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On the problem of constraints in minimally coupled relativistic wave equations for particles of unique mass

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Abstract. We study the problem of a possible change in the number of constraints in linear relativistic wave equations $(-i\beta_\mu \partial^\mu + m)\psi = 0$ for particles of unique mass, on introduction of minimal coupling to an external electromagnetic field. Complementing our earlier work in which we obtained conditions for non-loss of constraints in equations characterised by the minimal β-algebra $\beta_0^5 = \beta_0^3$ we derive here the conditions for such theories not to generate more constraints than in the free case. The results are illustrated by considering specific equations and a fallacy in certain conclusions of Kobayashi and Shamaly on this problem is pointed out.

Keywords. Relativistic wave equations; minimal electromagnetic coupling; constraints.

1. Introduction

In recent years several authors have discussed a variety of pathologies afflicting relativistic wave equations for high spin particles coupled to external fields. The types of difficulties that have come to light in the course of investigation of *specific* theories include (i) non-causal propagation in the presence of external fields (Velo and Zwanziger 1969a, b), (ii) occurrence of modes of complex frequencies when high magnetic fields are present (Tsai and Yildiz 1971; Mathews 1974; Seetharaman *et al* 1975), (iii) appearance of extra constraints at particular values of the external fields (Jenkins 1972, 1973), (iv) loss of constraints (Federbush 1961) and (v) unacceptable changes in the (anti-) commutation rules for field components on introduction of interactions (Johnson and Sudarshan 1961; Hagen 1971) in addition to (vi) the problem of possible indefiniteness of charge/energy in the free theory itself (Prabhakaran *et al* 1975; Khalil and Seetharaman 1978; Nagpal 1973; Krajcik and Nieto 1975a, b). It is also generally known that in every unique mass-spin equation of the Dirac-Bhabha form

$$(-i\beta^\mu \delta_\mu + m)\psi = 0,$$

(excepting the Dirac equation for spin $-\frac{1}{2}$), the wave function ψ must necessarily have more components than are required for the particle of given spin and it is the elimination of the unwanted components (through in-built constraints) that is at the heart of most of the pathologies. While one or more of the diseases have been demonstrated to be present in *specific* wave equations, very few general studies have been carried out to enunciate general theorems (applicable to a wide class of equations) for testing the occurrence or otherwise of one or more of the pathologies (Amar and Dozzio 1972, 1975; Khalil 1977). In a recent paper (Mathews *et al* 1979) we have carried out one such general study regarding the change in the number of constraints on introduction of minimal electro-magnetic coupling in unique mass theories.* We have established that in equations obeying Harish Chandra's unique-mass condition

$$\beta_0^n = \beta_0^{n-2},$$

with $n=4$ the number of constraints in the presence of external e.m. interaction would never be less than in the free case, but could be greater at a particular strength of the external field. In the next higher case, namely $\beta_0^5=\beta_0^3$, it turned out however that in general there is a possibility of loss of constraints and we obtained general criteria which would enable us to see from the free equation itself whether it is prone to loss of constraints pathology. It is one of our objectives in this paper to extend the above study and obtain conditions which ensure that extra constraints would not occur either. As a direct application of our results we demonstrate that the theory of Schwinger (1963) and Chang (1967) for spin-2 particles will not lose constraints (contrary to the recent claim of Kobayashi and Shamaly 1978) but will infact have extra constraints in certain critical fields.

The plan of the paper is as follows: In § 2 we consider the Schwinger-Chang theory (which has $\beta_0^5 = \beta_0^3$ as the minimal algebra) and demonstrate the gain in the number of constraints at a critical value of the external magnetic field. In § 3 we present a general analysis of equations with this algebra, leading to general criteria for such equations not to gain constraints at any external field strength. The familiar equations coming under this algebra are analysed afresh in § 4 in the light of our condition.

2. The Schwinger-Chang theory with minimal e.m. coupling

The Schwinger-Chang theory for spin-2 particles has a 30-component wave function, made up of symmetric tensor $h_{\mu\nu}$ and a third rank tensor $H_{\mu\nu\lambda}$ with the properties:

$$H_{\mu\nu\lambda} = - H_{\nu\mu\lambda},$$

$$H_{\mu\nu\lambda} + H_{\nu\lambda\mu} + H_{\lambda\mu\nu} = 0. \tag{1}$$

With minimal coupling to an external e.m. field the equations of motion are**

$$\pi_\lambda \left(H^{\lambda\nu\mu} + H^{\mu\nu\lambda}\right) - 2im^2 \left(h^{\mu\nu} - g^{\mu\nu} h\right) = 0, \tag{2a}$$

$$iH_{\nu\lambda\mu} - \tfrac{1}{2}i \left(g_{\mu\nu} H_\lambda - g_{\mu\lambda} H_\nu\right) - \left(\pi_\nu h_{\mu\lambda} - \pi_\lambda h_{\mu\nu}\right) = 0, \tag{2b}$$

*It is of interest to note that recently Takahashi and Kobayashi (1978) have drawn attention to the similarity that exists between Gribov ambiguity in non abelian gauge theories and the change in number of constraints in the presence of electromagnetic interaction.

