$

Note that  $E_q^{\alpha\pm}(\mathbf{k}; p)$  can be directly computed from the knowledge of the non-interacting boson spectrum  $\epsilon_\alpha^q(\mathbf{k}; p)$  derived in Sec. II. In particular, the minima  $E_q^{\alpha\pm}(\mathbf{k}; p)$  occur in the same position in the magnetic Brillouin zone as  $\epsilon_\alpha^q(\mathbf{k}; p)$ . Also, as noted in Ref. 7, the Mott gap  $E_q^{\alpha+}(\mathbf{k}; p) - E_q^{\alpha-}(\mathbf{k}; p)$  vanishes at the position of the minima of  $\epsilon_\alpha^q(\mathbf{k}; p)$  in the magnetic Brillouin zone provided we are at the tip of the Mott lobe where the SI transition takes place at constant density.

The momentum distribution can now be computed as  $n(\mathbf{k}) = -\lim_{T \rightarrow 0} (1/\beta) \sum_{\omega_n} \text{Tr} G^d(i\omega_n, \mathbf{k})$  and is given by<sup>16</sup>

$$n(\mathbf{k}) = \sum_{\alpha=0}^{q-1} \frac{E_q^{\alpha-}(\mathbf{k}; p) + \mu + U}{E_q^{\alpha+}(\mathbf{k}; p) - E_q^{\alpha-}(\mathbf{k}; p)}. \quad (17)$$

Eq. 17 shows that the peaks of  $n(\mathbf{k})$  occur when the Mott gap  $E_q^{\alpha+}(\mathbf{k}) - E_q^{\alpha-}(\mathbf{k})$  becomes small near the minima of  $\epsilon_\alpha^q(\mathbf{k}; p)$  as the SI transition is approached through the tip of the Mott lobe. The minima structure of the non-interacting bosons is therefore expected to be reflected in the peaks of the momentum distribution of the bosons in the Mott phase. In Fig. 4, we show a representative plot of  $n(\mathbf{k})$  as a function of  $\mathbf{k}$  for  $q = 3$  and  $p = 1$ . We find that the central peak of the momentum distribution lies at  $(0, 0)$  in accordance with the position of the minima of  $\epsilon_0^{(3)}(\mathbf{k}, 1)$ . Next, keeping in mind that the position of

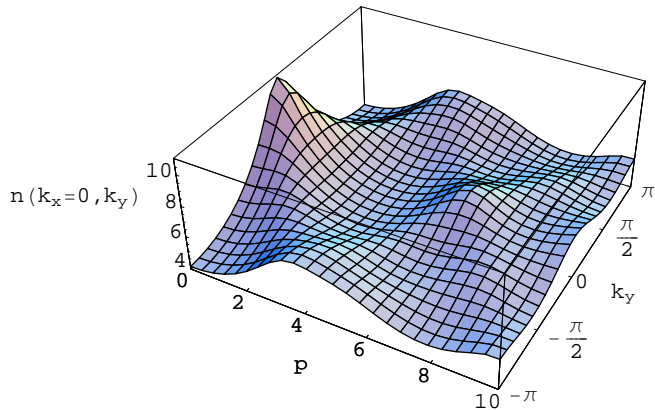


FIG. 7: (Color online) Same as Fig. 5 for  $q = 5$ .

the minima of  $\epsilon_\alpha^q(\mathbf{k}; p)$  always occur at  $k_x = 0$ , we plot the momentum distribution  $n(k_x = 0, k_y)$  as a function of  $k_y$  (for fixed  $t'(p)/t'_c(p) = 0.95$  and  $q = 3, 5$ ) for several representative values of  $p$  in Fig. 3. Fig. 3 clearly shows that as  $p$  increases, the peak structure of the momentum distribution changes from a single peak at  $k_y = 0$  to two split peaks at  $k_y = \pm k_y^{\min}(p)$  and finally to a single peak at  $k_y = \pi$ . Finally in Figs. 5, 6 and 7, we plot  $n(k_x = 0, k_y)$  for  $q = 3, 4$ , and  $5$ , as a function of  $k_y$  and  $p$  for a fixed  $t' = 0.04U$ . Note that for these plots, the proximity of the system to the tip of the Mott lobe changes with  $p$  since  $t'_c$  is a function of  $p$ . These plots again reveal the change in the peak structure of  $n(k_x = 0, k_y)$  as a function of  $p$ .

