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Covariant fields: Poincaré group representations and metric structure in the space of quantum states

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Abstract. The representations of the Poincaré group realized over the space of covariant fields transforming according to any irreducible representation $D^{(m,n)}$ of the Lorentz group are constructed explicitly with reference to a helicity basis. The representation is indecomposable in the massless case. The form of this representation together with the invariance of two-point Wightman functions of the field (which follows from a weak set of axioms) determines the metric structure in the space of quantum states of the field. This structure is explicitly determined for general $D^{(m,n)}$. Certain particular cases (especially the symmetric traceless tensor field) are discussed in detail. Finally we consider the representation pertaining to massive fields, and examine the passage to the limit of vanishing mass. We present a limiting procedure which leads from the unitary representation of the massive field to the indecomposable non-unitary representation of the massless field.

Keywords. Massless fields; Indecomposable representation, Poincaré group; Lorentz group; helicity basis; two-point Wightman functions.

1. Introduction

In the relativistic quantum theory of elementary particles, locally covariant fields play a central role. By *local* covariance we mean that at a *given space-time point* P, the field components as seen from two reference frames related by a Lorentz transformation bear the relation $\psi_P' = D(\Lambda)\,\psi_P$ where $D(\Lambda)$ is a numerical matrix independent of the coordinates of P, differential operators, etc. The set of matrices $D(\Lambda)$ for all Λ forms a (reducible or irreducible) representation of the Lorentz group (LG). When the representation is irreducible, it is by definition impossible to find any set of basis spinors* $u(\alpha)$ ($\alpha = 1, 2, \ldots, c$, where c is the number of components of the field) such that a subset of components of ψ_P with respect to the basis transforms (for all Λ) into the corresponding subset of components of ψ_P' with respect to the *same* basis. However this notion of irreducibility which applies to the *local* transformation property gives way to something different when the behaviour of ψ *as a field, i.e.*, as a function of the coordinates (in position or momentum space) is considered. While the local transformation relates the values of ψ and ψ' at x and Λx respectively (these being the coordinates of P in the two

* For each α, $u(x)$ is a column with c components. The term "spinor" for $u(\alpha)$ is appropriate since in $D^{(m,n)}$, the $u(\alpha)$ are indeed multispinors (of which vectors, tensors, etc. are special cases).

frames), what is most relevant when the transformation of ψ as a field function is to be examined, is a comparison of $\psi(x)$ and $\psi'(x)$ [or of $\psi(p)$ and $\psi'(p)$ in the momentum space version, which is what we shall be concerned with hereafter]. In this context, any spinor basis which is used must be the same for the directly compared quantities $\psi(p)$ and $\psi'(p)$, but one may well choose different bases at different points in momentum space. Then one would have to write $u(p;\alpha)$ for the basis spinors. With the resolution of ψ' and ψ in terms of the appropriate bases, the local transformation property $\psi'(\Lambda p) = D(\Lambda)\psi(p)$ now goes over into

$$\sum_{\alpha} a'(\Lambda p;\alpha)\,u(\Lambda p;\alpha) = \sum_{\alpha} a(p;\alpha)\,D(\Lambda)\,u(p;\alpha) \tag{1}$$

The relation between the "amplitudes" $a'(\Lambda p;\alpha)$ and $a(p;\alpha)$ is determined by the coefficients $\mathcal{D}_{\beta\alpha}$ in the expansion

$$D(\Lambda)\,u(p;\alpha) = \sum_{\beta} u(\Lambda p;\beta)\,\mathcal{D}_{\beta\alpha}(\Lambda;p) \tag{2}$$

The set of matrices $\mathcal{D}(\Lambda;p)$ provides a representation of the LG in the sense that*

$$\mathcal{D}(\Lambda_2\Lambda_1;p) = \pm\,\mathcal{D}(\Lambda_2;\Lambda_1 p)\,\mathcal{D}(\Lambda_1;p) \tag{3}$$

Taken together with the representatives $e^{ip\cdot a}$ of finite translations (by a four-vector a) it gives a representation of the Poincaré group (PG). This representation is in general reducible even when $D(\Lambda)$ is not: the freedom, which one has for choosing at will the dependence of the basis spinors $u(p;\alpha)$ on p, can be utilized to achieve the reduction. In fact for $p^2 > 0$ one accomplishes in this fashion the decomposition of ψ into parts transforming according to the Wigner irreducible representations (Wigner 1939) characterized by different values of spin; when $p^2 = 0$, the representation is again reducible, but it does not decompose into a direct sum of irreducible parts (Shaw 1965, McKerrell 1966).

Our main objectives in this paper are threefold. Firstly, we wish to present an elementary and direct determination of the matrices of the indecomposable representation of the PG realized over the space of covariant massless fields transforming locally according to an arbitrarily specified irreducible representation $D^{(m,n)}$ of the LG. We shall follow the procedure broadly outlined above, wherein the local transformation property, *i.e.*, the knowledge of $D(\Lambda)$ in eq. (1) as $D^{(m,n)}(\Lambda)$, will be in the foreground. The basic spinors $u(p;\alpha)$ will be taken to be characterized by definite values of helicity. Prescription of a procedure defining the normalization and phases of the $u(p;\alpha)$ for arbitrary p in terms of those for a standard vector $\mathring{p}$ will set the stage for the determination of the representation in terms of that of the "little group" which leaves $\mathring{p}$ invariant. The whole process is greatly facilitated by the use of a factorized form of the Lorentz group elements, wherein the factors operate independently in the "M- and N-subspaces" with which the quantum numbers m, n labelling $D^{(m,n)}$ are associated. A derivation employing basically the same method (but in a rather indirect way) was given recently by

* To be more precise, the "matrix" $\mathcal{D}(\Lambda)$ with elements $\mathcal{D}_{\beta\alpha}(\Lambda;p',p) \equiv \mathcal{D}_{\beta\alpha}(\Lambda;p)\,\delta(p'-\Lambda p)$, labelled by a pair of discrete indices (β,α) and continuous indices (p',p) provides a representation, it being understood that matrix multiplication includes integration over a continuous index too. The ambiguous signs arise in the case of double-valued representations, as is well known.

Mathews *et al* (1974 *a*, 1974 *b*). Considering the particular case of the massless vector field, they proved (Mathews *et al* 1974 *b*) from a general axiomatic approach that the indecomposable (and nonunitary) nature of the representation is the fundamental reason for the difficulties in quantization of the field. Our second aim in this paper is to generalize this result by proving that for massless fields* corresponding to any $D^{(m,n)}$, definite relations exist among the norms and scalar products of states with various helicities. The case $m = n = 1$ (symmetric traceless tensor field), which is of particular interest in connection with gravitation, will be treated in detail. Thirdly, we wish to consider representations corresponding to $m_0^2 \equiv p^2 \neq 0$, and display different limiting procedures whereby, as $m_0 \to 0$, one winds up with *either* the single component (unitary) irreducible representation associated with the unique invariant-helicity state (of helicity $n - m$) of $D^{(m,n)}$, *or* the nonunitary indecomposable representation referred to above.