**When equation (2) are written in the form $(\beta \cdot \pi + m)\,\psi=0$ the β_0 obeys the minimal equation $\beta_0^5-\beta_0^3=0$.

where $\quad H_\mu = H^\nu{}_{\mu\nu}$ and $h = h^\mu_\mu$.

The metric is $(-1, 1, 1, 1)$ and $\pi_\mu \equiv p_\mu - e\,A_\mu$, is the usual replacement for minimal e.m. coupling. The equations of motion following from (2) when written out explicitly have the form:

$$\pi_o\,(H^{olk} + H^{okl}) = -\,\pi_m\,(H^{mlk} + H^{mkl}) + 2\,im^2\,(h^{kl} - \delta^{kl}\,h), \tag{3a}$$

$$\pi_o\,H^{oko} = -\,\pi_m\,(H^{mok} + H^{mko}) + 2\,im^2\,h^{ok}, \tag{3b}$$

$$\pi_o\,h_{kl} = \pi_l\,h_{ko} + i\,H_{olk} + \frac{i}{2}\,\delta_{kl}\,H_o, \tag{3c}$$

$$\pi_o\,h_{ol} = \pi_l\,h_{oo} + i\,H_{olo} + \frac{i}{2}\,H_l. \tag{3d}$$

Since the spin-2 particle needs only 10 degrees of freedom the equation must lead to 20 constraints. On examining equation (3) we see that twelve primary constraints (Johnson and Sudarshan 1961) are already embodied therein. They are

$$\pi_m\,H^{moo} - i\,m^2\,h_{mm} = 0, \tag{4a}$$

$$i\,H_{klo} - \pi_k\,h_{ol} + \pi_l\,h_{ok} = 0, \tag{4b}$$

$$i\,H_{lmk} - \pi_l\,h_{mk} + \pi_m\,h_{kl} - \frac{i}{2}\,(\delta_{kl}\,H_m - \delta_{km}\,H_l) = 0. \tag{4c}$$

Further constraints (secondary, tertiary etc.) are to be obtained by differentiating (4) with time (i.e. operating on them with π_0) and eliminating the time derivatives through the equations of motion. This process applied to (4a) yields one (secondary) constraint namely

$$\pi_m\,\pi_k\,H^{kom} + \frac{ie}{2}\,F_{mk}\,H^{kmo} + i\,m^2\,\pi_m\,h_{om} + \frac{m^2}{2}\,H_o = 0. \tag{5a}$$

$$(F_{\mu\nu} = \delta_\mu\,A_\nu - \delta_\nu\,A_\mu).$$

Not all the remaining equations in (4) lead to secondary constraints as some of them involve time derivative terms which cannot be eliminated through the equations of motion contained in (2). It is not difficult to see that the equations (4b) and (4c) together give rise to just three more secondary constraint relations. These can be written as:

$$2i\,\pi_k\,H^{okm} + (2m^2\,\delta_{ml} + ie\,F_{ml})\,h_{lo} = 0. \tag{5b}$$

Thus we have altogether 16 constraints at this stage.

Differentiation of (5a) followed by elimination of time derivatives now leads to a tertiary constraint, which is of the form (**B** is the magnetic field)

$$\frac{3}{2}(m^4 + e^2 B^2)\, h_{oo} + \ldots . = 0, \tag{6a}$$

where the dots stand for terms involving only those parts of the wave function for which equations of motion are already available. But there is no equation of motion for h_{oo} and therefore no more constraint relation can be obtained from (6a). Turning to the equations (5b) we can see readily that they lead to three more (tertiary) constraints which for **B** along the z-axis may be written in the form

$$\frac{3}{2} e\, B\, H_{t1t} + i\, m^2\, H_{t2t} = \ldots ., \tag{6b}$$

$$i\, m^2\, H_{t1t} - \frac{3}{2} e\, B\, H_{t2t} = \ldots ., \tag{6c}$$

$$i\, m^2\, H_{t3t} = \ldots ., \tag{6d}$$

These equations supply the last three constraints needed to make up the required total of 20. The situation is therefore satisfactory *provided* further differentiation does not give rise to new constraints. In fact, differentiation of equations (6) lead in general to equations of motion for the quantities h_{oo} and H_{tkt} which appear on the l.h.s.