### B. Re-entrant SI transitions

The critical hopping  $t'_c$  for the MI-SF transition as a function of  $\mu$  can be determined from the condition<sup>16</sup>

$$r_q(p) = -G_0^{-1}(i\omega_n = 0) + \epsilon_{\min}^q(p) = 0. \quad (18)$$

The SI phase boundary so obtained is shown in Fig. 1 for  $q = 3$  and  $p = 1$  in Fig. 8 and displays the usual Mott lobes. The difference of the present case here with the SI transitions studied earlier<sup>3,7,9,16</sup> arises due to the non-monotonic  $p$  dependence of  $\epsilon_{\min}^q(p)$ . This point is demonstrated in Fig. 9 for  $q = 3, 4, 5$ , and  $6$  by plotting  $t'_c(p)$  as a function of  $p$  for  $n_0 = 1$  and  $\mu = \mu_{\text{tip}}$ . We find that  $t'_c(p)$  is a non-monotonic function of  $p$  and  $t'_c(p) > t'_c(0)$  for all  $p$ . Consequently, varying  $p$  at a fixed value of  $t' > t'_c(0)$  leads to a series of field-induced re-entrant SI transitions for any  $q$ . This is schematically marked by the red-dotted line in Fig. 9. We note that such re-entrant transitions as a function of the magnetic field strength are not present for SI transitions in a constant magnetic field<sup>16</sup>.

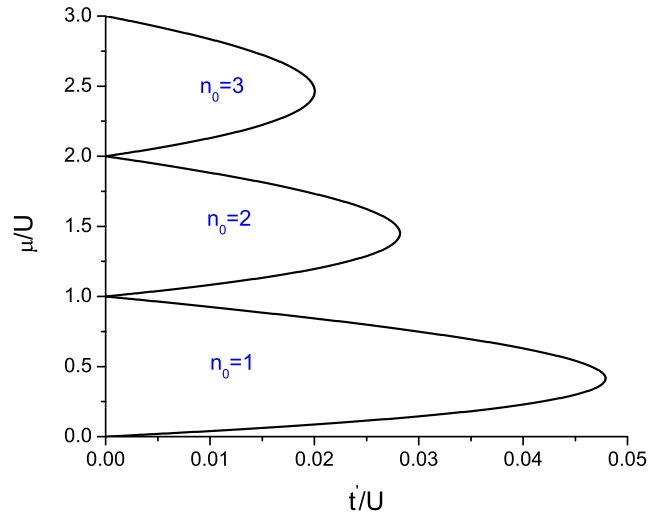


FIG. 8: (Color online) The MI-SF phase boundary for  $q = 3$  and  $p = 1$ .

## IV. THE SUPERFLUID PHASE

At  $t' = t'_c(p)$ , it becomes energetically favorable to create particles/holes at the minima of the energy dispersion of the bosons leading to the destabilization of the Mott phase. The Landau-Ginzburg theory of the resultant superfluid phase can be expressed by long-wavelength boson fields around these minima. In the present case, there are either one or two degenerate minima of the boson energy spectrum in the magnetic Brillouin zone leading to a Landau-Ginzburg theory of one or two low-energy boson fields<sup>3,7,9,16</sup>. We shall first consider the case with a single minima either at  $(0, 0)$  or  $(0, \pi)$  which occurs for specific ranges of  $p$  for all  $q$  as discussed in Sec. II. In either case, the boson field can be written as

