Hopf invariant for long-wavelength skyrmions in quantum Hall systems for integer and fractional fillings

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Abstract

We show that a Hopf term exists in the effective action of long-wavelength skyrmions in quantum Hall systems for both odd integer and fractional filling factors $\nu = 1/(2s+1)$, where s is an integer. We evaluate the prefactor of the Hopf term using Green function method in the limit of strong external magnetic field using model of local interaction. The prefactor (N) of the Hopf term is found to be equal to ν. The spin and charge densities and hence the total spin and charge of the skyrmion are computed from the effective action. The total spin is found to have a dominant contribution from the Berry term in the effective action and to increase with the size of the skyrmion. The charge and the statistics of the skyrmion, on the other hand, are completely determined by the prefactor of the Hopf term. Consequently, the skyrmions have charge νe and are Fermions (anyons) for odd integer (fractional) fillings. We also obtain the effective action of the skyrmions at finite temperature. It is shown that at finite temperature, the value of the prefactor of the Hopf term depends on the order in which the zero-momentum and zero-frequency

limits are taken.

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I. INTRODUCTION

Two-dimensional (2D) electron gases are known to exhibit a rich variety of interesting phenomena with variation of particle density or magnetic field. The most striking among these phenomena are the integer and fractional quantum Hall effects [1–5]. Since these phenomena take place at very high magnetic field, it may seem reasonable to think that the presence of a Zeeman term in the Hamiltonian would preclude any dynamics for the spin degree of freedom. This would have indeed been the case if the Lande g factor for the electrons in the sample were close to the free electron Lande g factor. The lowest-lying excitations of the system would then be quasihole-quasiparticle pairs with opposite spins. However, as observed by Halperin [6], the Lande g factor for electrons in Ga-As samples is much lower than the free-electron Lande g factor. The Zeeman splitting for the electrons in these systems is, therefore, small compared to both the cyclotron energy and the typical Coulomb energy. As a result, the spin degrees of freedom become dynamical, in spite of the presence of a strong external magnetic field. Nevertheless, for certain filling fractions ($\nu = 1, 1/3$), the ground state of the system is ferromagnetic even in the limit of vanishing Zeeman energy because of exchange interaction between the electrons. For these filling fractions, the lowest lying excitations are topologically non-trivial spin configurations called skyrmions [5,7,8]. The spatial extent of such configurations or skyrmions is determined by the relative strength of the Coulomb and Zeeman energies in the system. When the Zeeman energy is negligible compared to the Coulomb energy, the skyrmions have a large radius and are referred to as long-wavelength skyrmions. A small deviation from a ferromagnetic filling fraction creates such skyrmions with many electron spins reversed which strongly reduces the spin polarization of the system. A clear signature of such spin depolarization, suggesting skyrmionic spin configurations, has been observed experimentally at filling factor $\nu = 1$ [9,10]. The experimental realization of skyrmions for fractional filling factors is much more difficult since it requires a very low Lande g factor. Nevertheless, recently Leadly *et al.* [11] have observed signature of skyrmions at $\nu = 1/3$ by reducing the Lande g factor of the

sample by applying external pressure.

The spin configurations of the system can be characterized by a unit vector field $\mathbf{d}(x, y, t)$ that gives the direction of the local spin. For long-wavelength skyrmions, the space-time modulation of the spin configuration and hence the $\mathbf{d}$ field is slow. Thus it is possible to derive a low-energy effective action for the skyrmions in terms of $\mathbf{d}$ field gradients. It is well known [7,8,12] that in quantum Hall ferromagnets, such a low energy effective action $S_{\rm FM}$ would contain the Berry and the usual gradient terms:

$$S_{\rm FM} = S_{\rm Berry} + S_{\rm E}, \tag{1a}$$

$$S_{\rm Berry} = \frac{\rho_0}{2} \int d^2r\, dt\, B_0, \tag{1b}$$

$$S_{\rm E} = -\frac{\kappa}{2} \int d^2r\, dt\, (\nabla \mathbf{d})^2\,, \tag{1c}$$

where B_0 is defined in Eq. (3), ρ_0 is the ground state density and κ is the spin stiffness constant. This is to be contrasted with the effective action for antiferromagnetic systems, which contains a term quadratic in time derivative of the unit vector $\mathbf{d}$ as shown by Wen and Zee [13] and Haldane [14].

If the ground state of the system does not have time-reversal and parity invariance, the effective action may also contain the Hopf term given by

$$S_{\rm Hopf} = \frac{N}{32\pi} \int d^2\mathbf{r}\, dt\, \epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda}, \tag{2}$$

where $\epsilon^{\mu\nu\lambda}$ is the completely antisymmetric tensor, the prefactor N is a topological invariant and B_μ is an auxiliary gauge field given by

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = \mathbf{d} \cdot (\partial_\mu \mathbf{d} \times \partial_\nu \mathbf{d}). \tag{3}$$

For antiferromagnetic systems, Wilczek and Zee have shown that the topological invariant N determines the spin and the statistics of the skyrmion [15]. If N is an odd (even) integer, the skyrmion is a Fermion (boson) with spin $N/2$. For fractional N, the skyrmion is an

anyon. However, as we shall see in Sec. V, the method of Ref. [15] leads to quite different conclusions for the spin of the skyrmions in ferromagnetic systems due to the presence of the Berry term (1b) in the effective action. The charge and the statistics, on the other hand, are still completely determined by the topological invariant N.

The presence of a Hopf term in the effective action of systems described by a unit vector field $\mathbf{d}$ is well known in many different areas of condensed matter. It was initially conjectured by Dzyaloshinskii *et al*. that this term may exist in the effective action of planar antiferromagnets [16]. However, microscopic calculations showed that such a term is absent in the effective action of these systems because of symmetry requirements [13,14]. There are, nonetheless, various other systems where symmetry requirements permit a non-zero topological term in the low-energy effective action. A few examples are He^3-A films [17], lattice quantum Hall systems [18] and quasi-one-dimensional organic conductors [19]. Recently, it has been suggested that the superconducting phase in Sr_2RuO_4 has a spin-triplet p-wave pairing state with broken parity and time-reversal symmetry similar to He^3-A films [20]. Consequently, the corresponding low-energy effective action of Sr_2RuO_4 is also expected to have similar topological term [21].

Volovik and Yakovenko [17,18] have derived an explicit expression for the topological invariant N for a general class of mean-field Fermionic models, with the electron Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \boldsymbol{\sigma} \cdot \mathbf{d}(\mathbf{r}, t)\hat{H}_1, \tag{4}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices acting on the spin indices of the electrons. This class includes the above mentioned systems [17–20]. It was shown that the expression for the topological invariant N is

$$N = 2\pi \mathrm{Tr}\left(\frac{\partial G_0^{-1}}{\partial \omega} G_0 \frac{\partial G_0^{-1}}{\partial k_x} G_0 \frac{\partial G_0^{-1}}{\partial k_y} G_0\right), \tag{5}$$

where Tr denotes all integrations and matrix traces, k_x, k_y and ω are the electron momenta and frequency, and $G_0(k_x, k_y, \omega) = (\omega - \hat{H}_0 - \sigma_z \hat{H}_1 + i\eta \mathrm{Sgn}\omega)^{-1}$ is the unperturbed electron

Green function. In principle, this treatment should also hold for quantum Hall systems, since their Hamiltonians can be cast into the same class of mean-field models (4). However, since k_x and k_y are no longer good quantum numbers simultaneously in the presence of magnetic field, Eq. (5) cannot be directly applied to the present case. Nevertheless, if we compare the expression of the topological invariant N to that of Hall conductivity in quantum Hall systems [22],

$$\frac{h}{e^2}\sigma_{xy} = 2\pi \mathrm{Tr}\left(\frac{\partial G_0^{-1}}{\partial \omega} G_0 \frac{\partial G_0^{-1}}{\partial \phi_x} G_0 \frac{\partial G_0^{-1}}{\partial \phi_y} G_0\right), \tag{6}$$

we find that expressions (5) and (6) are similar except that the momenta (k_x, k_y) are replaced by boundary phase parameters (ϕ_x, ϕ_y) and the integration over momenta are to be replaced by average over the boundary phases [23]. In fact, as we shall see, N is the same topological invariant that determines the Hall conductivity in quantum Hall systems, and properly modified, the formalism developed in Ref. [17] gives exactly the same Eq. (6) for N.

Recently, there has been a lot of interest in finding the Hopf term in the effective action of long-wavelength skyrmions in quantum Hall systems at $\nu = 1$. Some of these works [24,25] use lowest Landau level (LLL) projection technique to derive the Hopf term. However, it has been pointed out [26] that the parameterization of the unit vector $\mathbf{d}$ in terms of the Euler angles in Ref. [24] is questionable. This gave rise to a subsequent controversy [27,28], which we aim to resolve. Further, the validity of the LLL projection in the present context was also questioned by Iordanskii and Plyasunov [29,30], who derived the prefactor of the Hopf term by explicit term by term evaluation of the effective action, starting from a mean-field Hamiltonian but without any LLL projection. A similar work, avoiding the LLL projection was later done by Ray [31]. Although the end result for the prefactor N is the same in all these works [24,25,29–31], it is not quite clear whether the result is an artifact of the simplicity of the models or assumptions used. The aim of the present work is to point out the robustness of this result and to generalize it for fractional filling factors $\nu = 1/(2s+1)$, where s is an integer. We also explicitly compute the spin and charge densities of the skyrmion, and hence its total spin and charge. It is shown that the dominant contribution to the

skyrmion spin comes from the Berry term in the action. The contribution to the skyrmion spin from the Hopf term comes equally from the bulk and the edge, but is small compared to the contribution of the Berry term. This result coincides with the results obtained by Baez *et al*. on a phenomenological basis [32]. The statistics of the skyrmion is also computed using the method of Ref. [15] and is found to be determined by the prefactor of the Hopf term. Our result regarding the statistics corroborates the result obtained by Yang and Sondhi [33] using a variational wave-function for the skyrmion, but differs from that of Dziarmaga [34].

The fate of the Hopf term at finite temperature is another very interesting question. At zero temperature, it is possible to carry out a derivative expansion of the polarization tensor and thus obtain an expression for the prefactor of the Hopf term in terms of the Green functions. However the situation is much more tricky at finite temperature. It is well known [35–38], at least in the case of U(1) gauge fields (electromagnetic field), that the polarization tensor becomes a non-analytic function of frequency and momentum at finite temperature and a derivative expansion can not be carried out unambiguously. The zero-frequency zero-momentum limit ($p_0 \to 0, \mathbf{p} \to 0$) depends on the order in which the limits are taken. This order is generally chosen from physical consideration and depends on the system of interest. In particular, in quantum Hall systems, the prefactor of the Chern-Simons term (the equivalent of the Hopf term for Abelian gauge fields) which determines the Hall conductivity depends on the order in which the zero-frequency and the zero-momentum limits are taken. In this work, we use imaginary time Matsubara formalism to derive an effective action for skyrmions at finite temperature and $\nu = 1$. We show that, analogous to the Chern-Simons term for the Abelian gauge fields, it is in general not possible to obtain a local Hopf term in position space at finite temperature. Nevertheless, one can still obtain a rather simple effective action for skyrmions in momentum space. We evaluate the prefactor of the Hopf term in both the dynamic ($\mathbf{p}/p_o \to 0$) and the static ($p_o/\mathbf{p} \to 0$) limit. In the static limit, the prefactor depends on temperature, while in the dynamic limit, it is independent of temperature and has the same value as at zero temperature.

The organization of the paper is as follows. In Sec.II, we derive the effective action at zero

temperature and obtain an expression for the prefactor N in terms of the Green functions. In Sec.III, the value of this prefactor is evaluated. In Sec.IV, we derive expressions for spin and charge densities and obtain the total spin and charge of the skyrmion. The statistics of the skyrmion is obtained in Sec.V. These results are generalized for fractional fillings in Sec.VI. In Sec.VII, we derive the effective action at finite temperature and obtain the prefactor of the Hopf term for both the static and the dynamic limits. This is followed by conclusion in Sec.VIII. Some details of the calculations are sketched in Appendices A-C while in Appendix D, we present a list of notations used, for clarity.

II. EXPRESSION FOR N AT $\nu = 1$

In this section, we derive the effective action at zero temperature and from there obtain an expression for the prefactor N in terms of the Green functions of the system at ν =1. Throughout the rest of the work, natural units $\hbar = c = 1$ are used. The convention of subscripts and superscripts are as follows. The Greek letters are used to denote space-time indices and these can take values (0,1,2) where 0 denotes time component and 1 and 2 denote the two space directions. The letters i, j, and k are used for the indices of the Pauli matrices σ_i and take values (1,2,3). The letters a, b, and c denote either space indices 1 and 2 or spinor indices $\uparrow$ and $\downarrow$. All repeated indices are summed over unless explicitly stated otherwise. We also use the following convention for vectors and operators whenever needed. All contravariant vectors are taken as $X^\mu = (X^0, \mathbf{X})$ and all operators as $\partial^\mu = (\partial^0, -\bigtriangledown)$. The covariant counterparts of the contravariant vectors and operators are obtained by applying the metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,\text{-}1,\text{-}1)$ [39].

The action for the system with a model local interaction $V(\mathbf{r}_1 - \mathbf{r}_2) = \gamma_0 \delta(\mathbf{r}_1 - \mathbf{r}_2)$ [40] can be written using Hartree-Fock approximation as [29,30]

$$S[\psi^\dagger, \psi] = \int d^2\mathbf{r}\, dt\, \psi_a^\dagger(\mathbf{r}, t)\, (i\partial_0 I - H + \epsilon_F I)_{ab}\, \psi_b(\mathbf{r}, t), \tag{7}$$

where the Fermionic field ψ is a two component spinor, I is the 2×2 unit matrix, ϵ_F is the Fermi energy and H is the mean-field Hamiltonian density of the system given by

$$H = \frac{[\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r})]^2}{2m} I - \gamma_0 \mathbf{d}(\mathbf{r},t)\cdot\boldsymbol{\sigma}. \tag{8}$$

Here $\mathbf{A}$ is the vector potential corresponding to the external magnetic field $\mathbf{H}_0$, $\mathbf{d}$ is the unit vector field that gives the direction of local spin, $\boldsymbol{\sigma}$ are the Pauli matrices, $\gamma_0 \sim e^2/l_B$ is the typical Coulomb energy of the system, and $l_B = \sqrt{1/eH_0}$ is the magnetic length. In this treatment we shall neglect Zeeman energy (E_z) and also assume that the characteristic cyclotron energy is much greater than all other energy scales in the problem, *i.e.* we consider the regime $\omega_c \gg \gamma_0 \gg E_z$. This is the relevant regime for long-wavelength skyrmions [24,29].