2. Helicity basis

Notation. We denote the generators of rotations and pure Lorentz transformations (boosts) by J and K respectively. We shall make extensive use of the combinations

$$M = \tfrac{1}{2}(J + \mathrm{i}K) \quad \text{and} \quad N = \tfrac{1}{2}(J - \mathrm{i}K) \tag{4}$$

The components of M (or N) have the same commutation relations as angular momentum operators, while $[M, N] = 0$. On account of this last property, any space on which the generators operate may be thought of as a direct product of M- and N-subspaces in which M and N act independently. Any finite transformation Λ of the Lorentz group can then be factorized in the form

$$\Lambda \equiv \exp \mathrm{i}(\theta \cdot J + \alpha \cdot K) = \exp(\mathrm{i}\zeta \cdot M) \exp(\mathrm{i}\zeta^* \cdot N), \quad \zeta = \theta - \mathrm{i}\alpha \tag{5}$$

The symbols R and B will be used for finite rotations and boosts respectively.

Basis Spinors.—We define the basis spinors $u(p; \alpha)$ for any p in terms of those for a particular standard momentum $\mathring{p}$ which we shall choose to be in the z-direction:

$$\mathring{p} = (\omega, 0, 0, \kappa), \quad \omega, \kappa > 0. \tag{6}$$

For brevity we shall write just $u(\alpha)$ instead of $u(p; \alpha)$ in the case of the standard momentum. For $p \neq \mathring{p}$ the spinor basis will be defined by

$$u(p; \alpha) = H(p, \mathring{p})\, u(\alpha) \tag{7}$$

where $H(p, \mathring{p})$ is any conveniently specified element of the Lorentz group which takes the vector $\mathring{p}$ into p, and is fixed once and for all for each p. With this definition, eq. (2) which defines the representation $\mathcal{D}$ can be rewritten as

$$[H^{-1}(\Lambda p, \mathring{p})\, D(\Lambda)\, H(p, \mathring{p})]\, u(\alpha) = \sum_{\beta} u(\beta)\, \mathcal{D}_{\beta\alpha}(\Lambda; p) \tag{8}$$

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* What we refer to as a "field of mass m_0^2" is the set $\psi(p)$ for all p on the positive sheet of the hyperboloid (or half-cone) $p^2 = m_0^2$. The question whether $\psi(p)$ exists for p^2 different from the specific m_0^2 under consideration does not arise at all in our treatment. So it is immaterial whether the field is free or interacting. The case of free tensor fields ($D^{(m,n)}$ with $m = n$) has been considered from a different point of view by Dürr and Rudolph (1970).

The square-bracketed operator takes $\mathring{p} \to p \to \Lambda p \to \mathring{p}$ so that it is an element of the little group which leaves $\mathring{p}$ invariant. Thus $\Phi(\Lambda; p)$ is identified as the representative of this particular element of the little group.

For our purposes it is most convenient to choose the $u(\alpha)$ to be eigen-spinors of the helicity operator $\boldsymbol{J} \cdot \mathring{\boldsymbol{p}} / |\mathring{\boldsymbol{p}}| \equiv J_3$. More particularly, we shall take them to be direct products of eigen-spinors of the subhelicity operators M_3 and N_3. So the notation $u(\alpha)$ will be elaborated to

$$u(\rho\sigma) = u_M(\rho)\, u_N(\sigma) \tag{9 a}$$

$$M_3 u_M(\rho) = \rho u_M(\rho); \quad N_3 u_N(\sigma) = \sigma u_N(\sigma) \tag{9 b}$$

Since $J_3 = M_3 + N_3$, we have $J_3 u(\rho\sigma) = \lambda u(\rho\sigma)$, where $\lambda = \rho + \sigma$. We can ensure that the $u(p; \alpha)$ for all p are also helicity eigenstates by choosing $H(p, \mathring{p})$ to be

$$H(p, \mathring{p}) = R(p, p')\, B(p', \mathring{p}) \tag{10}$$

where the boost B takes $\mathring{p}$ into another vector p' in the same direction but having $|\boldsymbol{p}'| = |\boldsymbol{p}|$ (*i.e.*, $p' = (\omega', 0, 0, \kappa')$, $\kappa' = |\boldsymbol{p}|$ and R brings p' into coincidence with p.

$$B(p', \mathring{p}) = e^{i\alpha K_3}; \quad e^{\alpha} = \frac{\omega + \kappa}{\omega' + \kappa'} \tag{11}$$

$$R(p, p') = \exp(i\boldsymbol{\theta} \cdot \boldsymbol{J}); \quad \boldsymbol{\theta} = \frac{\boldsymbol{p} \times \boldsymbol{p}'}{|\boldsymbol{p} \times \boldsymbol{p}'|} \arcsin \frac{|\boldsymbol{p} \times \boldsymbol{p}'|}{|\boldsymbol{p}| \cdot |\boldsymbol{p}'|} \tag{12}$$

Both B and R leave ρ and σ unchanged. Thus the $u(p; \alpha)$ are of the form

$$u(p; \rho\sigma) = u_M(p; \rho)\, u_N(p; \sigma) \tag{13 a}$$

$$u_M(p; \rho) = H_M(p, \mathring{p})\, u_M(\rho), \quad \boldsymbol{J} \cdot \boldsymbol{p}\, u_M(p; \rho) = \rho\, |\boldsymbol{p}|\, u_M(p; \rho) \tag{13 b}$$

and similarly for $u_N(p; \sigma)$, with H_M, H_N obtained by factorization of H as in (5).

Now we specialize to the irreducible representation $D^{(m,n)}$, wherein $\boldsymbol{M}$ and $\boldsymbol{N}$ are the $(2m+1)$-and $(2n+1)$-dimensional angular momentum matrices, and complete the definition of the helicity basis by fixing the relative phases and normalizations of the $u_M(\rho)$, $u_N(\sigma)$ for various ρ, σ as follows*:

$$u_M(\rho - 1) = [(m + \rho)(m - \rho + 1)]^{-\frac{1}{2}} M_- u_M(\rho) \tag{14 a}$$

$$u_N(\sigma + 1) = [(n - \sigma)(n + \sigma + 1)]^{-\frac{1}{2}} N_+ u_N(\sigma) \tag{14 b}$$

$$u^\dagger_M u_M = u^\dagger_N u_N = 1 \qquad \text{for all } \rho, \sigma. \tag{14 c}$$

These relations are in accordance with the standard conventions in angular momentum theory. The use of the lowering operator in (14 *a*) and the raising operator in (14 *b*) will be justified by later developments. It may be noted that the above considerations are independent of whether $p^2 = 0$ or $p^2 \neq 0$, though in the latter case the conventions (14) are not the most convenient.

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* The norm of any spinor as defined here is not Lorentz-invariant. It may be seen from eq. (13) that the norm of $u(p; \rho\sigma)$ is $[(\omega + \kappa)/(\omega' + \kappa')]^{\rho - \sigma}$. The Lorentz-invariant norm is $u^+(\rho\sigma)\, u(\rho\sigma) = (\omega + \kappa)^{-\rho + \sigma}$, but if this is used instead of (14 *c*) then (14 *a*) and (14 *b*) also have to be modified, and the standard representations for M_- and N_+ can no longer be maintained.