There is one value of the field $|\mathbf{B}|$, however at which this satisfactory situation breaks down. When $B = 2m^2/3e$ the left hand sides of equations (6b) and (6c) are linearly dependent, it is evident that these two constraints can then be written in such a way that one of them contains none of H_{tkt} and involves only quantities for which equations of motion are already known. Differentiation of this equation would then lead to an extra constraint.*

Our finding is in conflict with the claim of Kobayashi and Shamaly that when $B = 2m^2/3e$ there is a loss of constraints. In § 4 we discuss the source of this confusion.

3. General analysis of constraints

While it is possible always to carry out a constraint analysis as above for any linear wave equation, incorporating electromagnetic interactions, it would be of great advantage if one could find a general criterion which would reveal whether or not an arbitrary linear equation would suffer from the occurrence of extra constraints if minimal electromagnetic interactions were introduced. This section is devoted to the derivation of such a general criterion.

Consider the general equation

$$(\beta_\mu\, \pi^\mu - m)\, \psi(x) = 0, \tag{7}$$

*The constraint coming from (6b) and (6c) at the fourth stage can be verified to be independent of other constraints.

in which β's obey the algebra

$$\beta_0^5 = \beta_0^3. \tag{8}$$

For our purpose it is convenient to adopt for β_0 the canonical form*

$$\beta_0 = \begin{pmatrix} A & & \\ & O & \\ & & C \end{pmatrix}, \tag{9}$$

wherein A and C are blocks having the minimal equations $A^2=1$ and $C^3=0$. Making a corresponding partitioning of ψ and boost generators as

$$\psi = \text{Col}\,(\psi_1\,\psi_2\,\psi_3), \tag{10a}$$

$$\vec{K} = \begin{pmatrix} \vec{K}_{11} & \vec{K}_{12} & \vec{K}_{13} \\ \vec{K}_{21} & \vec{K}_{22} & \vec{K}_{23} \\ \vec{K}_{31} & \vec{K}_{32} & \vec{K}_{33} \end{pmatrix}, \tag{10b}$$

and using the well-known result that the β_i's can be written as

$$\vec{\beta} = i\,[\vec{K}, \beta_0], \tag{11}$$

the equations of motion for ψ_1, ψ_2, ψ_3 can be explicitly written down as follows

$$(A\pi_0 - m)\,\psi_1 - i\,[\vec{K}_{11}\,\vec{\pi}, A]\,\psi_1 + i\,A\,\vec{K}_{12}\,\vec{\pi}\,\psi_2$$

$$+ i\,(A\,\vec{K}_{13}\,\vec{\pi} - \vec{K}_{13}\,\vec{\pi}\,C)\,\psi_3 = 0, \tag{12a}$$

$$m\psi_2 + i\,\vec{K}_{21}\cdot\vec{\pi}\,A\,\psi_1 + i\,\vec{K}_{23}\cdot\vec{\pi}\,C\,\psi_3 = 0, \tag{12b}$$

$$(C\pi_0 - m)\,\psi_3 - i\,(\vec{K}_{2l}\,\vec{\pi}\,A - C\vec{K}_{3l}\,\vec{\pi})\,\psi_1 +$$

$$i\,C\vec{K}_{32}\,\vec{\pi}\,\psi_2 - i\,[\vec{K}_{33}\,\vec{\pi}, C]\,\psi_3 = 0. \tag{12c}$$

Equation (12b) is clearly a constraint, and C^2 times (12c) yields another. These are the primary constraints. Derivation of secondary and tertiary constraints proceeds in the usual fashion (see for example Mathews *et al* 1979). We obtain them to be

$$(1-\beta_0^2)\beta_0\,[m^2 + \beta_0\,\vec{K}\cdot\vec{\pi}\,\beta_0\,\vec{K}\cdot\vec{\pi} - \beta_0\,(\vec{K}\cdot\vec{\pi})^2\,\beta_0$$

$$+ i\,m\vec{K}\cdot\vec{\pi}\,\beta_0]\,\psi\,(x) = 0 \tag{13}$$

*The algebra (8) admits of one more diagonal block B with the property $B^2=0$ in the canonical form. But we do not know of any wave equations wherein such a block finds a place, and so we shall ignore this more general possibility.