To calculate the topological term in the action, it is convenient to introduce 2×2 local unitary SU(2) rotation matrix $U(\mathbf{r},t)$, that corresponds to the local rotation of $\mathbf{d}(\mathbf{r},t)$ from the homogeneous field $\mathbf{d} = \mathbf{e_z}$

$$U(\mathbf{r},t)\sigma_z U^{-1}(\mathbf{r},t) = \boldsymbol{\sigma}\cdot\mathbf{d}(\mathbf{r},t). \tag{9}$$

After the unitary transformation of the Fermi fields $\chi(\mathbf{r},t) = U^{-1}(\mathbf{r},t)\psi(\mathbf{r},t)$, the action becomes

$$S[\chi^\dagger,\chi,Q_\mu^{\rm int}] = \int d^2r\, dt\, \chi_a^\dagger(\mathbf{r},t)\Bigg(i\partial_0 I - Q_0^{\rm int}(\mathbf{r},t) - \frac{\left\{[\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r})]\, I - \mathbf{Q}^{\rm int}(\mathbf{r},t)\right\}^2}{2m} + \gamma_0\sigma_z + \epsilon_F I\Bigg)_{ab} \chi_b(\mathbf{r},t). \tag{10}$$

The new spinor fields $\chi(\mathbf{r},t)$ have their spin quantization axis along $\mathbf{e_z}$, and $Q_\mu^{\rm int}$ ($\mu = 0,1,2$) are the SU(2) gauge fields given by

$$Q_\mu^{\rm int} = -iU^{-1}(\partial_\mu U) = \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_\mu^{\rm int} = \frac{1}{2}\sigma_i\Omega_\mu^{i\,\rm int}. \tag{11}$$

The fields $\boldsymbol{\Omega}_\mu^{\rm int}$ are pure gauge fields satisfying the relation

$$f_{\mu\nu}^{\rm int} = \partial_\mu\boldsymbol{\Omega}_\nu^{\rm int} - \partial_\nu\boldsymbol{\Omega}_\mu^{\rm int} - \boldsymbol{\Omega}_\mu^{\rm int}\times\boldsymbol{\Omega}_\nu^{\rm int} = 0. \tag{12}$$

Further, the rotation matrix U satisfies the relation [17]

$$i(\partial_\mu U)U^{-1} = \frac{1}{2}\boldsymbol{\sigma}\cdot\left(-B_\mu\mathbf{d} + \mathbf{d}\times\partial_\mu\mathbf{d}\right), \tag{13}$$

where $B_\mu = \Omega_\mu^{3\,\mathrm{int}}$ is the auxiliary gauge field introduced in Eq. (3). The Hopf term can be expressed in terms of these auxiliary gauge fields B_μ (2) or equivalently in terms of the gauge fields $\mathbf{\Omega}_\mu^{\mathrm{int}}$ as

$$S_{\mathrm{Hopf}} = \frac{N}{96\pi^2}\int d^2r\, dt\, \epsilon^{\mu\nu\lambda}\, \mathbf{\Omega}_\mu^{\mathrm{int}} \cdot (\mathbf{\Omega}_\nu^{\mathrm{int}} \times \mathbf{\Omega}_\lambda^{\mathrm{int}}). \tag{14}$$

The effective action for the $\mathbf{Q}^{\mathrm{int}}$ fields can now be obtained by integrating out the spinor fields. For a slowly varying $\mathbf{d}$ field configuration, the $\mathbf{Q}^{\mathrm{int}}$ fields are small, and it is possible to carry out a gradient expansion of the effective action $S_{\mathrm{eff}}[\mathbf{Q}^{\mathrm{int}}]$ in the powers of $\mathbf{Q}^{\mathrm{int}}$ and its derivatives. It is clear from Eqs. (12) and (14) that the Hopf term originates from the $(\mathbf{Q}^{\mathrm{int}})^3$ and $\mathbf{Q}^{\mathrm{int}}\partial\mathbf{Q}^{\mathrm{int}}$ terms in the expansion of the effective action $S_{\mathrm{eff}}[\mathbf{Q}^{\mathrm{int}}]$.

To calculate the effective action, we first divide the action into two parts $S = S_0 + S_1$, where S_0 and S_1 are given by

$$\begin{aligned} S_0 &= \int d^2r\, dt\, \chi^\dagger(\mathbf{r},t)\left([i\partial_0 + \epsilon_F]I - \frac{[\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r})]^2}{2m}I + \gamma_0\sigma_z\right)\chi(\mathbf{r},t), \\ S_1 &= -\int d^2r\, dt\, \chi^\dagger(\mathbf{r},t)\frac{1}{2}\left(\Pi^\mu Q_\mu^{\mathrm{int}}(\mathbf{r},t) + Q_\mu^{\mathrm{int}}(\mathbf{r},t)\Pi^\mu\right)\chi(\mathbf{r},t), \end{aligned} \tag{15}$$

where $\Pi^a = [p^a - eA^a(\mathbf{r})]\, I/m$ and $\Pi^0 = I$. The effective action $S_{\mathrm{eff}}[\mathbf{Q}^{\mathrm{int}}]$ is then given by

$$e^{iS_{\mathrm{eff}}[\mathbf{Q}^{\mathrm{int}}]} = \frac{\int D\chi^\dagger D\chi e^{i(S_0+S_1)}}{\int D\chi^\dagger D\chi e^{iS_0}} = \langle e^{iS_1}\rangle_{S_0}. \tag{16}$$

At this stage, we introduce a unitary transformation on the field variables of the form

$$\chi(\mathbf{r},t) \to e^{\frac{i\phi_x x}{L_x}} e^{\frac{i\phi_y y}{L_y}} \chi(\mathbf{r},t), \tag{17}$$

where ϕ_x and ϕ_y are constant parameters and L_x and L_y are the dimensions of the system. Transformation (17) changes the boundary conditions on the single particle wavefunctions. We shall discuss its physical meaning in more details in the next section. At this point, we may consider it to be a mathematical trick used to facilitate computations. With this transformation, the action can be written as

$$\begin{aligned} S_0 &= \int d^2r\, \frac{d\omega}{2\pi}\, \chi^\dagger(\mathbf{r},\omega)\left([\omega + \epsilon_F]I - \frac{[\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r}) + \boldsymbol{\alpha}]^2}{2m}I + \gamma_0\sigma_z\right)\chi(\mathbf{r},\omega), \\ S_1 &= -\int d^2r\, \frac{d\omega dp_0}{(2\pi)^2}\, \chi^\dagger(\mathbf{r},\omega+p_0)\left(\frac{1}{2}\left[Q_\mu^{\mathrm{int}}(\mathbf{r},p_o), \frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right]_+\right)\chi(\mathbf{r},\omega), \end{aligned} \tag{18}$$

where $[...]_+$ means anticommutator, $\alpha^\mu = (\omega, \phi_x/L_x, \phi_y/L_y)$, and G_0 is the unperturbed Green function. Here we have omitted the quadratic term in Q_a in S_1. It is easy to see that this term does not contribute to the Hopf term in the effective action. The advantage of introducing the parameters ϕ_x and ϕ_y also becomes clear from the expression of S_1. The operators Π_x and Π_y can now be conveniently expressed as the derivatives of G_0^{-1} with respect to ϕ_x and ϕ_y. These parameters therefore take the place of momenta k_x and k_y which are no longer good quantum numbers in a magnetic field.

The unperturbed Green function G_0 is diagonal in the spin space. It is given by

$$G_0 = \begin{pmatrix} G_0^+ & 0 \\ 0 & G_0^- \end{pmatrix}, \tag{19}$$

where $G_0^\pm$ satisfy the following equations

$$\left(\omega + \epsilon_F - \frac{[\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r_1}) + \boldsymbol{\alpha}]^2}{2m} \pm \gamma_0\right) G_0^\pm(\mathbf{r_1}, \mathbf{r_2}, \omega) = \delta^2(\mathbf{r_1} - \mathbf{r_2}). \tag{20}$$

The perturbation expansion is now straightforward. The diagrams for the relevant terms in the effective action are shown in Fig. 1. The first term is the familiar polarization bubble and is given by

$$S_2 = \frac{1}{2}\int \frac{d^2p\, dp_0}{(2\pi)^3}\left(\Omega_\mu^{i\,\mathrm{int}}(\mathbf{p}, p_0) P_{ij}^{\mu\nu}(\mathbf{p}, p_0)\Omega_\nu^{j\,\mathrm{int}}(-\mathbf{p}, -p_0)\right), \tag{21}$$

where the tensor $P_{ij}^{\mu\nu}$ is given by

$$\begin{aligned} P_{ij}^{\mu\nu}(\mathbf{p}, p_0) = \frac{i}{4L_xL_y}\mathrm{Tr}\Bigg(\int d^2r_1\, d^2r_2 \frac{d\omega}{2\pi}\frac{1}{2}\left[\sigma_i e^{i\mathbf{p}\cdot\mathbf{r_1}}, \left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+ G_0(\mathbf{r_1}, \mathbf{r_2}, \omega) \\ \times\frac{1}{2}\left[\sigma_j e^{-i\mathbf{p}\cdot\mathbf{r_2}}, \left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2}, \mathbf{r_1}, \omega + p_0)\Bigg). \end{aligned} \tag{22}$$

It is clear from the structure of S_2 that to get the relevant terms of the form $\mathbf{Q}^{\mathrm{int}}\partial\mathbf{Q}^{\mathrm{int}}$, we need to expand $P_{ij}^{\mu\nu}$ to first order in external momenta $(\mathbf{p}, p_0)$. This expansion is carried out in Appendix A. The result is

$$\begin{aligned} S_2 &= \frac{1}{2} b_{ij}^{\mu\nu\lambda}\int d^2r\, dt\, \Omega_\nu^{j\,\mathrm{int}}(\mathbf{r}, t)\partial_\lambda\Omega_\mu^{i\,\mathrm{int}}(\mathbf{r}, t), \\ b_{ij}^{\mu\nu\lambda} &= \frac{1}{4L_xL_y}\mathrm{Tr}\left(\sigma_i\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0\sigma_j\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0\right). \end{aligned} \tag{23}$$

The other contribution to the Hopf term comes from the triangular diagram shown in Fig. 1. This diagram involves three $\mathbf{Q}$ fields. As $\mathbf{\Omega}_\mu^{\rm int}$ are pure gauge fields satisfying Eq. (12), a term with $\mathbf{\Omega}_\mu^{\rm int}\cdot(\mathbf{\Omega}_\nu^{\rm int}\times\mathbf{\Omega}_\lambda^{\rm int})$ is of the same order as a $\mathbf{\Omega}_\mu^{\rm int}(\partial_\lambda\mathbf{\Omega}_\nu^{\rm int})$ term. The contribution of these terms come from the triangular diagram which is given by

$$S_3 = \frac{1}{6}\int \frac{d^2p\,d^2q\,dp_0\,dq_0}{(2\pi)^6}\,T_{ijk}^{\mu\nu\lambda}(\mathbf{p},\mathbf{q},p_0,q_0)\Omega_\mu^{i\,\rm int}(\mathbf{p},p_0)\Omega_\nu^{j\,\rm int}(\mathbf{q},q_0) \\ \times\Omega_\lambda^{k\,\rm int}\left(-\mathbf{p}-\mathbf{q},-p_0-q_0\right), \tag{24}$$

where

$$T_{ijk}^{\mu\nu\lambda}(\mathbf{p},\mathbf{q},p_0,q_0) = \frac{i}{4L_xL_y}{\rm Tr}\left(\int d^2r_1\,d^2r_2\,d^2r_3\frac{d\omega}{2\pi}\,\frac{1}{2}\left[\sigma_i e^{i\mathbf{p}\cdot\mathbf{r_1}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+ G_0(\mathbf{r_1},\mathbf{r_2},\omega)\right. \\ \times\frac{1}{2}\left[\sigma_j e^{i\mathbf{q}\cdot\mathbf{r_2}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2},\mathbf{r_3},\omega+p_0+q_o) \\ \left.\times\frac{1}{2}\left[\sigma_k e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r_3}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}\right)_{\mathbf{r}_3}\right]_+ G_0\left(\mathbf{r_3},\mathbf{r_1},\omega+p_0\right)\right). \tag{25}$$

For the relevant order, here we need to consider the zeroth-order term in the expansion of $T_{ijk}^{\mu\nu\lambda}$ in powers of external momenta. This can be done following the method outlined in Appendix A, and one gets

$$S_3 = \frac{1}{6}c_{ijk}^{\mu\nu\lambda}\int d^2r\,dt\,\Omega_\mu^{i\,\rm int}(\mathbf{r},t)\Omega_\nu^{j\,\rm int}(\mathbf{r},t)\Omega_\lambda^{k\,\rm int}(\mathbf{r},t), \\ c_{ijk}^{\mu\nu\lambda} = \frac{i}{4L_xL_y}{\rm Tr}\left(\sigma_i\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0\sigma_j\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0\sigma_k\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0\right). \tag{26}$$

Having obtained the relevant terms in the effective action, we now evaluate $c_{ijk}^{\mu\nu\lambda}$ and $b_{ij}^{\mu\nu\lambda}$. Let us first consider $b_{ij}^{\mu\nu\lambda}$. We first note that G_0 is diagonal in the spinor space, since it contains I and σ_3. Further, it is shown in Appendix B that the coefficients $b_{ij}^{\mu\nu\lambda}$ satisfy $b_{ij}^{\mu\nu\lambda} = -b_{ij}^{\nu\mu\lambda}$. From these properties it follows that all the terms with $b_{ij}^{\mu\nu\lambda}$ (23) for $i\neq j$ either vanishes because of trace operation in spin indices or lead to total derivatives which can be neglected. Hence there are only three non-zero terms in $b_{ij}^{\mu\nu\lambda}$. These are

$$b_j^{\mu\nu\lambda} = \frac{1}{4L_xL_y}{\rm Tr}\left(\sigma_j\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0(\gamma_0)\sigma_j\frac{\partial\,G_0^{-1}}{\partial\alpha_\nu}G_0(\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0(\gamma_0)\right). \tag{27}$$

where j takes values from 1 to 3, and $G_0(\gamma_0)$ shows the dependence of G_0 on the parameter γ_0. Using properties of $\boldsymbol{\sigma}$ matrices, we get

$$\sigma_{1(2)} G_0(\gamma_0) = G_0(-\gamma_0)\sigma_{1(2)},$$
$$\sigma_3 G_0(\gamma_0) = G_0(\gamma_0)\sigma_3. \tag{28}$$

Using these relations, it is now easy to eliminate the σ matrices from the expression of $b_j^{\mu\nu\lambda}$. The result is

$$b_{1(2)}^{\mu\nu\lambda} = \frac{1}{4L_xL_y}\mathrm{Tr}\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0(-\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0(\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0(\gamma_0)\right), \tag{29}$$
$$b_3^{\mu\nu\lambda} = \frac{1}{4L_xL_y}\mathrm{Tr}\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0(\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0(\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0(\gamma_0)\right). \tag{30}$$

The contribution to the Hopf term from S_2 can therefore be written as

$$S_2 = \frac{1}{2}\int d^2r\,dt\,\left\{b_1^{\mu\nu\lambda}\left[\Omega_\nu^{1\,\mathrm{int}}(\mathbf{r},t)\partial_\lambda\Omega_\mu^{1\,\mathrm{int}}(\mathbf{r},t)+\Omega_\nu^{2\,\mathrm{int}}(\mathbf{r},t)\partial_\lambda\Omega_\mu^{2\,\mathrm{int}}(\mathbf{r},t)\right]\right.$$
$$\left.+b_3^{\mu\nu\lambda}\Omega_\nu^{3\,\mathrm{int}}(\mathbf{r},t)\partial_\lambda\Omega_\mu^{3\,\mathrm{int}}(\mathbf{r},t)\right\}. \tag{31}$$

Next, let us consider $c_{ijk}^{\mu\nu\lambda}$ given by Eq. (26). Here we notice that only those terms which have all (ijk) different produce non-zero contributions to S_3. The other terms, as sketched in Appendix B, vanish either under trace operations in spin indices or under exchange of space-time indices of $c_{ijk}^{\mu\nu\lambda}$. So there are six non-vanishing terms and a little algebra shows that their contributions are equal. Thus one may write

$$S_3 = c_{123}^{\mu\nu\lambda}\int d^2r\,dt\,\Omega_\mu^{1\,\mathrm{int}}(\mathbf{r},t)\Omega_\nu^{2\,\mathrm{int}}(\mathbf{r},t)\Omega_\lambda^{3\,\mathrm{int}}(\mathbf{r},t),$$
$$c_{123}^{\mu\nu\lambda} = \frac{i}{4L_xL_y}\mathrm{Tr}\left(\sigma_1\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0(\gamma_0)\sigma_2\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0(\gamma_0)\sigma_3\,\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0(\gamma_0)\right). \tag{32}$$