3. Representation of the little group for $\mathring{p}$ lightlike

When $\mathring{p}$ is lightlike ($\omega = \kappa$), the associated little group is generated by

$$L_1 \equiv J_1 + K_2, \quad L_2 = J_2 - K_1, \quad \text{and} \quad J_3. \tag{15}$$

It is isomorphic to the Euclidean group $E(2)$. Adopting a different parametrization* from that of eq. (5), we write a general element of the little group as $\mathring{\Lambda} = \exp(i\chi_1 L_1 + i\chi_2 L_2)\exp(i\phi J_3)$. Or in terms of M and N,

$$\mathring{\Lambda} = \exp(i\chi_+ M_-)\exp(i\phi M_3)\exp(i\chi_- N_+)\exp(i\phi N_3) \tag{16}$$

with $\chi_\pm = \chi_1 \pm i\chi_2$. It is noteworthy that M_+ and N_- do not appear, and it is this fact which is responsible for the peculiarities of the little group representation. We find immediately that

$$D(\mathring{\Lambda})\, u(\rho\sigma) = \sum_{\rho'\sigma'} u(\rho'\sigma')\, D_{\rho'\sigma',\rho\sigma}(\mathring{\Lambda}) \tag{17}$$

or more explicitly,

$$D(\mathring{\Lambda})\, u_M(\rho)\, u_N(\sigma) = \sum_{\rho'\sigma'} u_M(\rho')\, u_N(\sigma')\, d^{(m)}_{\rho'\rho}(\chi_+, \phi)\, d^{(n)}_{\sigma'\sigma}(\chi_-, \phi) \tag{18}$$

where

$$\mathcal{D}_{\rho'\sigma',\rho\sigma}(\mathring{\Lambda}) \equiv d^{(m)}_{\rho'\rho}\, d^{(n)}_{\sigma'\sigma} = \frac{[(i\chi_+ M_-)^{\rho-\rho'}]_{\rho'\rho}\, [(i\chi_- N_+)^{\sigma'-\sigma}]_{\sigma'\sigma}\, e^{i(\rho+\sigma)\phi}}{(\rho-\rho')!\,(\sigma'-\sigma)!} \tag{19}$$

The last step follows from the observation that only one term in the series expansion of $\exp(i\chi_+ M_-)$ connects ρ to ρ', namely that containing M_- to the power $(\rho-\rho')$, and similarly for $\exp(i\chi_- N_+)$. In $D^{(m,n)}$, the matrix elements of M_- and N_+ are given by eqs (14), and then $\mathcal{D}_{\rho'\sigma',\rho\sigma}$ is explicitly obtained as

$$\mathcal{D}_{\rho'\sigma',\rho\sigma} = \frac{(i\chi_+)^{\rho-\rho'}\,(i\chi_-)^{\sigma'-\sigma}\exp[i(\rho+\sigma)\phi]}{(\rho-\rho')!\,(\sigma'-\sigma)!}\left[\frac{(m+\rho)!\,(m-\rho')!\,(n+\sigma')!\,(n-\sigma)!}{(m-\rho)!\,(m+\rho')!\,(n-\sigma')!\,(n+\sigma)!}\right]^{\frac{1}{2}} \tag{20}$$

This is just the result obtained by Mathews *et al* (1974*b*). Since (20) vanishes for all $\rho' > \rho$ and $\sigma' < \sigma$, $u(-m, n) \equiv u_M(-m)\, u_N(n)$ transforms into itself and provides a one-dimensional invariant subspace with helicity $\lambda = n - m$. However, the complementary subspace spanned by the spinors with all other combinations of ρ and σ is not invariant, and the representation given by (20) is indecomposable.

4. Invariance of Wightman functions

The nature of the representation $\mathcal{D}$ associated with any p (whether lightlike or not) has definite implications in regard to the metric structure of the space in which

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* In the paper of Mathews *et al* (1974*b*), the parametrization $\mathring{\Lambda} = \exp i(\xi_1 L_1 + \xi_2 L_2 + \phi J_3)$ was used. M Seetharaman (Private communication) has given an elegant proof that the relation between the ξ_j and our χ_j is $\chi_j = i\xi_j\,(1 - e^{i\phi})/\phi$.

the quantized operator $\psi(p)$ acts. We shall now explore these, starting from the following axioms:

(*a*) In the linear vector space of quantum states $\Phi, \Phi', \ldots$ on which ψ operates, a hermitian or pseudo-hermitian scalar product

$$(\Phi, \Phi') \equiv \langle \Phi \mid \eta \mid \Phi' \rangle = (\Phi', \Phi)^* \tag{21}$$

is defined, wherein the metric η satisfies $\eta = \eta^\dagger$, $\eta^2 = 1$. If A is any operator in this space, its adjoint with respect to the scalar product will be denoted by $A^* = \eta A^\dagger \eta$.

(*b*) The elements (a, Λ) of the PG are realized in this space by operators $U(a, \Lambda)$ which are unitary with respect to the above scalar product:

$$U^* \equiv \eta U^\dagger \eta = U^{-1} \tag{22}$$

(*c*) There exists an invariant vector Φ_0 in the space:

$$U(a, \Lambda)\, \Phi_0 = \Phi_0 \text{ for all } a, \Lambda \tag{23}$$

(*d*) The field $\psi(p)$ is locally covariant:

$$\begin{aligned}\psi(p) \to \psi'(p) &= U(a, \Lambda)\, \psi(p)\, U^{-1}(a, \Lambda) \\ &= e^{-ip \cdot a} D(\Lambda)\, \psi(\Lambda^{-1} p)\end{aligned} \tag{24}$$

These axioms have the consequence that the expectation value of $\psi_\alpha(p)\, \psi_\beta^*(p)$ with respect to the invariant state Φ_0 is invariant under the PG:

$$(\Phi_0, \psi_\alpha(p)\, \psi_\beta^*(p)\, \Phi_0) = (\Phi_0, \psi_\alpha'(p)\, \psi_\beta'^*(p)\, \Phi_0) \tag{25}$$

We now take $p = \mathring{p}$, and make an expansion* of $\psi(\mathring{p})$ in terms of the $u(\rho\sigma)$:

$$\psi_\alpha(\mathring{p}) = \Sigma\, a_{\rho\sigma}\, u_\alpha(\rho\sigma) \tag{26}$$