$$(1-\beta_0^2)\,[m^3 + m\beta_0^2 M - m\beta_0^2 (\vec{K}\cdot\vec{\pi})^2 + m\beta_0 \vec{K}\cdot\vec{\pi}\,\beta_0 \vec{K}\cdot\vec{\pi} - i\,\beta_0^2 \vec{K}\cdot\vec{\pi}\,\vec{\beta}\cdot\vec{\pi}\,\vec{K}\cdot\vec{\pi}]\,\psi(x) = \ldots. \tag{14}$$

The dots stand for terms involving parts of ψ for which equations of motion are available at this stage. M is matrix differential operator defined by the relation

$$(1-\beta_0^2)\,\beta_0^2 \vec{K}\cdot\vec{\pi}\,\beta_0 \vec{K}\cdot\vec{\pi} = (1-\beta_0^2)\,\beta_0^2 M \beta_0. \tag{15}$$

It may be noted incidentally that the secondary constraints (13) leads to the tertiary constraint (14) only if an M satisfying (15) exists.* Operating on the right by $\beta_0^2(1-\beta_0^2)$ it becomes,

$$(1-\beta_0^2)\,\beta_0^2\,(K_i\,\beta_0\,K_j - K_j\beta_0 K_i)\beta_0^2\,(1-\beta_0^2) = 0. \tag{16}$$

While this condition ensures, as already shown in the paper of Mathews *et al* (1979) that no constraints are lost, the possibility still remains that extra constraints may appear in the presence of interactions. To see how this can come about, let us look at the tertiary constraint equations (14) once again. If further constraints are not to be generated from (14) then the matrix differential operator on the l.h.s. of this equation must be such that it should never be possible to write it as $N\beta_0$ for some N i.e.,

$$(1-\beta_0^2)\,[m^3 + m\beta_0^2 M - m\beta_0^2 (\vec{K}\cdot\vec{\pi})^2 + m\beta_0 \vec{K}\cdot\vec{\pi}\,\beta_0\vec{K}\cdot\vec{\pi} - i\,\beta_0^2 \vec{K}\cdot\vec{\pi}\,\vec{\beta}\cdot\vec{\pi}\,\vec{K}\cdot\vec{\pi}] \neq N\beta_0. \tag{17}$$

This condition is equivalent to

$$(1-\beta_0^2)\,[m^3 + m\beta_0^2 M - m\beta_0^2 (\vec{K}\cdot\vec{\pi})^2 + m\beta_0 \vec{K}\cdot\vec{\pi}\,\beta_0 \vec{K}\cdot\vec{\pi} - i\,\beta_0^2 \vec{K}\cdot\vec{\pi}\,\vec{\beta}\cdot\vec{\pi}\,\vec{K}\cdot\vec{\pi}]\,\beta_0^2\,(1-\beta_0^2) \neq 0. \tag{18}$$

It is not difficult to convince oneself that the last term on the l.h.s. of (18) can be reduced using the relation

$$[K_i, [\beta_j, K_k]] = -\,\delta_{jk}\,\beta_i. \tag{19}$$

One finds then with $P=(1-\beta_0^2)\,\beta_0^2$ that

$$P\,\vec{K}\cdot\vec{\pi}\,\vec{\beta}\cdot\vec{\pi}\,\vec{K}\cdot\vec{\pi}\,P = \frac{ie}{2}F_{ij}\,P[2\,K_l\,\beta_0\,K_j\,K_i - K_j\,K_l\,\beta_0\,K_i + 2K_j\,\beta_0\,K_l\,K_i + K_j\,\beta_0\,K_i\,K_l - K_l\,K_j\,\beta_0\,K_i]\,P\,\pi_l = \frac{ie}{2}F_{ij}\,R_{ijl}\,\pi_l \text{ (say)}. \tag{20}$$

*We wish to observe that eq. (15) is a restriction on the class of eqs. (7) under study. Also the solution of eq. (15) for M is not unique. But this does not affect our conclusions. We thank the referee for drawing our attention to this fact.