Using the relation $\sigma_1\sigma_2 = i\sigma_3$ and the cyclic property under trace operation, one can now eliminate the σ matrices to get

$$c_{123}^{\mu\nu\lambda} = -\frac{1}{4L_xL_y}\mathrm{Tr}\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0(-\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0(\gamma_0)\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0(\gamma_0)\right) = -b_1^{\mu\nu\lambda}, \tag{33}$$
$$S_3 = -b_1^{\mu\nu\lambda}\int d^2r\,dt\,\Omega_\mu^{1\,\mathrm{int}}(\mathbf{r},t)\Omega_\nu^{2\,\mathrm{int}}(\mathbf{r},t)\Omega_\lambda^{3\,\mathrm{int}}(\mathbf{r},t). \tag{34}$$

After some algebra, which is sketched in Appendix B, we find that S_3 (34) exactly cancels the first two terms of S_2 (31). It is shown in Appendix B that this cancelation is independent of the explicit expressions of the coefficients $b^{\mu\nu\lambda}_{1(2)}$ and $c^{\mu\nu\lambda}_{123}$. Also, it is easy to see that the tensor $b^{\mu\nu\lambda}_3$ is completely antisymmetric in space-time indices $(\mu\nu\lambda)$. As a result, we are left with

$$S_{\rm Hopf} = S_2 + S_3 = \frac{1}{2} b^{\mu\nu\lambda}_3 \int d^2r\, dt\, \Omega^{1\,\rm int}_\mu(\mathbf{r},t)\Omega^{2\,\rm int}_\nu(\mathbf{r},t)\Omega^{3\,\rm int}_\lambda(\mathbf{r},t). \tag{35}$$

Comparing Eqs. (14) and (35) and using Eq. (30) for $b^{\mu\nu\lambda}_3$, we find an expression for the prefactor N of the Hopf term:

$$N = \frac{4\pi}{3}\epsilon_{\mu\nu\lambda} b^{\mu\nu\lambda}_3 = 2\pi {\rm Tr}\left(\frac{\partial G_0^{-1}}{\partial\omega} G_0 \frac{\partial G_0^{-1}}{\partial\phi_x} G_0 \frac{\partial G_0^{-1}}{\partial\phi_y} G_0\right). \tag{36}$$

where we have substituted the expression for $b^{\mu\nu\lambda}_3$ (30) to obtain the last expression.

It is worthwhile to point out that the formal derivation of the Hopf term in this section (2,36) does not depend crucially on the model chosen. This is manifested in the fact that none of the cancelations of the various terms in the effective action depend on explicit expression of the unperturbed Green function, or equivalently on actual values of $b^{\mu\nu\lambda}$. The symmetry properties of $b^{\mu\nu\lambda}$ required for these cancelations are quite general, and so the result holds for any model of type (4).

III. EVALUATION OF N

The expression for the prefactor of the Hopf term N (36) is essentially the same as the expression for the Hall conductivity (6). Thus we follow the work of Niu *et al.* [22] to evaluate N.

We notice that since the unperturbed Green function G_0 is diagonal in spin indices, Eq. (36) can be written as

$$N = 2\pi \sum_{\alpha=\pm} {\rm Tr}\left(\frac{\partial G_0^{\alpha -1}}{\partial\omega} G_0^\alpha \frac{\partial G_0^{\alpha -1}}{\partial\phi_x} G_0^\alpha \frac{\partial G_0^{\alpha -1}}{\partial\phi_y} G_0^\alpha\right). \tag{37}$$

The expressions for the unperturbed Green functions $G_0^{\pm}$ can be obtained in terms of the single-particle eigenfunctions $|n\rangle$ and eigenenergies $\epsilon_n^{\pm}$ as

$$G_0^{\pm} = \sum_n |n\rangle\langle n| \left(\frac{\theta(\epsilon_n^{\pm} - \epsilon_F)}{\omega - (\epsilon_n^{\pm} - \epsilon_F) + i\eta} + \frac{\theta(\epsilon_F - \epsilon_n^{\pm})}{\omega - (\epsilon_n^{\pm} - \epsilon_F) - i\eta} \right), \tag{38}$$

where ϵ_F is the Fermi energy. These single-particle eigenfunctions and eigenvalues satisfy

$$\left(\sum_a \frac{\left(\hat{p}_a + \phi_a/L_a - eA_a(\mathbf{r})\right)^2}{2m} \mp \gamma_0 \right) |n\rangle = \epsilon_n^{\pm} |n\rangle, \tag{39}$$

where a can take values x and y. Substituting this form of the Green function in the expression for N, we get

$$\begin{aligned} N = 2\pi i \sum_{\alpha=\pm} \sum_{n,l} & \left(\frac{\partial G_0^{\alpha\,-1}}{\partial \phi_x} \right)_{nl} \left(\frac{\partial G_0^{\alpha\,-1}}{\partial \phi_y} \right)_{ln} \\ & \times \int \frac{d\omega}{2\pi i} \left(\frac{\theta(\epsilon_n^{\alpha} - \epsilon_F)}{\omega - (\epsilon_n^{\alpha} - \epsilon_F) + i\eta} + \frac{\theta(\epsilon_F - \epsilon_n^{\alpha})}{\omega - (\epsilon_n^{\alpha} - \epsilon_F) - i\eta} \right) \\ & \times \left(\frac{\theta(\epsilon_l^{\alpha} - \epsilon_F)}{\omega - (\epsilon_l^{\alpha} - \epsilon_F) + i\eta} + \frac{\theta(\epsilon_F - \epsilon_l^{\alpha})}{\omega - (\epsilon_l^{\alpha} - \epsilon_F) - i\eta} \right) \\ & \times \left(\frac{\theta(\epsilon_n^{\alpha} - \epsilon_F)}{\omega - (\epsilon_n^{\alpha} - \epsilon_F) + i\eta} + \frac{\theta(\epsilon_F - \epsilon_n^{\alpha})}{\omega - (\epsilon_n^{\alpha} - \epsilon_F) - i\eta} \right). \end{aligned} \tag{40}$$

The frequency integral can now be evaluated in a straightforward manner. Since at zero temperature only ϵ_0^+ lie below the Fermi energy, only the $\alpha = +$ term in the sum contributes. Further, it is important to notice that the only surviving terms in the frequency integral are the ones where ϵ_n^+ and ϵ_l^+ lie on the opposite sides of the Fermi energy ϵ_F, *i.e.* the poles of the integrand lie on different halves of the complex ω plane. Also, Eq. (40) contains matrix elements of the momentum operator $\mathbf{\Pi}_a = \partial G_0^{-1}/\partial \alpha_a$ [29]. These matrix elements connect different Landau levels, and vanish between the states of the same Landau level. This clearly points out that it is necessary to retain the wave functions for all Landau levels in the calculation. The use of the LLL-projected wave functions in this case would yield an erroneous zero value for N [29,30].

After evaluating the frequency integral, the expression for the integer N becomes

$$N = 2\pi i \sum_{\substack{n \\ \epsilon_n^+ < \epsilon_F}} \sum_{\substack{l \\ \epsilon_l^+ > \epsilon_F}} \frac{\left[\left(\frac{\partial G_0^{+\,-1}}{\partial \phi_x} \right)_{nl} \left(\frac{\partial G_0^{+\,-1}}{\partial \phi_y} \right)_{ln} - \left(\frac{\partial G_0^{+\,-1}}{\partial \phi_x} \right)_{ln} \left(\frac{\partial G_0^{+\,-1}}{\partial \phi_y} \right)_{nl} \right]}{(\epsilon_n^+ - \epsilon_l^+)^2}. \tag{41}$$

Using the relations

$$\left(\frac{\partial G_0^{+\,-1}}{\partial \phi_{x(y)}}\right)_{nl} = -(\epsilon_n^+ - \epsilon_l^+)\langle n|\frac{\partial l}{\partial \phi_{x(y)}}\rangle, \tag{42}$$

this can be further simplified to

$$N = 2\pi i \sum_{\substack{n \\ \epsilon_n^+ < \epsilon_F}} \left(\langle\frac{\partial n}{\partial \phi_x}|\frac{\partial n}{\partial \phi_y}\rangle - \langle\frac{\partial n}{\partial \phi_y}|\frac{\partial n}{\partial \phi_x}\rangle\right), \tag{43}$$

where $|\Phi_0\rangle = \prod_n |n\rangle$ is the unperturbed many-body ground state of the system.

So far, the derivatives in the expression of N are formal, and it is not clear why N has to be an invariant. To see this point more clearly, we now consider the physical meaning of the parameters ϕ_x and ϕ_y. The ground state of the system in this case is a Slater determinant of the single-particle states, which are solutions to Eq. (39). These states are constructed by Niu *et.al.* in a slightly different context [22], and are given by

$$\begin{aligned}
\psi_n(x,y) &= e^{\frac{-i\phi_x x}{L_x}} e^{\frac{-i\phi_y y}{L_y}} W_n(x,y), \\
W_n(x,y) &= \sum_{m=-\infty}^{\infty} e^{i\Lambda_n \beta_m l_B^2} e^{i\beta_m y} u_0(x - \beta_m l_B^2),
\end{aligned} \tag{44}$$

where $\Lambda_n = (2\pi n + \phi_x)/L_x$, $\beta_m = (2\pi m + \phi_y)/L_y$ and u_0 is the ground state wave function for a harmonic oscillator. From this, it is easy to check that ψ_n is the solution to the Hamiltonian H with eigenvalues ϵ_0^+. Furthermore, $W_n(x,y)$ also satisfies the boundary conditions

$$\begin{aligned}
W_n(x + L_x, y) &= e^{i\phi_x} e^{iyL_x/l_B^2} W_n(x,y), \\
W_n(x, y + L_y) &= e^{i\phi_y} W_n(x,y),
\end{aligned} \tag{45}$$

where L_x, L_y are the dimension of the system.

From these relations, it is clear that we can interpret ϕ_x and ϕ_y as the boundary phase parameters of the system. So, variation of these parameters means variation of the boundary conditions for the ground-state wavefunction. Following Niu *et al.* [22], we now argue that the value of the invariant N is independent of the boundary condition chosen, so that we can replace this expression by its average over all possible boundary conditions. This allows us to write

$$N = \frac{1}{2\pi i}\int_0^{2\pi} d\phi_x \int_0^{2\pi} d\phi_y \left(\langle \frac{\partial \Phi_0}{\partial \phi_y} | \frac{\partial \Phi_0}{\partial \phi_x} \rangle - \langle \frac{\partial \Phi_0}{\partial \phi_x} | \frac{\partial \Phi_0}{\partial \phi_y} \rangle \right). \tag{46}$$

It is easy to see from the condition (45) on the single-particle wavefunctions that $(2\pi, 2\pi)$ and (0,0) are same point in the ϕ space. As a result, we can interpret N as a surface integral over a closed surface in the parameter space, which can be expressed as a line integral. This line integral is

$$N = \frac{1}{2\pi i} \oint \mathbf{d}\vec{\phi} \cdot \langle \Phi_0 | \boldsymbol{\nabla}_{\vec{\phi}} \Phi_0 \rangle. \tag{47}$$

Since ϕ_x and ϕ_y are parameters in the Hamiltonian, the line integral in Eq. (47) can be interpreted as the Berry's phase [41] picked up by the ground-state wavefunction as it moves around a closed contour in the parameter space. But since in this case, the ground state is separated by a finite energy gap from the excited states and is non-degenerate, the many-body ground-state wavefunction must return to itself after traversing the contour. So, the value of the line integral must be 2πi times an integer, which immediately tells us that N must be an integer. The value of this integer can be explicitly evaluated by constructing the ground-state wavefunction as a Slater determinant of the single-particle states (44) and by choosing a rectangular contour in the parameter space. The calculation is straightforward [22,42] and for $\nu = 1$, it yields the value $N = 1$.

From the above discussion, it is clear that the above result is not an artifact of the simple model chosen to describe the system. The properties of the many-body ground state that we used to argue that N should be an integer are the presence of a finite gap between the ground state and the other excited states, and the non-degeneracy of the ground-state wavefunction. As long as these conditions are satisfied, the value of invariant N, which can not be a continuous function of the model parameters, must have the same value as that obtained from the calculation based on this simple model. It can therefore be argued that this result is robust against presence of weak disorder ($\omega_c \tau \gg 1$) in the system, which broadens the Landau levels to bands but do not lead to mixing of different Landau levels. A more mathematically rigorous justification of this issue is given by Niu *et al.* [22] and Ishikawa *et al.* in [43] in the context of Hall conductivity in quantum Hall systems.

IV. SPIN, CHARGE, AND GAUGE INVARIANCE

In this section, we compute the spin and charge densities for the skyrmion, as well as its total charge and spin. To do this, for reasons that will become clear later, we first consider the gauge invariance of our effective action. We have mentioned before that the B fields introduced in Eq. (3) are auxiliary gauge fields. The gauge transformation here corresponds to an arbitrary space-time-dependent rotation of the spin-quantization reference frame about the local $\mathbf{d}$ axis, since such a rotation does not change the physical state of the system. It can be easily seen that under such a transformation $U_\Lambda = e^{\frac{i}{2}(\boldsymbol{\sigma}\cdot\mathbf{d})\Lambda(x,y,t)}$, the auxiliary field B_μ transform as

$$B_\mu \to B_\mu + \partial_\mu \Lambda. \tag{48}$$

The action (2), however, is not invariant under this transformation. It acquires an additional surface term

$$\delta S = -\frac{N}{16\pi} \oint dl_a \, dt \, (B_a \partial_0 \Lambda - B_0 \partial_a \Lambda), \tag{49}$$

where dl_a is the length element along the edge, and $\oint$ denotes integration along the edge. From now on, we shall assume a rectangular boundary for simplicity.

Such a non-invariance of the bulk action under gauge transformation is well known for both Abelian and non-Abelian gauge theories with Chern-Simons term in non-compact space [44]. The most well-known example for the Abelian case is the effective action for integer and fractional quantum Hall states: $S_{\text{eff}} = (\sigma_{xy}/4) \int d^2r \, dt \, \epsilon^{\mu\nu\lambda} A^{\text{em}}_\mu F^{\text{em}}_{\nu\lambda}$. This non-invariance is merely a statement that in a space with boundary the bulk effective action is not the complete action for the problem, and we must add the edge action S_{edge}, so that the total action $S_{\text{total}} = S_{\text{bulk}} + S_{\text{edge}}$ is gauge invariant. In case of quantum Hall states, such a consideration leads to gapless edge excitations [45]. As we shall see, in our case, considering the edge action is absolutely crucial for determining the correct value of the contribution of the Hopf term to the total spin of the skyrmion.