Substituting the expression (24) for ψ' with $\Lambda = \mathring{\Lambda}$ in (25) and then using (26) we obtain

$$\begin{aligned}&\Sigma\, (\Phi_0, a_{\rho\sigma} a^*_{\rho'\sigma'}\, \Phi_0)\, u_\alpha(\rho\sigma)\, u^*_\beta(\rho'\sigma') \\ &\quad = \Sigma\, (\Phi_0, a_{\rho_1\sigma_1} a^*_{\rho_2\sigma_2} \Phi_0)\, [D(\mathring{\Lambda})\, u(\rho_1\sigma_1)]_\alpha\, [D(\mathring{\Lambda})\, u(\rho_2\sigma_2)]_\beta^* \\ &\quad = \Sigma\, (\Phi_0, a_{\rho_1\sigma_1} a^*_{\rho_2\sigma_2} \Phi_0)\, u_\alpha(\rho\sigma)\, \mathcal{D}_{\rho\sigma,\, \rho_1\sigma_1} u^*_\beta(\rho'\sigma')\, \mathcal{D}^*_{\rho'\sigma',\, \rho_2\sigma_2}\end{aligned} \tag{27}$$

where eq. (17) has been used in the last step. Introducing the notation

$$G_{\rho\sigma;\, \rho'\sigma'} \equiv (\Phi_0, a_{\rho\sigma} a^*_{\rho'\sigma'}\, \Phi_0) \tag{28}$$

and remembering that the u's are linearly independent, we conclude from (27) that

$$G_{\rho\sigma;\, \rho'\sigma'} = \Sigma\, G_{\rho_1\sigma_1;\, \rho_2\sigma_2} \mathcal{D}_{\rho\sigma,\, \rho_1\sigma_1} \mathcal{D}^*_{\rho'\sigma',\, \rho_2\sigma_2} \tag{29}$$

or in matrix notation (with rows and columns labelled by pairs of indices)

$$G = \mathcal{D}(\mathring{\Lambda})\, G\, \mathcal{D}^\dagger(\mathring{\Lambda}) \tag{30}$$

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* Weinberg (1964) has investigated fields containing *only* the invariant-helicity spinor $u(-m, n)$. The problems considered in this paper do not arise in this case because only a *unitary* irreducible representation of the little group is involved. The vector and tensor fields of electromagnetism and gravitation do not belong to this category. The resulting freedom of gauge and its bearing on the coupling of such fields to other fields have been considered by Weinberg (1965).

The important question now is this: What form is permitted for the matrix G by equation (30), which has to be valid for all elements $\mathring{\Lambda}$ of the little group, *i.e.*, for all complex $\chi_{\pm}$ and real ϕ in (16)? The question is important because the elements of G are nothing but scalar products between states $a^*_{\rho\sigma}\Phi_0$ and $a^*_{\rho'\sigma'}\Phi_0$. The matrix G thus determines the nature of the metric in the space spanned by these states. We proceed now to answer this question in the case when ψ transforms according to $D^{(m,n)}$ so that the expression (20) for $\mathcal{D}$ is applicable.

5. Determination of the metric matrix G for $D^{(m,n)}$

Consider first those transformations for which $\chi_+ = \chi_- = 0$. Then $\mathcal{D}_{\rho\sigma,\,\rho_1\sigma_1} = \delta_{\rho\rho_1}\delta_{\sigma\sigma_1}e^{i\,(\rho+\sigma)\,\phi}$ etc., and the condition (30) becomes

$$G_{\rho\sigma;\,\rho'\sigma'} = e^{i\,(\rho+\sigma)\,\phi}\,G_{\rho\sigma;\,\rho'\sigma'}\,e^{-i\,(\rho'+\sigma')\,\phi} \tag{31}$$

This can be satisfied for all ϕ only if

$$G_{\rho\sigma;\,\rho'\sigma'} = 0 \text{ whenever } \rho+\sigma-\rho'-\sigma' \neq 0 \tag{32}$$

Let us examine next the case of transformations for which $\phi = 0$ and χ_+ is arbitrary. It is adequate to consider the infinitesimal transformations, for which

$$\mathcal{D} \approx 1 + i\chi_+ M_- + i\chi_- N_+ \tag{33}$$

Equation (30) then reduces to $(\chi_+ M_- + \chi_- N_+)\,G - G\,(\chi_- M_-^{\dagger} + \chi_+ N_+^{\dagger}) = 0$. Since this is to be valid for arbitrary (infinitesimal) complex χ_+, the coefficients of χ_+ and $\chi_- \equiv \chi_+^*$ must separately vanish:

$$M_- G - G N_+^{\dagger} = 0,\quad N_+ G - G M_-^{\dagger} = 0 \tag{34}$$

Taking the $(\rho\sigma,\,\rho'\sigma')$ matrix element of this equation and noting that the matrix elements of M_- and N_+ are real, we obtain

$$(M_-)_{\rho,\,\rho+1} G_{\rho+1,\,\sigma;\,\rho'\sigma'} - G_{\rho\sigma;\,\rho',\sigma'-1}\,(N_+)_{\sigma',\,\sigma'-1} = 0 \tag{35 a}$$

$$(N_+)_{\sigma,\,\sigma-1} G_{\rho,\sigma-1;\,\rho'\sigma'} - G_{\rho\sigma;\,\rho'+1,\,\sigma'}\,(M_-)_{\rho',\,\rho'+1} = 0 \tag{35 b}$$

The first of these equations connects a chain of elements $G_{\rho\sigma;\,\rho'\sigma'}$ which have the values of σ, ρ' and $\rho-\sigma'$ in common, while the second equation involves a chain with ρ, σ' and $\sigma-\rho'$ constant. Considering eq. (35 *a*) first, we observe that if ρ is given its highest possible value, $\rho = m$, the first term does not exist, and so $G_{m\sigma;\,\rho'\sigma'-1} = 0$. Since the maximum value $(\sigma'-1)$ can take is $(n-1)$, this result does not say anything about $G_{m\sigma;\,\rho'n}$. The conclusion therefore is that for all σ and ρ',

$$\begin{aligned} &G_{m\sigma;\,\rho'n} \text{ is undetermined} \\ &G_{m\sigma;\,\rho'\sigma'} = 0 \text{ when } \sigma' < n \end{aligned} \tag{36}$$

If we now set $\rho = m-1$, we obtain $(M_-)_{m-1,\,m}\,G_{m\sigma;\,\rho'\sigma'} = G_{m-1,\,\sigma;\,\rho',\,\sigma'-1}\,(N_+)_{\sigma',\,\sigma'-1}$. This equation has nothing to say about $G_{m-1,\,\sigma;\,\rho'n}$; but it determines $G_{m-1,\,\sigma;\,\rho'\sigma'}$ for all $\sigma' \leqslant n-1$ in terms of the already evaluated quantities (36):

$$\left.\begin{aligned} &G_{m-1,\,\sigma;\,\rho'n} \text{ is undetermined,} \\ &G_{m-1,\,\sigma;\,\rho',\,n-1} = \frac{(M_-)_{m-1,\,m}}{(N_+)_{n,\,n-1}}\,G_{m\sigma;\,\rho'n} \\ &G_{m-1,\,\sigma;\,\rho'\sigma'} = 0 \text{ for } \sigma' < n-1 \end{aligned}\right\}. \tag{37}$$