$$\begin{aligned} \psi(\mathbf{r}, t) &= \chi_0(\mathbf{r}; p)\varphi(\mathbf{r}, t), \\ \chi_0(\mathbf{r}; p) &= \left[ \sum_{\alpha=0}^{q-1} \psi_\alpha(p) e^{2\pi i \alpha x / q} \right] e^{i k_y^{\min} y} \varphi(\mathbf{r}, t), \end{aligned} \quad (19)$$

where  $\psi_\alpha(p)$  denotes the components of eigenvectors of  $\Lambda_q(\mathbf{k}; p)$  at  $k_x = 0$ ,  $k_y = k_y^{\min}$  which can be either  $0$  or  $\pi$  for a fixed  $p$ , and  $\chi_0(\mathbf{r}; p)$  denotes the corresponding wavefunction in real space. Thus the superfluid density can be written as

$$\rho_s(\mathbf{r}) = |\langle \psi \rangle|^2 = \left| \sum_{\alpha=0}^{q-1} \psi_\alpha(p) e^{2\pi i \alpha x / q} \right|^2 |\varphi_0|^2, \quad (20)$$

where  $\varphi_0 = \langle \varphi(\mathbf{r}, t) \rangle \neq 0$  for  $t' > t'_c(p)$ . Note that  $\rho_s$  is independent of  $y$  irrespective of the value of  $k_y^{\min}$ , but displays spatial variation along  $x$ . Further, as discussed in Sec. II, for even  $q$ , only  $q/2$  of the components  $\psi_\alpha$  (corresponding to either even or odd integers  $\alpha$ ) will be non-zero. Consequently, we expect the period of  $\rho_s(x)$  to



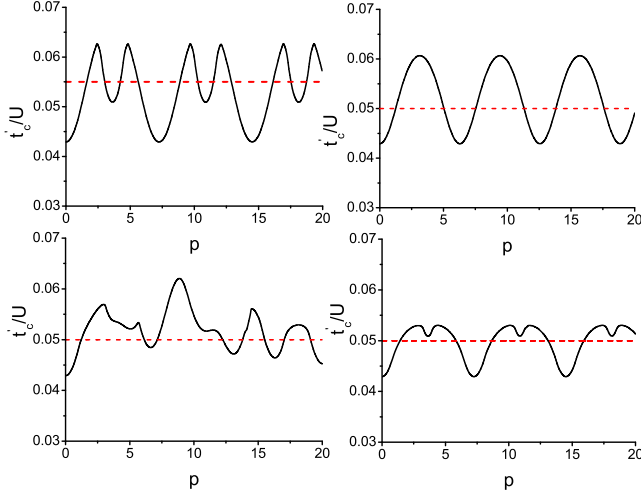


FIG. 9: (Color online) Plot of the critical hopping strength  $t'_c(p)$  as a function of  $p$  at the tip of the Mott lobe ( $\mu = \mu_{\text{tip}} = 0.414U$ ) for  $q = 3$  (left top panel), 4 (right top panel), 5 (left bottom panel), and 6 (right bottom panel). The red-dashed line is a guide to the eye showing reentrant SI transitions as  $p$  is varied at fixed  $t' > t'_c(p = 0)$ .

be halved. A plot of the renormalized superfluid density  $\rho_s(x)/\rho_s(0)$ , plotted in Fig. 10 for  $p = 0.5$  and  $q = 3, 4, 5$ , and 6, confirms this expectation. The presence of the periodic vector potential leads to a  $q$ -periodic pattern with  $q - 2$  small and one large peak in the superfluid density along  $x$  for all odd  $q$  as shown in the left panels of Fig. 10. In contrast, the superfluid density for even  $q$  displays a  $q/2$  periodic pattern. Note that this period halving leads to identical superfluid density patterns for vector potentials with periods  $q$  and  $2q$  for all odd  $q$ . This feature is clearly demonstrated in the top left ( $q = 3$ ) and the bottom right ( $q = 6$ ) panels of Fig. 10.