The 1+1D edge effective action that we need to construct, must transform under the gauge transformation (48) in such a manner so as to cancel the gauge non-invariant term of the bulk action:

$$\delta S_{\text{edge}} = \frac{N}{16\pi} \oint dl_a \, dt \, (B_a \partial_0 \Lambda - B_0 \partial_a \Lambda). \tag{50}$$

The simplest edge action that achieves this can be written in terms of a boson field η representing the edge excitations [32,44]:

$$S_{\text{edge}} = \oint dl_a \, dt \, \left[(D_t \eta)^2 - \kappa (D_a \eta)^2 \right] + \frac{N}{16\pi} \oint dl_a \, dt \, (B_a \partial_0 \eta - B_0 \partial_a \eta). \tag{51}$$

In Eq. (51), D_μ is the covariant derivative defined as $D_\mu = (\partial_\mu - B_\mu)$, κ is a real parameter, and the boson field η transforms under the gauge transformation (48) as $\eta \to \eta + \Lambda$. It is easy to check that under these conditions the first term of the edge action is invariant under gauge transformation, while the second term produces the correct extra term δS_{edge}, which cancels gauge contribution of the bulk action, so that the total action S_{total} remains gauge invariant.

Next, we proceed to compute the spin density of the skyrmion. The easiest way to do this is to consider the response of the system to arbitrary rotation by an infinitesimal solid angle $\boldsymbol{\theta}(\mathbf{r}, t)$. The action after such a rotation can be written as

$$\begin{aligned} S[\psi, \psi^\dagger, \mathbf{d}, Q_\mu^{\boldsymbol{\theta}}] = \int d^2r \, dt \, \psi_a^\dagger(\mathbf{r}, t) \Bigg(& [i\partial_0 + \epsilon_F] I - Q_0^{\boldsymbol{\theta}} \\ & - \frac{\left\{ [\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r})] I - \mathbf{Q}^{\boldsymbol{\theta}}(\mathbf{r}, t) \right\}^2}{2m} + \gamma_0 \boldsymbol{\sigma} \cdot \mathbf{d} \Bigg)_{ab} \psi_b(\mathbf{r}, t), \end{aligned} \tag{52}$$

where $Q_\mu^{\boldsymbol{\theta}} = -iU_{\boldsymbol{\theta}}^{-1}(\partial_\mu U_{\boldsymbol{\theta}}) = \boldsymbol{\sigma} \cdot \partial_\mu \boldsymbol{\theta}(\mathbf{r}, t)/2$ and $U_{\boldsymbol{\theta}} = e^{\frac{i}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\theta}(\mathbf{r}, t)}$. So, a variation of the effective action with respect to $\partial_0 \boldsymbol{\theta}(\mathbf{r}, t)$ gives the spin $\mathbf{J}$ of the skyrmion:

$$\mathbf{J} = \int d^2r \, dt \, \frac{\delta S_{\text{eff}}}{\delta[\partial_0 \boldsymbol{\theta}(\mathbf{r}, t)]}. \tag{53}$$

To compute the effective action, we again rotate the inhomogeneous $\mathbf{d}$ field to a homogeneous configuration ($\mathbf{d} = \mathbf{e_z}$) and proceed in the same manner as the previous section.

The difference is that now the internal gauge field $Q_\mu^{\rm int}$ is replaced by the total gauge field $Q_\mu^{\rm total} = Q_\mu^{\rm int} + U^{-1}Q_\mu^{\boldsymbol{\theta}}U$. The details of the computations are essentially the same as in the previous section. Specially, the cancelation of the different terms are exactly the same since the field tensor $f_{\mu\nu}$ (3) corresponding to $Q_\mu^{\rm total}$ is zero. After some algebra, it can be seen that the form of both the edge and the bulk effective actions remain the same, with the auxiliary fields B_μ and the scalar field η replaced by

$$\begin{aligned} B_\mu &\to B_\mu + \mathbf{d}\cdot\partial_\mu\boldsymbol{\theta}, \\ F_{\mu\nu} &\to F_{\mu\nu} + \partial_\mu\mathbf{d}\cdot\partial_\nu\boldsymbol{\theta} - \partial_\nu\mathbf{d}\cdot\partial_\mu\boldsymbol{\theta}, \\ \eta &\to \eta + \mathbf{d}(\infty)\cdot\boldsymbol{\theta}, \end{aligned} \tag{54}$$

where $\mathbf{d}(\infty) = \mathbf{e_z}$ is the constant unit vector at the edge. The total action $S_{\rm total}$ now is the sum of the bulk and the edge actions. There are two terms contributing to the spin of the skyrmion in the bulk. The first term is linear in B fields and has the same origin as the Berry term in Eq. (1b), while the second term is the Hopf term. The relevant part of the effective action is

$$\begin{aligned} S_{\rm bulk} &= \frac{\rho_0}{2}\int d^2r\,dt\,(B_0 + \mathbf{d}\cdot\partial_0\boldsymbol{\theta}) + \frac{N}{32\pi}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}\left(B_\mu + \mathbf{d}\cdot\partial_\mu\boldsymbol{\theta}\right)\left(F_{\nu\lambda} + 2\partial_\nu\mathbf{d}\cdot\partial_\lambda\boldsymbol{\theta}\right), \\ S_{\rm edge} &= \frac{N}{16\pi}\oint dl_a\,dt\,\{[\partial_0\eta + \mathbf{d}(\infty)\cdot\partial_0\boldsymbol{\theta}]\,[B_a + \mathbf{d}(\infty)\cdot\partial_a\boldsymbol{\theta}] \\ &\quad - [\partial_a\eta + \mathbf{d}(\infty)\cdot\partial_a\boldsymbol{\theta}]\,[B_0 + \mathbf{d}(\infty)\cdot\partial_0\boldsymbol{\theta}]\}. \end{aligned} \tag{55}$$

The variation of Eq. (55) with respect to $\partial_0\boldsymbol{\theta}(\mathbf{r},t)$ gives the spin of the skyrmion relative to the ferromagnetic ground state:

$$\mathbf{J} = \mathbf{J}^{\rm Berry} + \mathbf{J}^{\rm Hopf} + \mathbf{J}^{\rm edge}, \tag{56a}$$

$$\mathbf{J}^{\rm Berry} = \frac{\rho_0}{2}\int d^2r\,(\mathbf{d} - \mathbf{e}_z), \tag{56b}$$

$$\mathbf{J}^{\rm Hopf} = \frac{N}{16\pi}\int d^2r\,\epsilon^{ab0}\left[\partial_a\mathbf{d}\times\partial_b\mathbf{d} + \partial_a(\mathbf{d}B_b)\right], \tag{56c}$$

$$\mathbf{J}^{\rm edge} = \frac{N\mathbf{d}(\infty)}{16\pi}\oint dl_a\,\epsilon^{ab0}B_b. \tag{56d}$$

The first term in Eq. (56a) comes from the Berry term in the action (55), while the second and the third terms are contributions to the skyrmion spin from the Hopf and the edge terms. We first analyze the latter contribution. The first term in the integrand of $\mathbf{J}^{\rm Hopf}$ (56c) is the contribution to the local spin density of the skyrmion from the Hopf term and is given by $\mathbf{j}_0(\mathbf{r},t) = (N/16\pi)\epsilon^{ab0}\partial_a\mathbf{d}\times\partial_b\mathbf{d}$. This local spin density is directed along the local $\mathbf{d}$ vector and is a total derivative, so it produces no net contribution to the total spin of the skyrmion. The net contribution of the Hopf term to the total spin therefore comes from the second term in $\mathbf{J}^{\rm Hopf}$ and from $\mathbf{J}^{\rm edge}$ (56d). Using the expression for the topological charge $Q_{\rm top} = (1/4\pi)\int d^2r\,\mathbf{d}\cdot(\partial_x\mathbf{d}\times\partial_y\mathbf{d})$ and Eq. (3), it is easy to see that each of these terms contribute equally to the total skyrmion spin. The total contribution to the skyrmion spin from the Hopf term is therefore $N\mathbf{d}(\infty)Q_{\rm top}/2$. It is directed along $\mathbf{d}(\infty) = \mathbf{e}_z$ and for a skyrmion with unit topological charge $Q_{\rm top} = 1$ has the magnitude of $1/2$ at $\nu = 1$. Since the last term in the expression of the spin density comes from $S_{\rm edge}$, we see that it is crucial to include the edge action in the calculation of the contribution of the Hopf term to the total spin. Omitting the edge action would give only half of the actual result.

Next, we consider the contribution to the skyrmion spin from the Berry term (56b). This contribution is clearly proportional to the number of flipped spin in the skyrmionic state, and increases with the size of the skyrmion [12]. So in the case of a long-wavelength skyrmion, the Berry term produces dominant contribution to the skyrmion spin. The exact value of this contribution, unlike the contribution due to the Hopf term, is not a universal number, but depends on the details of the field configuration. So we conclude that the spin of the skyrmion in quantum Hall ferromagnets, as measured as a response to an external field, is not quantized to integral or fractional multiples of $1/2$. This is in contrast to the case of skyrmions in antiferromagnetic systems, which have a universal value of the spin completely determined by the coefficient of the Hopf term.

Finally, we compute the charge density and the total charge of the skyrmion. To do this, we compute the response of the system to an external electromagnetic potential $A_\mu^{\rm em}$. The action, in presence of an external electromagnetic potential can be written as

$$S[\psi,\psi^\dagger,\mathbf{d},A_\mu^{\rm em}]=\int d^2r\,dt\,\psi_a^\dagger(\mathbf{r},t)\Bigg(\left[i\partial_0-eA_0^{\rm em}(\mathbf{r},t)+\epsilon_F\right]I \\ -\frac{\left[\hat{\mathbf{p}}-e\mathbf{A}(\mathbf{r})-e\mathbf{A}^{\rm em}(\mathbf{r},t)\right]^2}{2m}I+\gamma_0\boldsymbol{\sigma}\cdot\mathbf{d}\Bigg)_{ab}\psi_b(\mathbf{r},t), \tag{57}$$

and the charge density ρ of the skyrmion can be obtained by varying the effective action, with respect to $A_0^{\rm em}$.

To calculate the effective action, we follow the procedure of Sec. II noting that $A_\mu^{\rm em}$ is a U(1) gauge field added to the SU(2) gauge field $\mathbf{Q}_\mu^{\rm int}$. Taking into account that the unperturbed Green function G_0 is diagonal in spin space, we find that any term in S_2 of the form $A_\mu^{\rm em}\partial_\nu\Omega_\lambda^{1(2)\,\rm int}$ vanishes due to the trace operation in spin indices. Similarly, it can be shown that all terms involving product of $A_\mu^{\rm em}$ and $\Omega^{\rm int}$ in S_3 also vanishes. So the cancelation of the terms in S_2 and S_3 follows exactly in the same way as in Sec.II, and finally we are left with the effective action

$$S_{\rm eff}[B_\mu,A_\mu^{\rm em}]=\frac{N}{32\pi}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda}+\frac{N_1e}{8\pi}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda}^{\rm em} \\ +\frac{Ne^2}{8\pi}\int d^2r\,dt\epsilon^{\mu\nu\lambda}A_\mu^{\rm em}F_{\nu\lambda}^{\rm em}, \tag{58}$$

where the coefficient N_1 is given by

$$N_1=2\pi{\rm Tr}\left(\sigma_3\frac{\partial G_0^{-1}}{\partial\omega}G_0\frac{\partial G_0^{-1}}{\partial\phi_x}G_0\frac{\partial G_0^{-1}}{\partial\phi_y}G_0\right). \tag{59}$$

Since the unperturbed ground state is completely spin polarized, it can be easily seen that $N_1=N$. The variation with respect to $A_0^{\rm em}$ now gives the charge density ρ

$$\rho(\mathbf{r},t)=\left(\frac{\delta S_{\rm eff}}{\delta A_0^{\rm em}}\right)_{A_\mu^{\rm em}=0}=\frac{Ne}{4\pi}\mathbf{d}\cdot(\partial_x\mathbf{d}\times\partial_y\mathbf{d}). \tag{60}$$

The electric charge density (60) is found to be proportional to the topological charge density. The space integral of this charge density gives the total charge of the skyrmion, given by

$$Q_{\rm em}=\frac{Ne}{4\pi}\int d^2r\mathbf{d}\cdot(\partial_x\mathbf{d}\times\partial_y\mathbf{d})=NeQ_{\rm top}. \tag{61}$$

For $\nu=1$, the skyrmion with $Q_{\rm top}=1$ therefore carries a charge e. For fractional filling, as we shall see in Sec. VI, the prefactor N has fractional value leading to fractional charge for the skyrmion.

V. STATISTICS

In this section, we outline the argument of Wilczek and Zee [15] to obtain spin, charge and statistics of skyrmions in quantum Hall ferromagnets. For this purpose, let us recall that the effective action of a skyrmion in quantum Hall ferromagnets is given by Eqs. (1a) and (2). First, we compute the z component of the spin of the skyrmion. For this purpose, we introduce an adiabatic time evolution which corresponds to a rotation of the $\mathbf{d}$ field in the XY plane about the center of the skyrmion, such that the wavefunction returns to itself after time T. The phase acquired by the skyrmion wavefunction ψ in the process is

$$\psi(\mathbf{r},T) = e^{i\int_0^T dt \int d^2r (L_{\rm Berry}+L_{\rm E}+L_{\rm Hopf})}\psi(\mathbf{r},0), \tag{62}$$

where $L_{\rm Berry} = \rho_0 B_0/2$, $L_E = -\kappa^2 \left(\nabla \mathbf{d}\right)^2$ and $L_{\rm Hopf} = N/32\pi\, \epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda}$ are the Lagrangians corresponding to $S_{\rm FM}$ (1a) and $S_{\rm Hopf}$ (2). But such a time evolution also corresponds to rotation of the skyrmion spin by 2π around the z axis in the spin-space [15,46]. So the phase picked up by the wavefunction in this process must be $\exp(2\pi i J_z)$, where J_z is the z component of the skyrmion spin. So J_z must be given by

$$J_z = \frac{1}{2\pi}\int_0^T dt \int d^2r (L_{\rm Berry} + L_{\rm E} + L_{\rm Hopf}). \tag{63}$$

To compute the phase picked up by the skyrmion wave-function, we now make a specific choice for the $\mathbf{d}$ field configuration [46]:

$$\mathbf{d}(\mathbf{r},t) = \cos[\gamma(|\mathbf{r}|)]\mathbf{e}_z + \sin[\gamma(|\mathbf{r}|)]\{\cos[\phi+\phi_0(t)]\mathbf{e}_r + \sin[\phi+\phi_0(t)]\mathbf{e}_\phi\}, \tag{64}$$

where ϕ is the azimuthal angle, $\cos[\gamma(|\mathbf{r}|)] = (r^2-\Lambda^2)/(r^2+\Lambda^2)$, Λ is a parameter depending on the skyrmion radius and $\phi(t)$ is chosen to be $2\pi t/T$ to ensure that the $\mathbf{d}$ field configuration returns to itself after a time T. With this choice of $\mathbf{d}$ field configuration, $Q_{\rm top} = 1$ and the fields B_μ can be represented as

$$B_\mu = \{1+\cos[\gamma(|\mathbf{r}|)]\}\partial_\mu(\phi+\phi_0(t)). \tag{65}$$

Using Eqs. (62) and (65), it can be seen that the phase picked up by the skyrmion has three distinct contribution. The contribution from the second term L_E gives the usual factor

$\exp(iET)$, where E is the skyrmion energy. The other two contributions to the phase comes from $L_{\rm Berry}$ and $L_{\rm Hopf}$ which are given by

$$\theta^s_{\rm Berry} = 2\pi\frac{\rho_0}{2}\int d^2r\,\{1+\cos[\gamma(|\mathbf{r}|)]\},$$
$$\theta^s_{\rm Hopf} = 2\pi\frac{N}{8\pi}\int d^2r\,\{1+\cos[\gamma(|\mathbf{r}|)]\}F_{xy}(\mathbf{r}), \tag{66}$$

where $F_{xy}(\mathbf{r}) = \nabla\times\mathbf{B}(\mathbf{r})$. From Eqs. (62), (63) and (66), we get the expression for the z component of the spin of the skyrmion, over and above the ferromagnetic background, to be

$$J_z = \frac{\rho_0}{2}\int d^2r\,\{\cos[\gamma(|\mathbf{r}|)]-1\} + \frac{N}{8\pi}\int d^2r\,\{1+\cos[\gamma(|\mathbf{r}|)]\}F_{xy}(\mathbf{r}). \tag{67}$$

The first term in the expression for the spin, represents the contribution of the Berry term to the skyrmion spin and its contribution can be seen to be same as Eq. (56b). The second term comes from the Hopf term in the action. The contribution of the second term can be directly evaluated by computing F_{xy} using the expression of B_μ in Eq. (65). After some algebra, one obtain this contribution to be $N/2$ and so the spin is given by

$$J_z = \frac{\rho_0}{2}\int d^2r\;[(\mathbf{d}-\mathbf{e}_z)\cdot\mathbf{e}_z] + \frac{N}{2}, \tag{68}$$

which agrees to the expression of the spin obtained in the last section. It is to be noted that the Berry term in the effective action of ferromagnets is linear in time derivative. So its contribution to the phase and hence to the skyrmion spin does not vanish in the adiabatic limit. In contrast, the effective action of antiferromagnets [13,14] has a term quadratic in time derivative. The contribution of this term vanishes as $1/T$ in the adiabatic limit [15], and hence the only contribution to the spin of the skyrmions in antiferromagnetic systems comes from the Hopf term.