It is easy to convince oneself that on continuing this process, reducing the values of ρ by one unit at a time, one finds that

$$G_{\rho\sigma;\,\rho' n} \text{ is undetermined;}$$

$$G_{\rho\sigma;\,\rho'\sigma'} = \frac{(M_-)_{\rho,\,\rho+1}(M_-)_{\rho+1,\,\rho+2}\cdots(M_-)_{\rho-\sigma'+n-1,\,\rho-\sigma'+n}}{(N_+)_{\sigma'+1,\,\sigma'}(N_+)_{\sigma'+2,\,\sigma'+1}\cdots(N_+)_{n,\,n-1}} \times G_{\rho-\sigma'+n,\,\sigma;\,\rho',\,n},\ (m-\rho \leqslant n-\sigma');$$

$$G_{\rho\sigma;\,\rho'\sigma'} = 0,\ (m-\rho > n-\sigma'). \tag{38}$$

A similar procedure using eq. (35 b), starting with $\sigma = -n$ would lead to the conclusion that

$$G_{\rho\sigma;-m\sigma'} \text{ is undetermined;}$$

$$G_{\rho\sigma;\,\rho'\sigma'} = \frac{(N_+)_{\sigma,\,\sigma-1}(N_+)_{\sigma-1,\,\sigma-2}\cdots(N_+)_{\sigma-\rho'-m+1,\,\sigma-\rho'-m}}{(M_-)_{\rho'-1,\,\rho'}(M_-)_{\rho'-2,\,\rho'-1}\cdots(M_-)_{-m,-m+1}} \times G_{\rho,\,\sigma-\rho'-m;\,-m,\sigma'},\ (m+\rho' \leqslant n+\sigma);$$

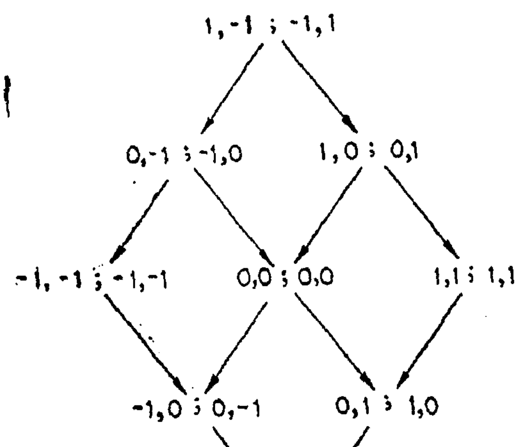
$$G_{\rho\sigma;\,\rho'\sigma'} = 0,\ (m+\rho' > n+\sigma). \tag{39}$$

On combining eqs (38) and (39) we see that the quantities which still remain undetermined are the $G_{\rho\sigma;\,-m,\,n}$. However, of these, all except the ones with $\rho+\sigma = n-m$ vanish on account of eq. (32). Thus the "free" parameters are

$$G_{-m+l,\,n-l;\,-m,\,n}\ (l = 0, 1, 2, \ldots) \tag{40}$$

Their number is either $(2m+1)$ or $(2n+1)$, whichever is smaller. The only other elements of G which can be non-zero are those which are linked to the elements (40) by eqs (38) and (39). Using the known forms of the matrix elements of M^- and N_+ one can easily show that

$$G_{-m+l+r,\,n-l+s;\,-m+s,\,n-r} = \left[\frac{l!\,(2m-l+r)!\,(2n-r)!}{(l-r)!\,(2m-l)!\,r!\,(2n)!} \times \frac{l!\,(2n-l+s)!\,(2m-s)!}{(l-s)!\,(2n-l)!\,s!\,(2m)!}\right]^{\frac{1}{2}} G_{-m+l,\,n-l;\,-m,\,n},\ (0 \leqslant r,\ s \leqslant l) \tag{41}$$



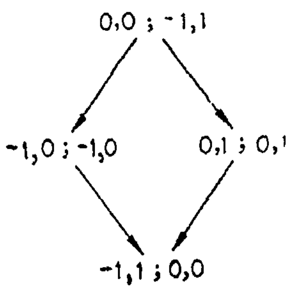


Fig. 1(b)

Table 1

	$G_{\rho\sigma;\,\rho'\sigma'}$								
$\rho\sigma$ $\rho'\sigma'$	1, 1	−1, −1	0, 0	−1, 1	1, −1	1, 0	0, 1	−1, 0	0, −1
1, 1	f	·	·	·	·	·	·	·	·
−1, −1	·	f	·	·	·	·	·	·	·
0, 0	·	·	f	g	·	·	·	·	·
−1, 1	·	·	g	h	f	·	·	·	·
1, −1	·	·	·	f	·	·	·	·	·
1, 0	·	·	·	·	·	·	f	·	·
0, 1	·	·	·	·	·	f	g	·	·
−1, 0	·	·	·	·	·	·	·	g	f
0, −1	·	·	·	·	·	·	·	f	·

The special case of the above results for a vector field ($m = n = 1/2$) was obtained in Mathews *et al* (1974 *b*). Here we shall consider the implications for fields transforming according to $D^{(1,\,1)}$ (a symmetric traceless tensor field) and $D^{(1,1/2)}$. The latter is presented as a nontrivial example of a half-integer spin field, while the former is of interest in connection with the theory of gravitation.

6. Special cases

6 *a*. *The symmetric traceless tensor field.* In this case ($m = n = 1$), the parameters of eq. (40) are $G_{-1,\,1;\,-1,\,1}$, $G_{0,\,0;\,-1,\,1}$, $G_{1,\,-1;\,-1,\,1}$. They are linked to other elements in the manner shown in figure 1. The linked elements in each of the diagrams here turn out to be equal, as may be seen from eq. (41). So the matrix made up of the $G_{\rho\sigma;\,\rho'\sigma'}$ has the explicit form shown in table 1, wherein f, g, h are arbitrary real parameters. (The dots stand for zeroes).

The diagonal elements give the norms of the various helicity states. The states (1, 1) and (− 1, − 1), of helicity ± 2, are the ones which one seeks to describe by the symmetric tensor field; to these one assigns a positive norm by choosing f positive. (It may be set equal to 1 without loss of generality).

But in doing so, one is forced to assign the same norm to the unwanted zero helicity state with $\rho = \sigma = 0$. The norms of all the other states (also unwanted) can be chosen to be zero by setting $g = h = 0$. What one has then are three states of norm f and three other pairs of " ghost " states of zero norm such that the scalar product of the two states of each pair is also f. (Alternatively, by taking symmetric and antisymmetric combinations of the paired states one can separate the pairs into uncoupled states of norm $+f$ and $-f$. It follows therefore that there are three negative-metric states in the theory). One member of each pair (namely the states* with $\rho, \sigma = -1, 1;\ 0, 1;\ -1, 0$) and the states

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* These are " good ghosts " in the terminology of Dürr and Rudolph (1970). The other members of these pairs are the " bad ghosts ". The state $\rho = \sigma = 0$ (which has norm f) is called " bad normal ". In comparing the present work with that of the above authors, the positions of the quantum numbers ρ and σ should be interchanged.

with helicity ± 2 ($\rho, \sigma = 1, 1; -1, -1$) satisfy the generalized Lorentz condition $p^\mu \psi_{\mu\nu} = p^\nu \psi_{\mu\nu} = 0$, as shown in the Appendix. These five states transform among themselves under Lorentz transformations, since the effect of the latter on any ($\rho\sigma$) is to admix with it other states having $\rho' > \rho$ and $\sigma' < \sigma$ only. Nevertheless it is not possible to define a nontrivial space of quantum states in which only these helicity states appear. Any attempt to do this is equivalent to equating to zero all those elements in table 1 for which either ($\rho\sigma$) or ($\rho'\sigma'$) is outside this set of five helicity states. But some of these elements are equal to f, and on setting this equal to zero we are reduced to the uninteresting situation where even the states of helicity ± 2 have zero norm. That this does indeed happen, can be verified directly by restricting G, M_- and N_+ eq. (30) to the relevant 5×5 submatrix and solving for G. One finds then that $f = g = 0$, leaving h arbitrary, (The state $\rho, \sigma = -1, 1$ to which h pertains is the invariant helicity state, for which $\psi_{\mu\nu} \propto p_\mu p_\nu$).