Similarly the terms with two π's in (18) can be simplified to

$$(1-\beta_0^2)\ [\beta_0^2\, M - \beta_0^2\, (\mathrm{K}\cdot\vec{\pi})^2 + \beta_0\, \mathrm{K}\cdot\vec{\pi}\, \beta_0\, \mathrm{K}\cdot\vec{\pi}]\ P$$

$$= i\, eF_{ij}\, (1-\beta_0^2)\ [\beta_0\, K_i\, \beta_0\, K_j - \beta_0\, K_i\, K_j\, \beta_0$$

$$+ K_i\, \beta_0\, K_j\, \beta_0]\ (1-\beta_0^2)\, \beta_0 = ieF_{ij}\, T_{ij}\ \text{(say)}. \tag{21}$$

The condition (18) for non-generation of further constraints now becomes,

$$m^3(1-\beta_0^2)\ \beta_0^2 + \frac{iem}{2} F_{ij}\, (T_{ij} - T_{ji}) + \frac{ie}{4} F_{ij}\, (R_{ijl} - R_{jil})\, \pi_l \neq 0. \tag{22}$$

The l.h.s. has a part involving the differential operator π_l, which evidently cannot cancel with the remainder of the operator. Therefore in order to satisfy (22) for arbitrary e.m. fields it is clearly necessary and sufficient that one of the two parts be non-zero i.e. either

$$R_{ijl} - R_{jil} \neq 0 \text{ for some } i, j, l, \tag{23a}$$

or

$$\beta_0^2\, (1-\beta_0^2) + \frac{ie}{2m^2} F_{ij}\, (T_{ij} - T_{ji}) \neq 0. \tag{23b}$$

Theories which satisfy condition (16) simultaneously with (23a) or (23b) are the ones in which the number of constraints will remain unaffected by the introduction of minimal e.m. interactions. We may note in passing that a sufficient condition for (23b) to be valid for all F_{ij} is that $(T_{ij} - T_{ji})$ must be a nilpotent matrix.

4. Discussion

The present work completes the analysis of the constraints problem of unique-mass theories with the minimal algebra $\beta_0^5 = \beta_0^3$ in the presence of e.m. interactions. Of the known relativistic wave equations, those due to Glass (1971), Schwinger and Chang (see also Hagen 1971) and Shamaly and Capri (1973) are characterised by this algebra. The Glass equation is already known to suffer loss of constraints (Mathews *et al* 1979).

As for the Schwinger-Chang theory one sees on applying the conditions of the last section that there could be no loss of constraints but that at a critical value of the magnetic field the number of constraints becomes excessive; the $(R_{ijl} - R_{jil})$ vanishes for all i, j, l in this theory, and

$$\beta_0^2\, (1-\beta_0^2) + \frac{ie}{2m^2} F_{ij}\, (T_{ij} - T_{ji}) \text{ vanishes when } B = \frac{2m^2}{3e}.$$

This finding raises the question as to why a different conclusion was arrived at by Kobayashi and Shamaly. Working in the second order formalism with second rank symmetric tensor components $h_{\mu\nu}$ as the basic variables they start with the field equations $L_{\mu\nu}=0$ (see for notation etc., Kobayashi and Shamaly (1978) and Velo (1972)). Of the ten constraints needed, eight are obtained as primary and secondary constraints

$$L_{\mu 0}=0,$$

and $$C_\nu \equiv \pi^\mu L_{\mu\nu}=0.$$

The ninth constraint

$$\chi=\pi^\mu \pi^\nu L_{\mu\nu}+m^2 L^\mu_\mu=0,$$

follows from $\pi^0 C_0=0$ after some manipulation. One more constraint is needed and it is to be sought from $\pi_0 \chi=0$. It turns out that $\pi_0 \chi$ involves the quantities $\pi_0{}^2 h_{ok}$ for which expressions in terms of lower order derivatives are not yet available. However, such expressions are obtainable from $\pi_0 C_k=0$ and can be used to eliminate $\pi_0{}^2 h_{ok}$ from $\pi_0 \chi=0$ (thus reducing the latter to a constraint) *except* when the external field is a magnetic field of magnitude $B=2m^2/3e$. In this special situation, only two of the three $\pi_0{}^2 h_{ok}$ can be recovered from the equations $\pi_0 C_k=0$, and so $\pi_0 \chi=0$ is no longer a constraint. From this fact Kobayashi and Shamaly conclude that there is a deficiency in the number of constraints for this critical value of the field. However in drawing this conclusion they have overlooked the fact that in place of this particular lost constraint there is now one linear combination of the $\pi_0 C_k=0$ which is a constraint. In fact it may be verified that differentiation of this new-constraint leads to an extra constraint—one more than in the free case. This extra constraint of the second order formalism is the equivalent of the two extra constraints we have found in the first order formulation.

We may conclude with the observation that to our knowledge this is the first time a general matrix algebraic analysis of the constraint problem, applicable to a wide variety of equations and encompassing earlier known results has been given.

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