Next, we compute the charge of the skyrmion. To do this, we rotate the skyrmion adiabatically around a circle of radius R_0 in the XY plane and compute the phase picked up by the skyrmion. Since the skyrmion is a charged object and we have a magnetic field $\mathbf{H}_0$ perpendicular to the plane, the phased picked up by the skyrmion wavefunction must be given by $\exp(ie^*\Phi/\Phi_0)$, where Φ is the magnetic flux through the circle, e^* is the skyrmion

charge, and $\Phi_0 = \pi/e$ is the flux quantum. On the other hand, this rotation can again be thought as an adiabatic time evolution of the skyrmion wavefunction. So using Eq. (62), we can obtain an expression for the charge of the skyrmion to be

$$e^* = \frac{\Phi_0}{\Phi} \int_0^T dt \int d^2r (L_{\text{Berry}} + L_{\text{Hopf}}), \tag{69}$$

where $\mathbf{r}$ denotes coordinates as measured from the center of the circle.

To evaluate the integral in Eq. (69), we first rewrite the expression of the B_μ fields in terms of complex coordinates as

$$B_\mu = \{1 + \cos[\gamma(|z - Z_0(t)|)]\} \partial_\mu \left[\ln(z - Z_0(t)) - \ln(|z - Z_0(t)|)\right]. \tag{70}$$

where $z = x + iy$, $Z(t) = X_0 + iY_0$ represent the coordinates of the center of the skyrmion and the time evolution is such that the skyrmion rotates once around the circle in time T, *i.e.* $Z(T) = Z(0)$. Then the contribution of the Berry term to the phase is

$$\begin{aligned} \theta^c_{\text{Berry}} &= \frac{\rho_0}{2} \int d^2z \int_0^T \{1 + \cos[\gamma(|z - Z_0(t)|)]\} \partial_t \ln(z - Z_0(t)) \\ &= \rho_0 i \int d^2z \oint dZ_0 \frac{1}{z - Z_0(t)} + \frac{\rho_0}{2} i \int d^2z \int_0^T \{\cos[\gamma(|z - Z_0(t)|)] - 1\} \frac{\partial_t Z_0(t)}{z - Z_0(t)}. \end{aligned} \tag{71}$$

The evaluation of the first term on the right-hand side of Eq. (71) is straightforward and gives $2\pi^2 R_0^2 \rho_0$. To evaluate the second term, we interchange the order of integration, and shift the integration variable to $z' = z - Z_0$. Then the space integral can be written as $\int d|z'| \{\cos[\gamma(|z'|)] - 1\} \int d\phi' \exp(-i\phi')$. Since the integrand $\{\cos[\gamma(|z'|)] - 1\}$ vanishes at infinity, the z' integral converges, and hence integration over ϕ' yields a zero result. So, the net contribution from the Berry term to the charge comes from the first term. The contribution from the Hopf term to the phase can be written as

$$\theta^c_{\text{Hopf}} = \frac{N}{16\pi} \int d^2z \int_0^T \{1 + \cos[\gamma(|z - Z_0(t)|)]\} F_{xy}(|z - Z_0(t)|) \frac{1}{z - Z_0(t)} \partial_t Z_0(t). \tag{72}$$

Since F_{xy} is a localized function which vanishes at infinity, it can be shown, following similar logic as in the case of the Berry term, that the contribution of the Hopf term to the phase vanishes. So we finally get

$$e^* = \frac{\Phi_0}{\Phi} 2\pi^2 R_0^2 \rho_0 = Ne \tag{73}$$

where in the last step we have used the relation $\rho_0 = N/2\pi l_B^2$. This result agrees to that of Yang and Sondhi [33], obtained by explicitly computing the Berry phase using a variational wavefunction.

It is to be noted, that the contribution to the charge of the skyrmion, within this prescription, comes from the Berry term in the action. On the other hand, when we calculated the charge of the skyrmion as a response of the effective action to an external electromagnetic field in Sec.IV, we found that the charge of the skyrmion is a consequence of the coupling of the Hopf term to the external electromagnetic field. This apparent contradiction can be resolved in the following way. Skyrmions in quantum Hall ferromagnets have a non-zero electric charge and consequently couple to electromagnetic fields. Now, let us consider the effective action of a skyrmion in presence of a constant magnetic field H_0 in the z direction. The coupling of the Hopf term in the effective action to this field would give us a term

$$\begin{aligned} S_{\rm em} &= \frac{Ne}{8\pi} \int d^2r\, dt\, \epsilon^{\mu\nu\lambda} B_\mu F^{\rm em}_{\nu\lambda}, \\ &= \frac{NeH_0}{4\pi} \int d^2r\, dt\, B_0 = \frac{\rho_0}{2} \int d^2r\, dt\, B_0 = S_{\rm Berry} \end{aligned} \tag{74}$$

So, we see that the Berry term in the effective action of skyrmions in quantum Hall ferromagnets is a consequence of the non-zero electric charge of the skyrmions and the presence of a constant magnetic field H_0 in the system. Therefore it is not surprising that the contribution to the charge, obtained by rotating the skyrmion, comes from the Berry term. It is to be noted that in antiferromagnetic system like He^3-A films [17], the skyrmions are electrically neutral and hence we do not have a Berry term in their effective action.

Finally, we compute the statistical phase picked up due to exchange of two skyrmions. To calculate this statistical phase, the first skyrmion is adiabatically rotated around a circle of radius R_0 with the second skyrmion at the center. The statistical phase is then defined as the *additional* phase picked up by the wave-function of the first skyrmion due to presence of the second skyrmion at the center. The radius of the circle is chosen to be much larger than

radii of both the skyrmions, so that these skyrmions can be assumed to be non-interacting. In this case, the total $\mathbf{d}$ field due to the skyrmions are given by $\mathbf{d} = \mathbf{d}^1 + \mathbf{d}^2 - \mathbf{z}$, where superscripts 1 and 2 refer to the first and the second skyrmion respectively. Consequently the Lagrangian of the two-skyrmion system can be shown to be given by

$$\begin{aligned} L &= L_{\text{Berry}} + L_{\text{Hopf}}, \\ L_{\text{Berry}} &= \frac{\rho_0}{2}[B_0^1(\mathbf{r}-\mathbf{R}_0) + B_0^2(\mathbf{r})], \\ L_{\text{Hopf}} &= \frac{N}{32\pi}\epsilon^{\mu\nu\lambda}[B_\mu^1(\mathbf{r}-\mathbf{R}_0) + B_\mu^2(\mathbf{r})][F_{\nu\lambda}^1(\mathbf{r}-\mathbf{R}_0) + F_{\nu\lambda}^2(\mathbf{r})]. \end{aligned} \tag{75}$$

From the Lagrangian, it is clear that the additional phase appears due to the cross term $N/32\pi\,\epsilon^{\mu\nu\lambda}B_\mu^1(\mathbf{r}-\mathbf{R}_0)F_{\nu\lambda}^2(\mathbf{r})$ in the Lagrangian. The Berry term, being linear in B fields, do not contribute to the statistical phase. The contribution to the statistical phase therefore comes from the Hopf term and is given by

$$\theta_{\text{stat}} = \frac{Ni}{16\pi}\int d^2z\{1+\cos[\gamma(|z-Z_0(t)|)]\}F_{xy}^2(z)\oint\frac{1}{z-Z_0(t)}. \tag{76}$$

Since the integrand in Eq. (76) is non-zero only within the radius of the second skyrmion at the center, where $\{1+\cos[\gamma(|z-Z_0(t)|)]\}$ is a constant, the contour integral can be easily evaluated and we finally get $\theta_{\text{stat}} = N\pi$. The statistics of the skyrmion is, therefore, completely determined by the coefficient of the Hopf term. Consequently, the skyrmions would be Fermions for odd integer fillings and, as we shall see in the next section, anyons for fractional filling factors $\nu = 1/(2s+1)$. Our result for the skyrmion statistics here agrees to that of Yang and Sondhi [33], but contradicts that of Dziarmaga [34].

The spin and the statistics obtained for the skyrmion is apparently not in accordance with the spin-statistics theorem. However, it must be noted that the skyrmions in quantum Hall ferromagnets move against a non-trivial spin background. As a result, measurement of skyrmion spin, as a response to the external magnetic field, always have a contribution from the background. This renders the definition of the spin quantum number for the skyrmions somewhat arbitrary. This situation is to be contrasted with the skyrmion in antiferromagnetic systems like He^3-A films, where the ground state has no net spin and the

skyrmion has a well defined spin quantum number in accordance with the spin-statistics theorem.

VI. FRACTIONAL FILLING

The results of the previous section can be generalized to the case of fractional fillings. This is most easily done within the framework of Chern-Simons (CS) theory for fractional quantum Hall states [47,48]. Within this framework, the electrons, apart from their mutual interaction, are also coupled to a CS gauge field and the coupling constant of the CS field is chosen so that an even number of flux quanta gets attached to each electron. The resultant composite Fermions then fill up an integer number of Landau level at the mean field level. Thus fractional quantum Hall effect for the electrons turns out to be integer quantum Hall effect for the composite Fermions. In this work, we shall consider only those filling fractions where the unperturbed ground state is known to be fully spin-polarized. Such filling fractions are given by $\nu = 1/(1+2s)$ [49] where s is an integer, $2s$ flux quanta are attached to each electron and the resultant composite Fermions fill up one Landau level. In what follows, we neglect the correlation effects between the composite Fermions.

We shall now generalize the results of the earlier section. The procedure is mostly similar to that of the earlier sections, and we shall only point out the essential differences. We start with the mean field effective action (7) which now reads

$$S[\psi^\dagger,\psi,A^{\rm cs}_\mu] = \int d^2r\,dt\,\psi^\dagger_a(\mathbf{r},t)\Bigg([i\partial_0 - A^{\rm cs}_0(\mathbf{r},t)+\epsilon_F]I - \frac{\left[\hat{\mathbf{p}}-\mathbf{A}^{\rm eff}(\mathbf{r})-\mathbf{A}^{\rm cs}\right]^2}{2m}I \\ +\gamma_0\mathbf{d}(\mathbf{r},t)\cdot\boldsymbol{\sigma}\Bigg)_{ab}\psi_b(\mathbf{r},t) + \frac{1}{16\pi s}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}A^{\rm cs}_\mu(\mathbf{r},t)F^{\rm cs}_{\nu\lambda}(\mathbf{r},t), \qquad (77)$$

where ψ are two component composite Fermion spinor fields. Here $\mathbf{A}^{\rm cs}$ means fluctuations of the CS fields from the mean value $\langle\mathbf{A}^{\rm cs}\rangle$. The effective vector potential $\mathbf{A}^{\rm eff}$ is then given by $\mathbf{A}^{\rm eff} = e\mathbf{A} + \langle\mathbf{A}^{\rm cs}\rangle$ and the last term in the action is the usual CS term with $F^{\rm cs}_{\mu\nu} = \partial_\mu A^{\rm cs}_\nu - \partial_\nu A^{\rm cs}_\mu$. The effective magnetic field experienced by the composite Fermions are $\mathbf{H}_{\rm eff} = \mathbf{H_0}/(1+2s)$.

We now introduce SU(2) rotation matrix $U(\mathbf{r},t)$ and the parameters ϕ_x and ϕ_y exactly in the same fashion. After these transformations the action $S[\chi^\dagger,\chi,Q_\mu,A_\mu^{\rm cs}]$ can be written as

$$\begin{aligned}
S &= S_0 + S_1 + S_{\rm cs},\\
S_0 &= \int d^2r\,\frac{d\omega}{2\pi}\,\chi_a^\dagger(\mathbf{r},\omega)\left((\omega+\epsilon_F)I - \frac{[\hat{\mathbf{p}} - e\mathbf{A}^{\rm eff}(\mathbf{r}) + \boldsymbol{\alpha}]^2}{2m}I + \gamma_0\sigma_z\right)_{ab}\chi_b(\mathbf{r},\omega),\\
S_1 &= -\int d^2r\,\frac{d\omega dp_0}{(2\pi)^2}\,\chi_a^\dagger(\mathbf{r},\omega+p_0)\,\left(\frac{1}{2}\left[(Q_\mu^{\rm int} + A_\mu^{\rm cs} I), \frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right]_+\right)_{ab}\chi_b(\mathbf{r},\omega),\\
S_{\rm cs} &= \frac{1}{16\pi s}\int d^2\mathbf{r}\,dt\,\epsilon^{\mu\nu\lambda}A_\mu^{\rm cs}(\mathbf{r},t)F_{\nu\lambda}^{\rm cs}(\mathbf{r},t).
\end{aligned}\tag{78}$$

So, we find that the effect of fluctuations of the CS gauge fields is to modify the basic interaction vertex S_1 by changing $Q_\mu^{\rm int}$ to $Q_\mu^{\rm eff} = Q_\mu^{\rm int} + A_\mu^{\rm cs} I$. To obtain the effective action for the skyrmion, we therefore integrate out the Fermion fields and expand in powers of $Q_\mu^{\rm eff}$. The relevant terms contributing to the Hopf term are again given by the same diagrams in Fig. 1 with $\mathbf{Q}^{\rm int}$ replaced by $\mathbf{Q}^{\rm eff}$. The crucial point which makes the calculation of these relevant terms simple is that the unperturbed Green functions are diagonal in the spin space. As a result, any term in S_2 which has the form $A_\mu^{\rm cs}\partial_\nu\Omega_\lambda^{1(2)\,\rm int}$ vanishes due to the trace operation in spin indices. Similarly, it can be shown that all terms involving product of $A_\mu^{\rm cs}$ and $\Omega^{\rm int}$ in S_3 also vanishes. So, the cancelation of the terms in S_2 and S_3 follows exactly in the same way as in Sec.II, and finally we are left with the effective action

$$\begin{aligned}
S_{\rm eff}[B_\mu, A_\mu^{\rm cs}] = &\frac{N_2}{32\pi}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} + \frac{N_3}{8\pi}\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}A_\mu^{\rm cs}F_{\nu\lambda}\\
&+(\frac{N_2}{8\pi} + \frac{1}{16\pi s})\int d^2r\,dt\,\epsilon^{\mu\nu\lambda}A_\mu^{\rm cs}F_{\nu\lambda}^{\rm cs},
\end{aligned}\tag{79}$$

where the coefficients N_2 and N_3 are given by

$$\begin{aligned}
N_2 &= 2\pi{\rm Tr}\left(\frac{\partial G_0^{-1}}{\partial\omega}G_0\frac{\partial G_0^{-1}}{\partial\phi_x}G_0\frac{\partial G_0^{-1}}{\partial\phi_y}G_0\right),\\
N_3 &= 2\pi{\rm Tr}\left(\sigma_3\frac{\partial G_0^{-1}}{\partial\omega}G_0\frac{\partial G_0^{-1}}{\partial\phi_x}G_0\frac{\partial G_0^{-1}}{\partial\phi_y}G_0\right).
\end{aligned}\tag{80}$$

Since the unperturbed ground state for the composite Fermions is completely spin polarized, it can be easily seen that $N_2 = N_3$. Also since the composite Fermions fill up exactly

one effective Landau level for the considered filling fractions, the arguments of Sec.III can again be carried through to get $N_2 = N_3 = 1$. Using these results and integrating out the CS gauge fields [48], we finally get

$$S_{\rm Hopf} = \frac{\nu}{32\pi} \int d^2r\, dt\, \epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda}, \tag{81}$$

where $\nu = 1/(1+2s)$ is the filling fraction.