It is clear then that in any quantization scheme for $\psi_{\mu\nu}$, one has to keep all the nine different helicity states, and achieve the reduction to a " physical " subspace by a Gupta-Bleuler-type subsidiary condition $(p^\mu \psi_{\mu\nu})^+ \mid \Phi \rangle_{\text{phys}} = (p^\nu \psi_{\mu\nu})^+ \mid \Phi \rangle_{\text{phys}} = 0$. It may be recalled that of the four " unphysical " helicity states ($\rho, \sigma = 0, 0$; $1, 0$; $0, -1$; $1, -1$), the last three are ghost states of zero norm; but the first is not incorporeal, having a norm exactly equal to that of the physically interesting states of helicity ± 2.

The generalization of the above results to any " spin s " tensor field (tranforming according to $D^{(m,m)}$, $m = \frac{1}{2}s$) is straightforward. The norms of the states of leading helicity ($\pm s$) in this case belong to the linked set of elements of G characterized by $l = 2m$ in (41), and all these elements turn out to be equal. One finds then that the leading helicity states are accompanied by ($s - 1$) other states ($\rho = \sigma = m - 1, m - 2, \ldots, -m + 1$) of the same norm f. The wave functions associated with these latter do not obey the generalized Lorentz condition $p^\mu u_{\mu\nu} \ldots = p^\nu u_{\mu\nu} \ldots = \ldots = 0$. The states which satisfy this condition are characterized by either $\rho = -m$ or $\sigma = +m$ (or both). They are the physically important leading helicity ($\pm s$) states together with their ($2s - 1$) "gauge partners". Each of the latter can be chosen to be of zero norm, but has scalar product f with its " mirror state ". (The mirror state $\bar{\rho}\bar{\sigma}$ of $\rho\sigma$ has $\bar{\rho} = \sigma$, $\bar{\sigma} = \rho$). Besides these there are also other mirror pairs (of zero norm and unit scalar product) of which neither member is a gauge state, if $s > 2$.

6 *b*. *Fields transforming according to* $D^{(1, \frac{1}{2})}$. In this case elements $G_{\rho\sigma;\ \rho'\sigma'}$ whose values are arbitrary are, according to eq. (40), $G_{-1, \frac{1}{2};\ -1, \frac{1}{2}}$ and $G_{0, -\frac{1}{2};\ -1, \frac{1}{2}}$. The former is not linked to anything else, while the chain of links arising from the latter is as in figure 2. The difference from the case of tensor fields is striking. There is no longer any symmetry between states whose helicities differ just by the

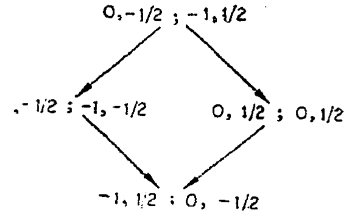

Figure 2

sign. In particular it may be noted that while the state with helicity $-3/2$, ($\rho = -1$, $\sigma = -\frac{1}{2}$), can be normalized to unity, that with helicity $+3/2$, ($\rho = 1$, $\sigma = \frac{1}{2}$) is forced to have zero norm.

It is evident that similar results obtain in any $D^{(m,n)}$ with $m \neq n$. Symmetry between positive and negative helicities may be restored by taking a reducible representation $D^{(m,n)} + D^{(n,m)}$ (as in the case of the Dirac field). The generalization of our treatment to cover the reducible case is straightforward and will not be pursued here.

7. Non-zero mass

The nature of the results obtained in the last two sections was a consequence of the indecomposability of the little group representation appearing in the problem. It is of interest to carry out a parallel development for $p^2 \equiv m_0^2 \neq 0$. In order to facilitate a direct comparison with the case $p^2 = 0$, we shall choose the standard vector $\overset{\circ}{p}$ as $(\omega, 0, 0, \kappa)$ with $\omega, \kappa > 0$ and $\omega^2 - \kappa^2 = m_0^2$, instead of the usual choice $(m_0, 0, 0, 0)$. It is easy to verify that our standard vector is left invariant by the following combinations of Lorentz generators:

$$J_1' \equiv \frac{\omega}{m_0} J_1 + \frac{\kappa}{m_0} K_2, \quad J_2' = \frac{\omega}{m_0} J_2 - \frac{\kappa}{m_0} K_1, \quad J_3' = J_3 \tag{42}$$

They obey an algebra isomorphic to that of the rotation group*. However, rather than deal with this group directly, we shall express the J_i' in terms of $\boldsymbol{M}$ and $\boldsymbol{N}$ in accordance with the practice followed in the previous sections.

$$\begin{aligned} J_1' &= (2m_0)^{-1} [(\omega - \kappa) M_+ + (\omega + \kappa) M_- + (\omega + \kappa) N_+ + (\omega - \kappa) N_-] \\ J_2' &= (2im_0)^{-1} [(\omega - \kappa) M_+ - (\omega + \kappa) M_- + (\omega + \kappa) N_+ - (\omega - \kappa) N_-] \\ J_3' &= M_3 + N_3 \end{aligned} \tag{43}$$

The two sets

$$\begin{aligned} M_+' &= \frac{\omega - \kappa}{m_0} M_+, \qquad M'_- = \frac{\omega + \kappa}{m_0} M_-, \qquad M_3' = M_3; \\ N_+' &= \frac{\omega + \kappa}{m_0} N_+, \qquad N'_- = \frac{\omega - \kappa}{m_0} N_-, \qquad N_3' = N_3 \end{aligned} \tag{44}$$

independently have the same commutation relations as the third component of angular momentum and the associated raising and lowering operators. Further $\boldsymbol{M}'^2 = \frac{1}{2}(M_+' M_-' + M_-' M_+') + M_3'^2 = \boldsymbol{M}^2$, and similarly $\boldsymbol{N}'^2 = \boldsymbol{N}^2$, so that in $D^{(m,n)}$, these are $m(m+1)$ and $n(n+1)$ times the respective unit matrices. However M_+' is not the hermitian conjugate of M_-'. Consequently the relation between the norm of any eigenspinor $u_M'(\rho)$ of $\boldsymbol{M}'^2$ and M_3', and the norm of $M_-' u_M'(\rho)$ is

* If $m_0^2 < 0$, J_1' and J_2' are not real linear combinations of J and K. In this case, the generators of the little group are to be taken as iJ_1', iJ_2', J_3'. Their algebra is isomorphic to that of $O(2, 1)$. We shall not pursue this case further here.

$$u'_M{}^\dagger(\rho)\, M_-'^\dagger\, M'_-\, u_M'(\rho) = \left(\frac{\omega+\kappa}{\omega-\kappa}\right) u'_M{}^\dagger(\rho)\, M_+'\, M_-'\, u_M'(\rho)$$

$$= \left(\frac{\omega+\kappa}{\omega-\kappa}\right) u'_M{}^\dagger(\rho)\, [M'^2 - M_3'^2 + M_3']\, u_M'(\rho)$$

$$= \left(\frac{\omega+\kappa}{\omega-\kappa}\right) [m(m+1) - \rho(\rho-1)]\, u_M'^\dagger(\rho)\, u'_M(\rho) \tag{45}$$