Next, we derive the effective low-energy Landau-Ginzburg theory. To this end, we substitute Eq. 19 in Eq. 13 and obtain the effective low-energy Landau-Ginzburg action in terms of the  $\varphi$  fields. The details of this procedure is charted out in Ref. 16. The resultant action is given by

$$S_1^{\text{LG}} = \int d^2r dt \left[ \varphi^*(\mathbf{r}, t) [K_0 \partial_t^2 + iK_1 \partial_t + r_q(p) - v_q(p)^2 (\partial_x^2 + \partial_y^2)] \varphi(\mathbf{r}, t) + \frac{g'}{2} |\varphi(\mathbf{r}, t)|^4 \right], \quad (21)$$

where  $K_0 = 1/2 \partial^2 G_0^{-1} / \partial \omega^2 |_{\omega=0} = n_0(n_0 + 1)U^2 / (\mu + U)^3$ ,  $K_1 = \partial G_0^{-1} / \partial \omega |_{\omega=0} = 1 - n_0(n_0 + 1)U^2 / (\mu + U)^2$ , and  $v_q(p)^2 = \nabla_{\mathbf{k}}^2 \epsilon_{\min}(\mathbf{k}; p) / 2$ ,  $r_q(p)$  is given by Eq. 18, and  $g' = g \sum_{x,y=0}^{q-1} |\chi_0(\mathbf{r}; p)|^4 / q^2$ . At the tip of the Mott lobe, where  $\mu = \mu_{\text{tip}} = U(\sqrt{n_0(n_0 + 1)} - 1)$ ,  $K_1 = 0$ . Thus we have a critical theory with dynamical critical exponent  $z = 1$ . Away from the tip,  $K_1 \neq 0$  rendering  $z = 2$ . Thus the critical theory turns out to have similar exponent as in the case without magnetic field<sup>4</sup>.

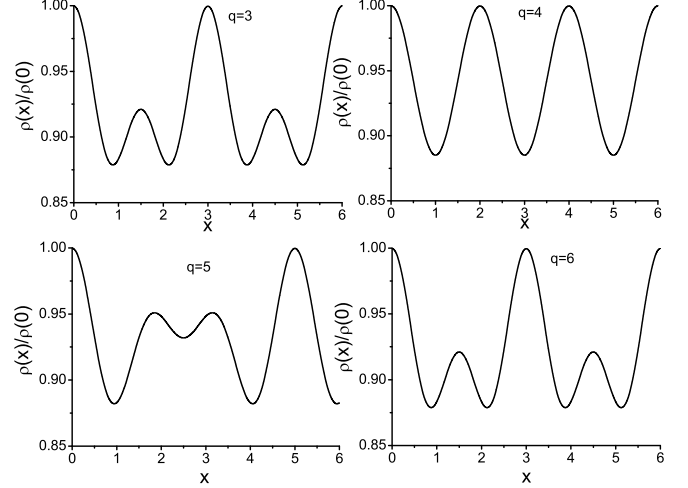


FIG. 10: (Color online) Plot of the superfluid density  $\rho_s(x)/\rho_s(0)$  as a function of  $x$  for  $q = 3, 5$  (left panels) and  $q = 4, 6$  (right panels). Note that the superfluid density displays a  $q$  periodic pattern for odd  $q$ s and a  $q/2$  periodic pattern for even  $q$ s.  $p$  is set to 0.5 for all plots.