So, all the results of the previous sections can be generalized by replacing the value of N by ν. The skyrmions are anyons carrying a charge $e\nu Q_{\rm top}$. This coincides with the results obtained by Baez *et al*. [32] using a phenomenological approach.

There are a couple of points that are worth mentioning here. First, the same results may be obtained using the formalism of spin-allowed CS theory [49]. The calculations follow exactly the same lines for the filing fractions discussed here, and are therefore not repeated. Second, although the theoretical derivations are exactly the same as in the case of integer filling, fractional fillings are achieved at much stronger external fields which means that experimental observation of skyrmions for fractional fillings will require samples with very low Lande g factor. Recently, however, Leadly *et al*. [11] have observed evidence of spin depolarization near $\nu = 1/3$ by reducing the Lande g factor of the sample by applying external pressure.

VII. FINITE TEMPERATURE

In this section, we derive the effective action for the skyrmions at finite temperature at $\nu = 1$. We shall also assume that disorder is negligible *i.e.* $\omega_c\tau \gg 1$ and the temperature $T = 1/\beta$ is small enough so that one can neglect phonons. Throughout this section, we shall use the Matsubara formalism. We use natural units $\hbar = c = k_B = 1$ and all other conventions listed in Sec. III remain the same except for the replacement $t \to -i\tau$.

The action for the system can be written using the same model of local interaction as in Sec II (7,8), but with $i\int dt$ replaced by $-\int_0^\beta d\tau$. Following steps identical to those in Sec.II, we then integrate out the Fermionic fields to obtain the effective action given by

$$
\begin{aligned}
S_{\rm eff} &= S_2 + S_3, \\
S_2 &= \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\,\frac{1}{\beta}\sum_{ip_n}\Big[\Omega_\mu^{1\,\rm int}(\mathbf{p},ip_n)P_1^{\mu\nu}(\mathbf{p},ip_n)\Omega_\nu^{1\,\rm int}(-\mathbf{p},-ip_n) \\
&\quad +\Omega_\mu^{2\,\rm int}(\mathbf{p},ip_n)P_2^{\mu\nu}(\mathbf{p},ip_n)\Omega_\nu^{2\,\rm int}(-\mathbf{p},-ip_n)+\Omega_\mu^{3\,\rm int}(\mathbf{p},ip_n)P_3^{\mu\nu}(\mathbf{p},ip_n)\Omega_\nu^{3\,\rm int}(-\mathbf{p},-ip_n)\Big] \\
S_3 &= \int \frac{d^2p\,d^2q\,d^2l}{(2\pi)^6}\,\frac{1}{\beta^2}\sum_{ip_n,iq_n} T_{123}^{\mu\nu\lambda}(\mathbf{p},\mathbf{q},ip_n,iq_n)\Omega_\mu^{1\,\rm int}(\mathbf{p},ip_n)\Omega_\nu^{2\,\rm int}(\mathbf{q},iq_n) \\
&\quad \times\Omega_\lambda^{3\,\rm int}\left(-\mathbf{p}-\mathbf{q}\right),-ip_n-iq_n)\,, \qquad (82)
\end{aligned}
$$

where p_n and q_n are the Matsubara frequencies and $P_i^{\mu\nu}$ and $T_{123}^{\mu\nu\lambda}$ can be expressed in terms of the unperturbed Green functions G_0 as

$$
\begin{aligned}
P_i^{\mu\nu}(\mathbf{p},ip_n) &= \frac{-1}{4L_xL_y}\mathrm{Tr}\left(\int d^2r_1d^2r_2\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{2}\left[\sigma_i e^{i\mathbf{p}\cdot\mathbf{r_1}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+ G_0(\mathbf{r_1},\mathbf{r_2},i\omega_n)\right. \\
&\quad \left.\times\frac{1}{2}\left[\sigma_i e^{-i\mathbf{p}\cdot\mathbf{r_2}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2},\mathbf{r_1},i\omega_n+ip_n)\right), \\
T_{123}^{\mu\nu\lambda}(\mathbf{p},\mathbf{q},ip_n,iq_n) &= \frac{i}{4L_xL_y}\mathrm{Tr}\left(\int d^2r_1d^2r_2d^2r_3\frac{1}{\beta}\sum_{i\omega_n}\frac{1}{2}\left[\sigma_1 e^{i\mathbf{p}\cdot\mathbf{r_1}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+\right. \\
&\quad \times G_0(\mathbf{r_1},\mathbf{r_2},i\omega_n)\frac{1}{2}\left[\sigma_2 e^{i\mathbf{q}\cdot\mathbf{r_2}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2},\mathbf{r_3},i\omega_n+iq_n) \\
&\quad \left.\times\frac{1}{2}\left[\sigma_3 e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r_3}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}\right)_{\mathbf{r}_3}\right]_+ G_0\left(\mathbf{r_3},\mathbf{r_1},i\omega_n-ip_n\right)\right). \qquad (83)
\end{aligned}
$$

First, we consider the terms in the effective action which contain $T_{123}^{\mu\nu\lambda}$, $P_1^{\mu\nu}$ and $P_2^{\mu\nu}$. It can be easily shown that each of these terms are analytic functions of $\mathbf{p}$ and p_0. The reason for this analyticity is simple. Each of these functions contain a product of two or three Green functions in combination of $G_0^{\pm}G_0^{\mp}$ or $G_0^{\pm}G_0^{\pm}G_0^{\mp}$. As a result, the energy eigenvalues in the denominator of these Green functions are different even when their Landau level indices are the same and this restores the analyticity of these functions. Since these functions are analytic, we can carry out a derivative expansion similar to the zero temperature case. This leads to cancelation of the first two terms of S_2 and S_3 up to the required order.

The procedure for obtaining the effective action has been, till now, similar to that in the zero temperature case. However, at this stage, we come across an important difference. At finite temperature, $P_3^{\mu\nu}$ turns out to be a non-analytic function of frequency and momenta,

so that we can not carry out a derivative expansion analogous to the zero temperature case to obtain the Hopf term. This non-analyticity of the polarization bubble is well known in finite temperature field theory and many-body theory [35–37]. To see exactly where this non-analyticity appear, we compute $P_3^{\mu\nu}(\mathbf{p}, p_0)$. The method of this computation is sketched in App.C. The result is

$$\begin{aligned}
P_3^{00}(\mathbf{p}, p_0) &= |\mathbf{p}|^2 \Pi_0(\mathbf{p}, p_0), \\
P_3^{i0}(\mathbf{p}, p_0) &= p_i p_0 \Pi_0(\mathbf{p}, p_0) + i\epsilon^{i0j} p_j \Pi_1(\mathbf{p}, p_0), \\
P_3^{ij}(\mathbf{p}, p_0) &= p_0^2 \Pi_0(\mathbf{p}, p_0) + i\epsilon^{ij0} p_0 \Pi_1(\mathbf{p}, p_0) \\
&\quad + (|\mathbf{p}|^2 \delta_{ij} - p_i p_j) \Pi_2(\mathbf{p}, p_0),
\end{aligned} \tag{84}$$

where the functions Π_0, Π_1 and Π_2 are given by

$$\begin{aligned}
\Pi_0(\mathbf{p}, p_0) &= \frac{-l_B^2}{4} \sum_{n=0}^{\infty} \sum_{n'=0}^{n} \sum_{\alpha=\pm} C_{nn'}^{\alpha}(p_0) \frac{n'!}{n!} e^{\frac{-|\mathbf{p}|^2 l_B^2}{2}} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right)^{n-n'-1} L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right), \\
\Pi_1(\mathbf{p}, p_0) &= \frac{-l_B \omega_c^{\frac{1}{2}}}{8\sqrt{m}} \sum_{n=0}^{\infty} \sum_{n'=0}^{n} \sum_{\alpha=\pm} C_{nn'}^{\alpha}(p_0) \frac{n'!}{n!} e^{\frac{-|\mathbf{p}|^2 l_B^2}{2}} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right)^{n-n'-1} L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \\
&\quad \times \left[\frac{|\mathbf{p}|^2 l_B^2}{2} \left\{ L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) + 2 L_{n'-1}^{n-n'+1} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \right\} - (n-n') L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \right], \\
\Pi_2(\mathbf{p}, p_0) &= \frac{-l_B^2}{16m} \sum_{n=0}^{\infty} \sum_{n'=0}^{n} \sum_{\alpha=\pm} C_{nn'}^{\alpha}(p_0) \frac{n'!}{n!} e^{\frac{-|\mathbf{p}|^2 l_B^2}{2}} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right)^{n-n'-1} \\
&\quad \times \left\{ L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) + 2 L_{n'-1}^{n-n'+1} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \right\} \\
&\quad \times \left[\frac{|\mathbf{p}|^2 l_B^2}{2} \left\{ L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) + 2 L_{n'-1}^{n-n'+1} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \right\} - (n-n') L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right) \right],
\end{aligned} \tag{85}$$

where n and n' are Landau level indices, f is the Fermi distribution function, f_n^{α} is shorthand notation for $f(\epsilon_n^{\alpha})$, ϵ_n^{α} are the energy eigenvalues given by $\epsilon_n^{\alpha} = (n + \frac{1}{2})\omega_c + \alpha\gamma_0$, $L_{n'}^{n-n'}$ is the associated Laugurre polynomial and $C_{nn'}^{\alpha}$ is given by

$$\begin{aligned}
C_{nn'}^{\alpha}(p_0) &= \left(\frac{(f_n^{\alpha} - f_{n'}^{\alpha})\omega_c (n-n')}{p_0^2 - \omega_c^2 (n-n')^2} \right) \quad \text{if } p_0 \neq 0, \\
&= -\delta_{nn'} \left(\frac{\partial f(z)}{\partial z} \right)_{z=\epsilon_n^{\alpha}} \quad \text{if } p_0 = 0.
\end{aligned} \tag{86}$$

From Eq. (85), it can be shown that the functions Π_1 and Π_2 are non-analytic at the origin. The value of these functions in the limit $\mathbf{p} \to 0$ depends on whether p_0 is set to zero before or after the limit is taken. This non -analytic nature arises from the contribution of the $n = n'$ term in the expression of $C^{\alpha}_{nn'}(p_0)$. Further, Π_0, Π_1 and Π_2 do not either diverge or approach zero as $\mathbf{p}$ and $p_0 \to 0$ irrespective of how the limit $\mathbf{p} \to 0, p_0 \to 0$ is taken. So, it can be inferred from Eqs. (82), (84) and (85) that in the long-wavelength low-frequency limit, the relevant leading order term in the effective action is given by

$$S_{\text{eff}} = \frac{1}{2} \int \frac{d^2p\, dp_0}{(2\pi)^3}\, \epsilon^{\mu\nu\lambda} B^{\text{int}}_{\mu}(\mathbf{p}, p_0) i p_\lambda \Pi_1(\mathbf{p}, p_0) B^{\text{int}}_{\nu}(-\mathbf{p}, -p_0), \tag{87}$$

where we have expressed $\Omega^{3\,\text{int}}_{\mu}$ fields in terms of the gauge fields B^{int}_{μ} using Eq. (13). If we could have taken an unambiguous $\mathbf{p} \to 0, p_0 \to 0$ limit for Π_1, Eq. (87) would reduce to a local effective action in position space similar Eq. (2). However, such a limit does not exist and in general it is not possible to obtain a simple effective action in position space. Nevertheless, as shown above, it is still possible to obtain a rather simple effective action in momentum space.

Although, it is not possible to take an unambiguous zero frequency zero momentum limit of the effective action (87), one can still infer the fate of the prefactor N at finite temperature from physical considerations. To elucidate this point, let us consider the Hall conductivity (which is the Chern number) in these systems at finite temperature and in absence of disorder. If we attempt to compute the term in the effective action giving rise to the Hall conductivity, we find that we obtain an identical non-analytic polarization tensor and the effective action is again given by

$$S^{\text{Hall}}_{\text{eff}} = \frac{e^2}{2} \int \frac{d^2p\, dp_0}{(2\pi)^3}\, \epsilon^{\mu\nu\lambda} A^{\text{em}}_{\mu}(\mathbf{p}, p_0) i p_\lambda \Pi_1(\mathbf{p}, p_0) A^{\text{em}}_{\nu}(-\mathbf{p}, -p_0). \tag{88}$$

To obtain the d.c Hall conductivity as a response of the system to an external field from this effective action, we must now specify the dc limit. There are two different limits possible. In the dynamic limit, we measure the response of the system to a spatially homogeneous but very slowly time varying external field. In this case, to compute the response, we first set

$\mathbf{p} = 0$ and then take the limit $p_0 \to 0$. On the other hand, in the static limit, we measure the response of the system with respect to a static but weakly spatially inhomogeneous field and consequently first set $p_0 = 0$ and then take the $\mathbf{p} \to 0$ limit. It is well known that at finite temperature the conductivities in the static and the dynamic limits are different [50–52].

Similarly, the prefactor of the Hopf term, which contributes in part to the spin of the skyrmion, can be looked upon as the response of the system to external magnetic field [17]. In particular, part of the z component of the spin ($J_z^{\rm Hopf}$) can be obtained from the Hopf term as a response to an external magnetic field in the z direction. It can be easily seen, that the Hopf term in the effective action of the system with the external magnetic field ($H_z^{\rm ext}$) is given by Eq. (87) with $B_\mu^{\rm int}$ replaced by $B_\mu^{\rm total} = B_\mu^{\rm int} + H_z^{\rm ext}$. The contribution from the Hopf term to the z component of the spin of the skyrmion can then be computed from this action as a response of the system to the external magnetic field $H_z^{\rm ext}$. To compute this, we must again specify the dc limit. In the dynamic limit, relevant for spatially homogeneous magnetic field with weak temporal variation, a simple calculation leads to $J_z^{\rm Hopf} = N_{\rm dynamic}/2$, where

$$N_{\rm dynamic} = 16\pi \lim_{p_0 \to 0} \Pi_1(0, p_0) = N = 1. \tag{89}$$

The prefactor of the Hopf term remain the same as the zero temperature value in the dynamic limit. In the static limit, appropriate for static but weakly spatially varying magnetic field, we get $J_z^{\rm Hopf} = N_{\rm static}/2$, where

$$N_{\rm static} = 16\pi \lim_{\mathbf{p} \to 0} \Pi_1(\mathbf{p}, 0) = 1 - \beta\omega_c \sum_{n=0}^{\infty} \sum_{\alpha=\pm} (n + \frac{1}{2}) f_n^\alpha (1 - f_n^\alpha). \tag{90}$$

The prefactor of the Hopf term becomes a function of temperature in the static limit. The temperature dependence of $N_{\rm static}$ is shown in Fig. 2. As expected, $N_{\rm static} = N_{\rm dynamic}$ at $T = 0$ and reduces to zero at high temperature ($T/\omega_c \gg 1$). The temperature dependence of $N_{\rm dynamic}$ and $N_{\rm static}$ is similar to that of the dynamic and static Hall conductivities in these systems [50–52].