Then if the relation

$$u_M'(\rho-1) = [m(m+1) - \rho(\rho-1)]^{-\frac{1}{2}}\, M_-'\, u_M'(\rho) \tag{46 a}$$

of the usual form is to hold, the norm of $u_M'(\rho)$ will have to be made dependent on ρ. Eqs (45) and (46) are mutually consistent if we adopt the normalization

$$u_M'^\dagger(\rho)\, u_M'(\rho') = a \left(\frac{\omega-\kappa}{\omega+\kappa}\right)^\rho \delta_{\rho\rho'} \tag{46 b}$$

where a is a constant independent of ρ. It is easy to verify then that we also have

$$u_M'(\rho+1) = [m(m+1) - \rho(\rho+1)]^{-\frac{1}{2}}\, M_+'\, u_M'(\rho) \tag{46 c}$$

as usual. In a similar way, with respect to a basis $u_N'(\sigma)$ defined by

$$u_N'(\sigma \pm 1) = [n(n+1) - \sigma(\sigma \pm 1)]^{-\frac{1}{2}}\, N_\pm'\, u_N'(\sigma) \tag{47 a}$$

$$u_N'^\dagger(\sigma)\, u_N'(\sigma') = b \left(\frac{\omega+\kappa}{\omega-\kappa}\right)^\sigma \delta_{\sigma\sigma'} \tag{47 b}$$

we obtain a representation of N' in the form of the standard angular momentum matrices of dimension $(2n+1)$.

Returning now to eqs (43), we observe that in terms of $M_1' \equiv \frac{1}{2}(M_+ + M_-)$, $M_2' \equiv -\frac{1}{2}i(M_+ - M_-)$ and M_3', the generators of the little group are just $J_i' = M_i' + N_i'$. For finite transformations we have $\exp(i\boldsymbol{\xi}\cdot\boldsymbol{J}') = \exp(i\boldsymbol{\xi}\cdot\boldsymbol{M}')\exp(i\xi\cdot\boldsymbol{N}')$. The representation $\mathcal{D}$ of the little group defined over the basis $u_M'(\rho)\, u_N'(\sigma)$ through eqs (46) and (47) is thus a direct product of spin m and spin n representations of the rotation group. This can be decomposed in the standard manner into a direct sum of spin s irreducible representations, $|m-n| \leqslant s \leqslant m+n$. Taking $\mathcal{D}$ in this decomposed form, we can carry out an analysis similar to that of section 4 for two point functions of fields $\psi(p)$ associated with $p = \mathring{p}$, $\mathring{p}^2 = m_0^2 \neq 0$. We are once again led to eq. (30). However, the implications of this equation are now quite different. Here $\mathcal{D}$ is *unitary* and so we can rewrite eq. (30) as $G\mathcal{D} = \mathcal{D}G$ and conclude therefrom (by Schur's lemma) that G is a diagonal matrix which is a direct sum of multiples of unit matrices. To each irreducible part of $\mathcal{D}$ associated with a definite spin s, there corresponds a segment of G which is an unspecified multiple of a unit matrix of the same dimension. In the special case of $D^{(1,1)}$ for example, there are three independent constants associated with $s = 0, 1, 2$, as is to be expected. This is to be contrasted with the situation when $p^2 = 0$ (section 6) where also there are three independent constants, but not associated with disjoint sets of states. In particular, the constant f appears in association with every one of the nine states, either as the norm or through a scalar product.

8. The limit $m_0 \to 0$

Since the results of the section apply for any $m_0 \neq 0$, however small it may be, it is of interest to examine the limit $m_0 \to 0$ in order to understand how the qualitative difference between the little-group representations for $m_0 \equiv 0$ and $m_0 \to 0$ can be sustained. The choice of the standard vector $\mathring{p}$ as $(\omega, 0, 0, \kappa)$ for both $\mathring{p}^2 = 0$ and $\mathring{p}^2 \neq 0$ enables this limit to be taken simply by making $\omega \to \kappa$. If, in doing this, we retain the standard representations for M' and N', then the isomorphism of the little group to the rotation group would appear to be maintained even in the limit. However, this is not the full story. If one looks at the basis spinors $u_M'(\rho)$ and $u_N'(\sigma)$ over which the representation is defined, one finds that as $\omega \to \kappa$, the norms go to zero or infinity depending on the signs of ρ and σ. The situation can be made less singular by choosing the hitherto irrelevant common factors a and b in eq. (46 *b*) and (47 *b*) suitably. The choice $a = (m_0^2/4\kappa^2)^m$, $b = (m_0^2/4\kappa^2)^{-n}$ leaves $u_M'(-m)$ and $u_N'(n)$ with unit norm while making the norms of all other spinors (and hence the spinors themselves) zero. The rotation group involved here is therefore purely nominal: the spinors which it admixes with $u_M'(-m)\, u_N'(n)$ are all zero. In other words $u_M'(-m)\, u_N'(n)$ transforms into itself, providing effectively a one-dimensional representation, the representation "matrix" being $d^{(m)}_{-m,-m}\, d^{(n)}_{-n,-n}$. It may be recalled that in the theory with $\mathring{p}^2 \equiv 0$ exactly the same spinor occurs as the invariant helicity spinor in $D^{(m,n)}$. What we have obtained by the limiting procedure is just the unitary irreducible representation associated in the massless case with this single spinor.

These still remains the question as to whether the reducible but indecomposable representation realized over the set of all helicity states (section 3) cannot be recovered by taking the limit $m_0 \to 0$ suitably. The answer is that it can be. To accomplish this we have to do two things: Redefine the norms of the $u_M'(\rho)$ and $u_N'(\sigma)$ is such a way that the norms tend to a finite limit for *all* ρ and σ as $m_0 \to 0$; and change the scale of the parameters ξ_1 and ξ_2 of the little group by a factor $(2\kappa/m_0)$. The second step is equivalent to taking the generators as $(m_0/2\kappa)\, J_1'$, $(m_0/2\kappa)\, J_2'$ and J_3'. Correspondingly, in the M and N subspaces one has, instead of (44)

$$M_+'' = \frac{\omega - \kappa}{2\kappa} M_+, \qquad M_-'' = \frac{\omega + \kappa}{2\kappa} M_-, \qquad M_3'' = M_3$$

$$N_+'' = \frac{\omega + \kappa}{2\kappa} N_+, \qquad N_-'' = \frac{\omega - \kappa}{2\kappa} N_-, \qquad N_3'' = N_3 \tag{48}$$

It is evident that in the limit $m_0 \to 0$ $(\omega \to \kappa)$,

$$M_+'' \to 0, \quad N_-'' \to 0, \quad M_-'' \to M_-, \quad N_+'' \to N_+ \tag{49}$$

This is just the kind of situation we had in section 3 where, for $m_0 \equiv 0$, M_+ and N_- did not appear in the theory. It may be verified now that by defining the relation between spinors for different ρ through

$$M_-''\, u_M''(\rho) = \frac{m_0}{2\kappa} \left(\frac{\omega + \kappa}{\omega - \kappa}\right)^{\frac{1}{2}} [m(m+1) - \rho(\rho - 1)]^{\frac{1}{2}}\, u_M''(\rho - 1) \tag{50}$$

one ensures that the normalization of $u_M''(\rho)$ is independent of ρ. In the limit $m_0 \to 0$, this equation goes over into $M_- u_M(\rho) = [m(m+1) - \rho(\rho-1)]^{\frac{1}{2}} u_M(\rho-1)$ which means that we have the standard representation for M_- in this limit, exactly as in section 3. Similar statements can be made about N_+, starting from

$$N_+'' u_N''(\sigma) = \frac{m_0}{2\kappa}\left(\frac{\omega+\kappa}{\omega-\kappa}\right)^{\frac{1}{2}} [n(n+1) - \sigma(\sigma+1)]^{\frac{1}{2}} u_N''(\sigma+1) \qquad (51)$$

With this, the transition from the decomposable unitary representation of the massive case to the indecomposable non-unitary representation of the massless case is complete.