Finally, we briefly comment on the case where there are two degenerate minima either at  $(0, \pm k_y^{\min})$  or at  $(0, 0)$  and  $(0, \pi)$ . In this case,  $\psi(\mathbf{r}, t) = \chi_0^+(\mathbf{r}; p)\varphi_+(\mathbf{r}, t) + \chi_0^-(\mathbf{r}; p)\varphi_-(\mathbf{r}, t)$  where  $\chi_0^\pm(\mathbf{r})$  denotes the eigenfunctions of  $\Lambda(\mathbf{k}; p)$  in real space at  $(0, \pm k_y^{\min})$  and  $\varphi_\pm(\mathbf{r}, t)$  denotes low-energy fluctuating fields about the minima. Substituting this expression of  $\psi$  in Eq. 13, and following the coarse-graining procedure detailed in Ref. 16, we find that for all  $q$  and  $p$ , the superfluid phase corresponds to the condensation of only one of the low-energy fields:  $\langle \varphi_+ \rangle = 0, \langle \varphi_- \rangle \neq 0$  or  $\langle \varphi_- \rangle = 0, \langle \varphi_+ \rangle \neq 0$ . Thus the effective Landau-Ginzburg action in these cases is qualitatively similar to Eq. 21. The superfluid density, plotted in Fig. 11 for  $q = 3, 5$  and  $p = 2.5$ , shows similar  $q$  periodic pattern as observed in Fig. 10 for odd  $q$ .

## V. DISCUSSION

There are several possible experimental verifications of our theory. First, we suggest measurement of  $n(\mathbf{k})$  for the bosons in the Mott phase near the transition as done earlier in Ref. 8 for 2D optical lattices without the synthetic magnetic field. Our prediction is that the peak structure of the momentum distribution along  $k_x = 0$  at a fixed  $t'/U$  near  $t'_c$  would be similar to those shown in Figs. 5..7. In particular the shift in the peak position of  $n(0, k_y)$  with  $p$  and change from a single to double peak structure as a function of  $p$  should be observable in such experiments. Second, the re-entrant SI transition can also be verified by measuring  $n(\mathbf{k})$  as a function of  $p$  by fixing  $t' > t'_c(p = 0)$  as shown in Fig. 9. Finally, the spatial variation of the superfluid density can also be

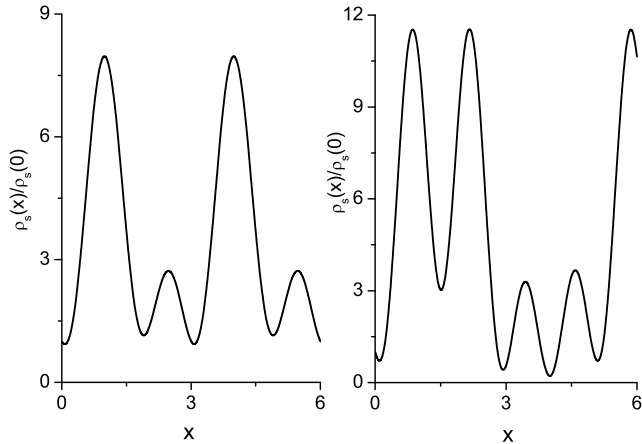


FIG. 11: (Color online) Same plot as Fig. 10 for  $q = 3$  ( $k_y^{\min} = 2.2$ ) (left panel) and  $q = 5$  ( $k_y^{\min} = 1.89$ ) (right panel) showing  $q$  periodic patterns.  $p = 2.5$  for both cases.

observed by measuring  $n(\mathbf{k})$  in the superfluid phase.

In conclusion, we have analyzed the MI-SF transition of ultracold bosons in a 2D optical lattice in the presence of a synthetic periodic magnetic field. We have shown that the precursor peaks of the momentum distribution in the Mott phases can be tuned by the strength  $p$  of the synthetic field. We have also demonstrated that the bosons, in the presence of such a periodic synthetic magnetic field, show a series of field-induced re-entrant SI transitions, and that the superfluid density in the SF phase near criticality shows  $q$  ( $q/2$ ) periodic spatial pattern for odd (even)  $q$ . We have suggested several experiments which can test our theory.

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