VIII. CONCLUSION

The present work generalizes the derivation of the Hopf term in earlier works [17–19] to quantum Hall systems. We find that although k_x and k_y are no longer good quantum numbers simultaneously, it is possible to replace the integration over these wave vectors by averaging over phases of the boundary conditions. This procedure is well known and is very useful in determining the quantized Hall conductivity which is essentially given by the same topological invariant [22,42]. The prefactor of the Hopf term (N) is found to be the filling factor (ν).

The value of N obtained agrees to the previous result [24,25,29] for the integer filling case. The method used here is, however, quite different and more powerful. It does not need either the assumption of lowest Landau level projection [24,25] or laborious term by term evaluation of the effective action [29]. Furthermore it provides a simple physical argument in terms of Berry's phase as to why the prefactor the Hopf term has to be an integer. It is pointed out that this is a general result, and not a consequence of the simplicity of the model assumed. Our method also does not require parameterization of the SU(2) rotation matrix U in terms of the Euler angles and therefore avoids any ambiguity which may arise from such a procedure [26,28]. The generalization of this method to the case of fractional filling factors $\nu = 1/(2s+1)$ is straightforward. The results obtained are in agreement with the results of Baez *et al*. [32] obtained using phenomenological approach.

The expressions for the spin and charge densities of the skyrmion are obtained from the effective action. The contribution to the spin density comes from both the Berry term and the Hopf term in the effective action. In the case of the long-wavelength skyrmions, the contribution due to the former dominates. The total spin of the skyrmion, therefore, depends on the system details. The charge and the statistics of the skyrmion, on the other hand, is completely determined by the prefactor of the Hopf term. The charge for a skyrmion with $Q_{\rm top} = 1$ is shown to be νe and the statistical phase is found to be $\nu\pi$. Consequently, the skyrmions are Fermions (anyons) for odd integer (fractional) fillings.

We also obtain an effective action for the skyrmion at finite temperature. It is shown that it is not, in general, possible to obtain a local Hopf term in position space at finite temperature because of the non-analyticity of $P_3^{\mu\nu}(\mathbf{p}, p_0)$. However, it is possible to derive a rather simple effective action for the system in the momentum space. One can then choose static or dynamic limits based on physical reasoning and find the value of the Hopf invariant in these limits. In the dynamic limit, the prefactor of the Hopf term is independent of temperature and has the same value as at zero temperature while in the static limit it becomes a function of temperature.

To conclude, we derive the prefactor of the Hopf term in the effective action of skyrmions in quantum Hall systems for both integer and fractional fillings. This prefactor (N) is found to be ν. This suggests that the skyrmion with $Q_{\rm top} = 1$ have charge νe and statistical phase $\nu\pi$. On the other hand, the dominant contribution to the spin of the skyrmion comes from the Berry term in the effective action. As a result, the spin of the skyrmion depends on system parameters and increases with the size of the skyrmion. We also discuss the fate of the Hopf term at finite temperature. At finite temperature, it is in general not possible to obtain a local Hopf term in position space. However, it is possible to obtain an effective action in the momentum space and to obtain the value of the prefactor of the Hopf term from this action in the static and the dynamic limit.

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IX. APPENDIX A

In this section, we briefly outline the calculation of the relevant term in the action from the expression of $P_{ij}^{\mu\nu}(\mathbf{p}, p_0)$. For the relevant term in the effective action, we need to retain the first order term in expansion of $P_{ij}^{\mu\nu}(\mathbf{p}, p_0)$. We start from the expression of $P_{ij}^{\mu\nu}$ in Eq. (22) which is

$$P_{ij}^{\mu\nu}(\mathbf{p}, p_0) = \frac{i}{4L_xL_y}\mathrm{Tr}\left(\int d^2r_1\, d^2r_2 \frac{d\omega}{2\pi}\frac{1}{2}\left[\sigma_i e^{i\mathbf{p}\cdot\mathbf{r_1}}, \left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+ G_0(\mathbf{r_1}, \mathbf{r_2}, \omega)\right.$$

$$\times\frac{1}{2}\left[\sigma_j e^{-i\mathbf{p}\cdot\mathbf{r_2}},\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2},\mathbf{r_1},\omega+p_0)\Bigg), \tag{91}$$

where G_0 is the unperturbed Green function. Using the Fourier transform of the Green functions,

$$G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)=\int d^2r_1\,d^2r_2 e^{i(\mathbf{k}_1\cdot\mathbf{r}_1-\mathbf{k}_2\cdot\mathbf{r}_2)}G_0(\mathbf{r}_1,\mathbf{r}_2,\omega),$$

Eq. (91) can be written in Fourier space as

$$P_{ij}^{\mu\nu}(\mathbf{p},p_0)=\frac{i}{4L_xL_y}\mathrm{Tr}\int\frac{d^2k_1\,d^2k_2\,d\omega}{(2\pi)^5}\Bigg(\sigma_i\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{k}_1+\frac{\mathbf{p}}{2}}G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)$$
$$\times\sigma_j\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{k}_2+\frac{\mathbf{p}}{2}}G_0(\mathbf{k}_2+\mathbf{p},\mathbf{k}_1+\mathbf{p},\omega+p_0)\Bigg). \tag{92}$$

Next, we carry out a Taylor expansion of $P^{\mu\nu}$ in powers of the external frequency and momentum p_λ, where λ can take values (0,1,2). After some tedious but straightforward manipulations we finally obtain

$$P_{ij}^{\mu\nu}(\mathbf{p},\mathbf{q},p_0)=\frac{-ip_\lambda}{4L_xL_y}\int\frac{d^2k_1\,d^2k_2\,d^2k_3\,d\omega}{(2\pi)^7}\mathrm{Tr}\Bigg(\sigma_i\left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{k}_1}G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)\sigma_j$$
$$\times\left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{k}_2}G_0(\mathbf{k}_2,\mathbf{k}_3,\omega)\left(\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}\right)_{\mathbf{k}_3}G_0(\mathbf{k}_3,\mathbf{k}_1,\omega)\Bigg), \tag{93}$$

where we have used the identities following from Eq. (20)

$$\frac{\partial G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)}{\partial k_1^a}+\frac{\partial G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)}{\partial k_2^a}=\int\frac{d^2k_3}{(2\pi)^2}G_0(\mathbf{k}_1,\mathbf{k}_3,\omega)\left(\frac{\partial G_0^{-1}}{\partial\alpha_a}\right)_{\mathbf{k}_3}G_0(\mathbf{k}_3,\mathbf{k}_2,\omega),$$
$$\frac{\partial G_0(\mathbf{k}_1,\mathbf{k}_2,\omega)}{\partial\omega}=-\int\frac{d^2k_3}{(2\pi)^2}G_0(\mathbf{k}_1,\mathbf{k}_3,\omega)\left(\frac{\partial G_0^{-1}}{\partial\alpha_0}\right)_{\mathbf{k}_3}G_0(\mathbf{k}_3,\mathbf{k}_2,\omega). \tag{94}$$

Substituting the expression of $P_{ij}^{\mu\nu}$ in Eq. (21), and transforming back to real space, we obtain for S_2

$$S_2=\frac{1}{2}b_{ij}^{\mu\nu\lambda}\int d^2r\,dt\,\Omega_\nu^{j\,\mathrm{int}}(\mathbf{r},t)\partial_\lambda\Omega_\mu^{i\,\mathrm{int}}(\mathbf{r},t), \tag{95}$$

where the tensor $b_{ij}^{\mu\nu\lambda}$ is given by

$$b_{ij}^{\mu\nu\lambda}=\frac{1}{4L_xL_y}\mathrm{Tr}\left(\sigma_i\frac{\partial G_0^{-1}}{\partial\alpha_\mu}G_0\sigma_j\frac{\partial G_0^{-1}}{\partial\alpha_\nu}G_0\frac{\partial G_0^{-1}}{\partial\alpha_\lambda}G_0\right). \tag{96}$$

This procedure can be similarly carried out to obtain the zeroth order term in expansion of $T_{ijk}^{\mu\nu\lambda}$ which contribute to S_3. The manipulations are straightforward and directly yield Eq. (26).

X. APPENDIX B

In this section, we sketch the derivation of some of the properties of the tensors $b_{ij}^{\mu\nu\lambda}$ and $c_{ijk}^{\mu\nu\lambda}$ which lead to cancelation of the terms S_2 and S_3 in the effective action. We begin with the tensor $c_{ijk}^{\mu\nu\lambda}$. It is easy to see from the expression of $c_{ijk}^{\mu\nu\lambda}$ in Eq. (26), that all components of this tensor that have odd number of σ_1 or σ_2 matrices vanish by trace operation. So the only possible non-zero terms in S_3 are of the form

$$S_3 = \frac{1}{6} c_{ij3}^{\mu\nu\lambda} \int d^2r\, dt\, \Omega_\mu^{i\,\mathrm{int}}(\mathbf{r},t)\Omega_\nu^{j\,\mathrm{int}}(\mathbf{r},t)\Omega_\lambda^{3\,\mathrm{int}}(\mathbf{r},t), \tag{97}$$

where the indices i and j are such that $\sigma_i\sigma_j = I$ or $\pm i\sigma_3$. We first consider the case where $i = j$. Let us perform a gauge transformation on S_3, which corresponds, as we have seen in Sec.(IV), to a rotation about the local $\mathbf{d}$ axis. Under such a transformation, $\Omega_\lambda^{3\,\mathrm{int}} \to \Omega_\lambda^{3\,\mathrm{int}} + \partial_\lambda\Lambda$ and the action S_3 picks up an extra term

$$\delta S_3 = \frac{1}{6} c_{ii3}^{\mu\nu\lambda} \int d^2r\, dt\, \Omega_\mu^{i\,\mathrm{int}}(\mathbf{r},t)\Omega_\nu^{i\,\mathrm{int}}(\mathbf{r},t)\partial_\lambda\Lambda(\mathbf{r},t). \tag{98}$$

It therefore follows that for the action to be gauge-invariant, δS_3 must vanish for all i and consequently we must have $c_{ii3}^{\mu\nu\lambda} = -c_{ii3}^{\nu\mu\lambda}$. The property $c_{ii3}^{\mu\nu\lambda} = -c_{ii3}^{\nu\mu\lambda}$ can also be checked by explicit evaluation of the coefficient $c_{ii3}^{\mu\nu\lambda}$. As a result, the terms in S_3 with identical spin indices vanish. This can be seen by interchanging the indices μ and ν in Eq. (97) for terms with $i = j$ and $k = 3$. So the only non-vanishing terms in S_3 are those with all different spin indices. Furthermore, since $c_{123}^{\mu\nu\lambda} = -b_1^{\mu\nu\lambda}$ (34), S_3 can be expressed as

$$S_3 = -\frac{1}{2} b_1^{\mu\nu\lambda} \int d^2r\, dt\, \Omega_\lambda^{3\,\mathrm{int}} \left(\Omega_\mu^{1\,\mathrm{int}}\Omega_\nu^{2\,\mathrm{int}} - \Omega_\mu^{2\,\mathrm{int}}\Omega_\nu^{1\,\mathrm{int}}\right). \tag{99}$$

Next, we demonstrate the cancelation of the terms in S_2 and S_3. We begin with the first term of S_2 in Eq. (31). Using Eq. (12), this term can be expressed as

$$\begin{aligned} S_2^1 &= \frac{1}{2} b_1^{\mu\nu\lambda} \int d^2r\, dt\, \left[\Omega_\nu^{1\,\mathrm{int}} \left(\partial_\mu\Omega_\lambda^{1\,\mathrm{int}} - \Omega_\lambda^{2\,\mathrm{int}}\Omega_\mu^{3\,\mathrm{int}} + \Omega_\lambda^{3\,\mathrm{int}}\Omega_\mu^{2\,\mathrm{int}}\right)\right] \\ &= \frac{1}{2} b_1^{\mu\nu\lambda} \int d^2r\, dt\, \left[-\Omega_\lambda^{1\,\mathrm{int}} \left(\partial_\mu\Omega_\nu^{1\,\mathrm{int}}\right) - \Omega_\nu^{1\,\mathrm{int}}\Omega_\lambda^{2\,\mathrm{int}}\Omega_\mu^{3\,\mathrm{int}} + \Omega_\nu^{1\,\mathrm{int}}\Omega_\lambda^{3\,\mathrm{int}}\Omega_\mu^{2\,\mathrm{int}}\right], \end{aligned} \tag{100}$$

where in the last step we have integrated the first term in S_2^1 by parts. Using Eq. (12), it can be seen after some algebra that Eq. (100) can be expressed as

$$S_2^1 = \frac{1}{2}\int d^2r\, dt \Big\{ b_1^{\mu\nu\lambda}\left[\Omega_\mu^{3\,\rm int}\Omega_\nu^{2\,\rm int}\Omega_\lambda^{1\,\rm int} + \Omega_\mu^{3\,\rm int}\Omega_\nu^{1\,\rm int}\Omega_\lambda^{2\,\rm int} - \Omega_\mu^{2\,\rm int}\Omega_\nu^{1\,\rm int}\Omega_\lambda^{3\,\rm int}\right] \\ -(b_1^{\mu\nu\lambda}+b_1^{\nu\mu\lambda})\left[\Omega_\lambda^{1\,\rm int}\left(\partial_\nu\Omega_\mu^{1\,\rm int}\right) + \frac{1}{2}\Omega_\mu^{3\,\rm int}\Omega_\nu^{2\,\rm int}\Omega_\lambda^{1\,\rm int}\right]\Big\}. \tag{101}$$

An identical calculation for the second term in Eq. (31) yields

$$S_2^2 = \frac{1}{2}\int d^2r\, dt \Big\{ b_1^{\mu\nu\lambda}\left[\Omega_\mu^{1\,\rm int}\Omega_\nu^{3\,\rm int}\Omega_\lambda^{2\,\rm int} + \Omega_\mu^{1\,\rm int}\Omega_\nu^{2\,\rm int}\Omega_\lambda^{3\,\rm int} - \Omega_\mu^{3\,\rm int}\Omega_\nu^{2\,\rm int}\Omega_\lambda^{1\,\rm int}\right] \\ -(b_1^{\mu\nu\lambda}+b_1^{\nu\mu\lambda})\left[\Omega_\lambda^{2\,\rm int}\left(\partial_\nu\Omega_\mu^{2\,\rm int}\right) + \frac{1}{2}\Omega_\mu^{1\,\rm int}\Omega_\nu^{3\,\rm int}\Omega_\lambda^{2\,\rm int}\right]\Big\}. \tag{102}$$

Using Eqs. (101), (102) and (99), after some straightforward algebra, one obtains

$$S_2^1 + S_2^2 + S_3 = -\frac{1}{2}(b_1^{\mu\nu\lambda}+b_1^{\nu\mu\lambda})\int d^2r\, dt \Big[\Omega_\lambda^{1\,\rm int}\left(\partial_\nu\Omega_\mu^{1\,\rm int}\right) + \Omega_\lambda^{2\,\rm int}\left(\partial_\nu\Omega_\mu^{2\,\rm int}\right) \\ - \quad \frac{1}{2}\left(\Omega_\mu^{3\,\rm int}\Omega_\nu^{2\,\rm int}\Omega_\lambda^{1\,\rm int} + \Omega_\mu^{1\,\rm int}\Omega_\nu^{3\,\rm int}\Omega_\lambda^{2\,\rm int} - \Omega_\mu^{3\,\rm int}\Omega_\nu^{1\,\rm int}\Omega_\lambda^{2\,\rm int}\right)\Big] \tag{103}$$

The gauge invariance of the effective action then requires that the terms on the right-hand side of Eq. (103) be invariant under the gauge transformation $\Omega_\mu^{3\,\rm int} \to \Omega_\mu^{3\,\rm int} + \partial_\mu\Lambda$, which leads to the condition $b_1^{\mu\nu\lambda} = -b_1^{\nu\mu\lambda}$. Consequently, the first two terms in S_2 exactly cancel S_3. The property $b_1^{\mu\nu\lambda} = -b_1^{\nu\mu\lambda}$ can also be checked by explicit evaluation of the coefficient $b_1^{\mu\nu\lambda}$. However, it is to be noted that this property and the consequent cancelation of terms in S_2 and S_3 follows from general gauge invariance requirement and does not depend on the details of the explicit expressions of the coefficients $b_1^{\mu\nu\lambda}$.