9. Discussion

Wightman and Gårding (1964) appear to have been the first to give a formal proof for the fact that local covariance of a massless quantum field implies an indefinite metric in the space of quantum states. They employed the analyticity properties of two-point Wightman functions to prove this result for the massless vector field. The present work makes use of weaker assumptions and quite elementary methods in arriving at a detailed picture of the metric structure of the state vector space of an arbitrary covariant quantum field. In the special case of tensor fields, the essential features of this structure have been already observed by Dürr and Rudolph (1970), though their procedure based on the use of specific commutation rules seems more restrictive than ours.

As regards the passage from the massive to the massless case, it is very interesting that a straight limit $m_0 \to 0$ destroys all but the invariant spinor characterized by $\rho = -m, \sigma = n$ (helicity $\lambda = n - m$). In the case of tensor representations $(n = m)$ this means that the unique helicity which survives is $\lambda = 0$, and this is of little interest. The rescaling of the parameters of the little group, which is necessary if all the spinors are to be safeguarded, is similar in spirit to the rescaling employed in the Inönü-Wigner (1953) contraction procedure (*see* also Talman 1968), which leads from the irreducible representations of the rotation group to the unitary (infinite-dimensional) irreducible representations of $E(2)$. In the present work, the correspondence is between a group element with parameters ξ_1, ξ_2, ξ_3 of $E(2)$ (pertaining to the massless case) and the element with parameters $(m_0/2\kappa)\,\xi_1$, $(m_0/2\kappa)\,\xi_2$, ξ_3 of the rotation group (massive case). In the limit $m_0 \to 0$ the latter set goes to 0, 0, ξ_3 for all (finite) values of ξ_1, ξ_2, making the correspondence between the massive and massless cases a singular one. The precise nature of this correspondence does not appear to have been understood before, though comparison of the $m_0 \to 0$ limit with the situation when $m_0 \equiv 0$ has been studied in the context of specific problems, as for example by van Dam and Veltman (1970), Deser (1973), Seetharaman *et al* (1972).

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Appendix

The effect of M_3 and N_3 on a contravariant four-vector whose components are labelled by the usual four-vector index is represented in matrix form by

$$\tfrac{1}{2}(J_3 + \mathrm{i}K_3) \rightarrow \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \mp 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \mp 1 & 0 & 0 & 0 \end{pmatrix} \tag{A·1}$$

In the case of a second rank tensor $u^{\mu\nu}$, adding up the effects on the two indices, one finds the elements $\frac{1}{2}[(J_3 \pm \mathrm{i}K_3)\,u]^{\mu\nu}$ to be given by

$$\begin{pmatrix} \mp u^{30} \mp u^{03} & \mp u^{31} - \mathrm{i}u^{02} & \mp u^{32} + \mathrm{i}u^{01} & \mp u^{33} \mp u^{00} \\ -\mathrm{i}u^{20} \pm u^{13} & -\mathrm{i}u^{21} - \mathrm{i}u^{12} & -\mathrm{i}u^{22} + \mathrm{i}u^{11} & -\mathrm{i}u^{23} \mp u^{10} \\ \mathrm{i}u^{10} \pm u^{23} & \mathrm{i}u^{11} - \mathrm{i}u^{22} & \mathrm{i}u^{12} + \mathrm{i}u^{21} & \mathrm{i}u^{13} \mp u^{20} \\ \mp u^{00} \mp u^{33} & \mp u^{01} - \mathrm{i}u^{32} & \mp u^{02} + \mathrm{i}u^{31} & \mp u^{03} \mp u^{30} \end{pmatrix} \tag{A·2}$$

If $u^{\mu\nu}$ is characterized by M-helicity ρ and N-helicity σ, with $p = \overset{\circ}{p}$, the above expression taken with the upper signs should be equal to $\rho u^{\mu\nu}$ and with the lower signs, to $\sigma u^{\mu\nu}$. Hence the nonvanishing elements of $u^{\mu\nu}$ in the various cases are as follows. (The values of ρ, σ are indicated at the left of each line).

$$\begin{aligned}
&(1, 1): \quad u^{11} = -u^{22} = -\mathrm{i}u^{12} = -\mathrm{i}u^{21} \\
&(-1, -1): \quad u^{11} = -u^{22} = \mathrm{i}u^{12} = \mathrm{i}u^{21} \\
&(0, 0): \quad u^{00} = -u^{33} \overset{*}{=} u^{11} = u^{22}; \\
&\qquad\qquad u^{30} = -u^{03},\ u^{12} = -u^{21} \text{ (These vanish in } D^{(1,1)}\text{)}. \\
&(-1, 1): \quad u^{00} = u^{33} = u^{03} = u^{30} \\
&(1, -1): \quad u^{00} = u^{33} = -u^{03} = -u^{30} \\
&(1, 0): \quad u^{01} = -u^{31} = -\mathrm{i}u^{02} = \mathrm{i}u^{32} \overset{*}{=} u^{10} = -u^{13} = -\mathrm{i}u^{20} = \mathrm{i}u^{23} \\
&(0, 1): \quad u^{01} = u^{31} = -\mathrm{i}u^{02} = -\mathrm{i}u^{32} \overset{*}{=} u^{10} = u^{13} = -\mathrm{i}u^{20} = -\mathrm{i}u^{23} \\
&(-1, 0): \quad u^{01} = u^{31} = \mathrm{i}u^{02} = \mathrm{i}u^{32} \overset{*}{=} u^{10} = u^{13} = \mathrm{i}u^{20} = \mathrm{i}u^{23} \\
&(0, -1): \quad u^{01} = -u^{31} = \mathrm{i}u^{02} = -\mathrm{i}u^{32} \overset{*}{=} u^{10} = -u^{13} = \mathrm{i}u^{20} = -\mathrm{i}u^{23}
\end{aligned} \tag{A·3}$$

The starred equalities in the above are required for symmetry and tracelessness.

From the above, it is easy to verify that the condition $p_\mu u^{\mu\nu} = p_\nu u^{\mu\nu} = 0$ is satisfied for $(\rho, \sigma) = (1, 1)\,(-1, -1),\ (-1, 1),\ (-1, 0),\ (0, 1)$.

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