XI. APPENDIX C

In this section, we sketch the method of computation of the finite temperature polarization tensor $P_3^{\mu\nu}$ which is given by

$$P_3^{\mu\nu}(\mathbf{p},p_0) = \frac{-1}{4L_xL_y}\mathrm{Tr}\left(\int d^2r_1\, d^2r_2 \frac{1}{\beta}\sum_{i\omega_n}\frac{1}{2}\left[\sigma_3 e^{i\mathbf{p}\cdot\mathbf{r_1}}, \left(\frac{\partial G_0^{-1}}{\partial\alpha_\mu}\right)_{\mathbf{r}_1}\right]_+ G_0(\mathbf{r_1},\mathbf{r_2},i\omega_n) \right. \\ \left. \times\frac{1}{2}\left[\sigma_3 e^{-i\mathbf{p}\cdot\mathbf{r_2}}, \left(\frac{\partial G_0^{-1}}{\partial\alpha_\nu}\right)_{\mathbf{r}_2}\right]_+ G_0(\mathbf{r_2},\mathbf{r_1},i\omega_n+ip_n)\right). \tag{104}$$

The evaluation of $P_3^{\mu\nu}$ is tedious but straightforward. Here we shall only show the derivation of P_3^{00} explicitly. After evaluating the matrix trace, we can write P_3^{00} as

$$P_3^{00} = \frac{-1}{4}\int d^2r_1\, d^2r_2 e^{i\mathbf{p}\cdot(\mathbf{r_1}-\mathbf{r_2})}\frac{1}{\beta}\sum_{i\omega_n}\Big(G_0^+(\mathbf{r_1},\mathbf{r_2},i\omega_n)G_0^+(\mathbf{r_1},\mathbf{r_2},i\omega_n+ip_n) + G_0^-(\mathbf{r_1},\mathbf{r_2},i\omega_n)G_0^-(\mathbf{r_1},\mathbf{r_2},i\omega_n+ip_n)\Big). \tag{105}$$

The Green Functions $G_0^{\pm}$ is easily obtained as solution of Eq. (20) exactly as in the zero temperature case in terms of the single-particle eigenfunctions and energies (39) by replacing $\omega \to i\omega_n$:

$$G_0^{\pm} = \frac{|n\rangle\langle n|}{i\omega_n - \epsilon_n^{\pm}}. \tag{106}$$

Using Eq. (106), it is straightforward to perform the frequency sums in Eq. (105). After carrying out the frequency sum, we perform an analytic continuation to real frequency ($ip_n \to p_o + i\eta$). It is crucial that the analytic continuation is performed after the frequency sums are carried out. As a result, one gets

$$\begin{aligned} P_3^{00} &= -\frac{1}{4}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{\alpha=\pm}\frac{f_n^\alpha - f_{n'}^\alpha}{p_0 - \omega_c(n-n') + i\eta}\langle n|e^{i\mathbf{p}\cdot\mathbf{r_1}}|n'\rangle\langle n'|e^{-i\mathbf{p}\cdot\mathbf{r_2}}|n\rangle \quad \text{if } p_0 \neq 0, \\ &= \frac{1}{4}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{\alpha=\pm}\delta_{nn'}\left(-\frac{\partial f(z)}{\partial z}\right)_{z=\epsilon_n^\alpha}\langle n|e^{i\mathbf{p}\cdot\mathbf{r_1}}|n'\rangle\langle n'|e^{-i\mathbf{p}\cdot\mathbf{r_2}}|n\rangle \quad \text{if } p_0 = 0, \end{aligned} \tag{107}$$

where f_n^α is shorthand notation for $f(\epsilon_n^\alpha)$ and ϵ_n^α are the energy eigenvalues given by $\epsilon_n^\alpha = \epsilon_n + \alpha\gamma_0$.

Using the expression for the Landau wavefunctions $\langle n|\mathbf{r}\rangle$ [48,50], we get for $n \geq n'$

$$\langle n|e^{i\mathbf{p}\cdot\mathbf{r_1}}|n'\rangle = \sqrt{\frac{n'!}{n!}}e^{-\frac{|\mathbf{p}|^2 l_B^2}{4}}(p_x+ip_y)^{n-n'}L_{n'}^{n-n'}(\frac{|\mathbf{p}|^2 l_B^2}{2}), \tag{108}$$

where $L_{n'}^{n-n'}$ are the associated Laugurre polynomials.

Using Eq. (108), it is now a matter of straightforward algebraic manipulations to obtain an expression for P_3^{00}. It is to be noted that in Eq. (107) one needs to separate the terms in the sum with $n \geq n'$ from the terms with $n \leq n'$ and interchange n and n' in the latter. This gives the final expression for P_3^{00} as

$$P_3^{00} = |\mathbf{p}|^2 \Pi_0(\mathbf{p}, p_0),$$
$$\Pi_0(\mathbf{p}, p_0) = \frac{-l_B^2}{4} \sum_{n=0}^{\infty} \sum_{n'=0}^{n} \sum_{\alpha=\pm} C_{nn'}^{\alpha}(p_0) \frac{n'!}{n!} e^{\frac{-|\mathbf{p}|^2 l_B^2}{2}} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right)^{n-m-1} L_{n'}^{n-n'} \left(\frac{|\mathbf{p}|^2 l_B^2}{2} \right), \quad (109)$$

where $C_{nn'}^{\alpha}(p_0)$ is defined in Eq. (86). The other components of $P_3^{\mu\nu}$ can be computed in an identical manner and in this way we obtain Eqs. (84) and (85).

XII. APPENDIX D

In this appendix, we list the notations used in this paper for the sake of clarity.

α_μ :- (ω, ϕ_x, ϕ_y).

ϕ_x, ϕ_y :- Dimensionless boundary value parameters.

$\mathbf{H}_0$:- Constant external magnetic field along z axis.

$\mathbf{H}_{\text{eff}}$:- Effective magnetic field experienced by the composite Fermions.

$\mathbf{A}$:- Vector potential corresponding to $\mathbf{H}_0$.

A_μ^{cs} :- Chern-Simons fields.

A_μ^{em} :- External electromagnetic field.

$U(\mathbf{r}, t)$:- 2×2 SU(2) rotation matrices.

γ_0 :- Typical Coulomb energy of the system $\sim e^2/l_B$.

ν :- Filling fraction.

$2s$:- Number of flux quanta attached to electrons.

I :- 2×2 unit matrix.

$\boldsymbol{\sigma}$:- Pauli matrices.

$\mathbf{J}$:- Spin of the skyrmion.

$\mathbf{J}^{\text{Berry}}$:- Contribution to the spin of the skyrmion from the Berry term.

$\mathbf{J}^{\text{Hopf}}$:- Contribution to the spin of the skyrmion from the bulk Hopf term.

$\mathbf{J}^{\text{edge}}$:- Contribution to the spin of the skyrmion from the edge.

$\mathbf{j}_0$:- Spin density of the skyrmion.

ρ :- Charge density of the skyrmion.

ρ_0 :- Uniform ground-state particle density.

Q_μ^{int} :- Internal gauge field $= -iU^{-1}(\partial_\mu U) = \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_\mu^{\text{int}}$.

Q_μ^{θ}:- Gauge fields due to rotation by an arbitrary solid angle $\boldsymbol{\theta} = \frac{1}{2}\boldsymbol{\sigma}\cdot\partial_\mu\boldsymbol{\theta}$.

Q_μ^{total} :- Total gauge field $= Q_\mu^{\text{int}} + UQ_\mu^{\boldsymbol{\theta}}U^{-1}$.

Q_μ^{eff} :- Effective gauge field for fractional filling $= Q_\mu^{\text{int}} + A_\mu^{\text{cs}}I$.

Q_{em} :- Electromagnetic Charge of the skyrmion.

Q_{top} :- Topological Charge of the skyrmion.

$\mathbf{d}$:- Unit vector field.

$f_{\mu\nu}^{\text{int}}$:- Field tensor for the internal gauge field $= 0$.

B_μ :- Auxiliary gauge field.

$F_{\mu\nu}$:- Field tensor corresponding to the $\mathbf{B}$ field $= \partial_\mu B_\nu - \partial_\nu B_\mu = \mathbf{d}\cdot(\partial_\mu\mathbf{d}\times\partial_\nu\mathbf{d})$.

$F_{\mu\nu}^{\text{cs}}$:- Field tensor corresponding to the $\mathbf{A}^{\text{cs}}$ field.

$F_{\mu\nu}^{\text{em}}$:- Field tensor corresponding to the $\mathbf{A}^{\text{em}}$ field.

G_0 :- Unperturbed Green function.

$P_{ij}^{\mu\nu}$:- The Polarization Bubble.

$T_{ijk}^{\mu\nu\lambda}$:- The Triangle Diagram.

$b_{ij}^{\mu\nu\lambda}$:- Coefficient of the second order term in the effective action corresponding to i and j spin indices and μ ν and λ space-time indices.

$b_j^{\mu\nu\lambda}$:- $b_{ij}^{\mu\nu\lambda}$ with i = j.

$c_{ijk}^{\mu\nu\lambda}$:- Coefficient of the third order term in the effective action corresponding to i j and k spin indices and μ ν and λ space-time indices.

N :- prefactor of the Hopf term.

N_0 :- Number of particles in the ground state.

Λ :- Arbitrary space-time dependent angle corresponding rotation about the $\mathbf{d}$ axis.

$\boldsymbol{\theta}$:- Arbitrary space-time dependent solid angle.

η :- Boson fields for the edge action.

L_x, L_y :- Dimensions of the sample.

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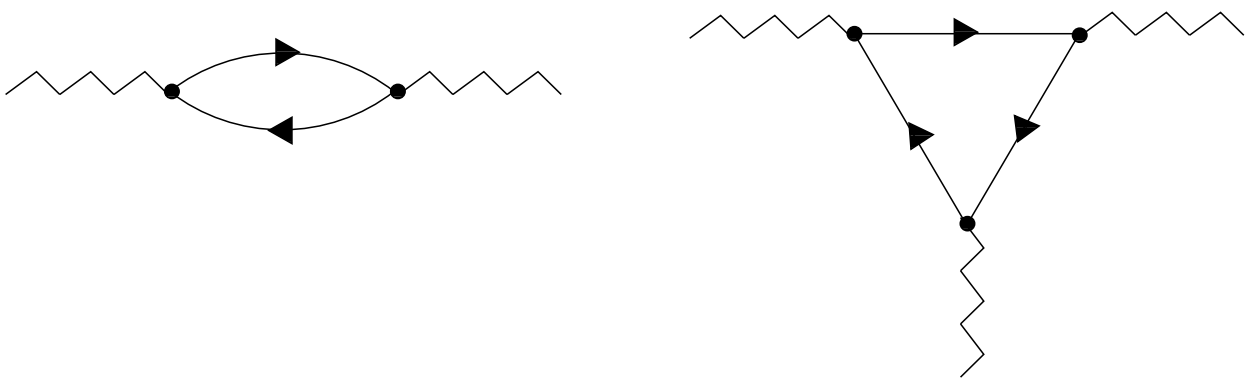

FIG. 1. Polarization Bubble and Triangular Diagram. The wavy lines represent the gauge fields Q_μ, the solid lines represent the Green function G_0 and the dots represent the vertices $\frac{\partial G_0^{-1}}{\partial \alpha_\mu}$.

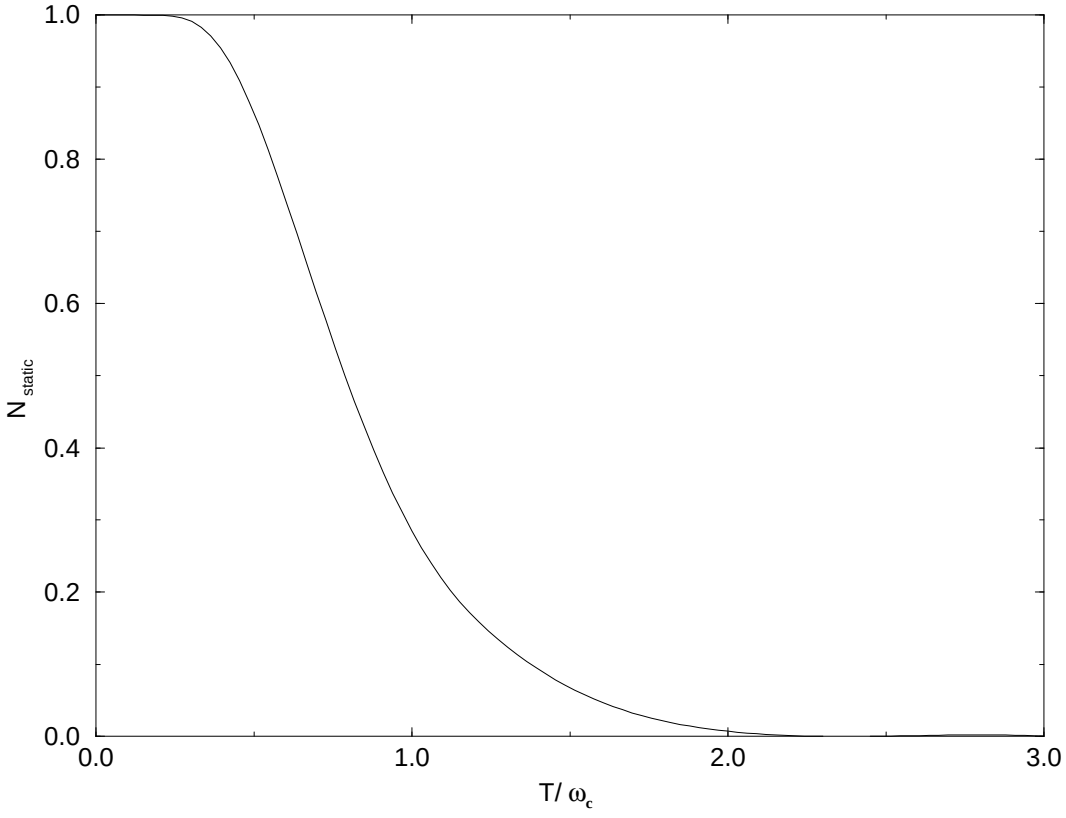

FIG. 2. Temperature dependence of $N_{\rm static}$.