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Wigner-Weisskopf atom and quantum Stochastic dilations

KALYAN B SINHA
Indian Statistical Institute, 7, SJS Sansanwal Marg, New Delhi 110 016, India

Abstract. Quantum stochastic processes are operator processes in Fock space adapted in a natural way with respect to the tensor product factorization of the space. One can describe the usual classical stochastic processes (e.g., Brownian, Poisson) as well as new variety of non-commuting processes in this language. In this framework, quantum stochastic calculus (like Ito calculus in the classical case) has been developed by Hudson and Parthasarathy. Using this calculus, one can rewrite the simple Wigner-Weisskopf model of a decaying atom. In this model, the system (atom) is coupled linearly to the environment (a bath of infinite degrees of freedom represented by a reducible bosonic quantum stochastic process at non-zero temperature). The quantum stochastic differential equation of evolution is solved, and the bath variables are averaged out by taking vacuum expectation values leading to a law of relaxation of the state of the system.

Keywords. Quantum stochastic calculus; two-level atom; singular coupling limit.

1. Introduction

Examples of unstable systems or relaxation models abound in nature, e.g., decaying particles, radiation of an excited atom, measurement process in quantum mechanics and relaxation of a macrosystem to its equilibrium, to name a few. All these examples have a unifying backdrop, viz., there is a system where our interest lies and there is an environment or bath or apparatus. In a suitable sense the environment is 'large' compared to the system and the two are 'coupled'.

Almost all attempts to understand such phenomena in the quantum mechanical framework can be divided into two broad categories. The first is model-independent general analysis and has been summarized in the recent book of Exner (1985). The second is a somewhat more detailed, sometimes model-dependent study, typical examples of which are the works of Fonda *et al* (1973), Davies (1976), and Gorini *et al* (1976).

In the second group, one usually takes a Hamiltonian consisting of a part as the Hamiltonian of the system, and another for the environment and then the so-called coupling or interaction Hamiltonian. The Schrödinger or Heisenberg equation is then written down and in an appropriate weak coupling or singular coupling limits, the irreversible master equation results. Typically this involves a rescaling of time and one usually argues that the time scale of observation is very large compared to the intrinsic time-scale of relaxation of the system so that one has the final effect of wiping out the memory terms inherent in the original equations of motion.

More recently one has a new theory, that of the quantum stochastic processes, pioneered by Hudson and Parthasarthy (1984) and Accardi (1980), which allows one to describe the situation more satisfactorily though the general theory of relaxation phenomena is still missing. We shall be content here with a brief description of the simple Wigner-Weisskopf two-level atom, 'stochastically' coupled to the radiation field basing on the work of Applebaum (1986).

2. The model

The model is that of a two-level atom represented by $\mathbb{C}^2$ with the algebra of observables $M_2(\mathbb{C})$ generated by $a = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $a^\dagger = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ so that the CAR (canonical anti-commutation relations) are satisfied:

$$a^2 = 0, \; aa^\dagger + a^\dagger a = I. \tag{1}$$

The atomic Hamiltonian H_S:

$$H_S = \tfrac{1}{2} w_0 (a^\dagger a - aa^\dagger), \quad w_0 > 0, \text{ and}$$

its Gibbs state at inverse temperature $\beta > 0$ is given by the density matrix

$$\rho_S = [\text{tr}\, \exp(-\beta H_S)]^{-1} \exp(-\beta H_S)$$

$$= [\exp(1/2\beta w_0) + \exp(-1/2\beta w_0)]^{-1} \begin{bmatrix} \exp(1/2\beta w_0) & 0 \\ 0 & \exp(-1/2\beta w_0) \end{bmatrix}. \tag{2}$$

The bath or the environment of the atom is composed of n independent harmonic oscillators, generated by $\{a_j, 1 \le j \le n\}$ satisfying the CCR (canonical commutation relations)

$$[a_j, a_k] = 0, \; [a_j, a_k^\dagger] = \delta_{jk} I. \tag{3}$$

The Hamiltonian for the environment H_E is given as $H_E = \sum_1^n w_j a_j^\dagger a_j$ ($w_j > 0$), and the associated state by the density matrix $\rho_E = [\text{tr}\, \exp(-\beta H_E)]^{-1} \exp(-\beta H_E)$. Then it is easy to see that

$$\text{tr}(a_j \rho_E) = \text{tr}(a_j^\dagger \rho_E) = 0, \; \text{tr}(\rho_E a_j^\dagger a_k) = \delta_{jk} \mu_j^2,$$

where

$$\mu_j^2 = [1 - \exp(-\beta w_j)]^{-1} \exp(-\beta w_j) > 0.$$

Following Araki and Woods (1963) (see also Parthasarathy and Sinha 1986), we write the cyclic *-representation (though reducible) on $\mathcal{H}_n = \Gamma(\mathbb{C}^n) \otimes \Gamma(\mathbb{C}^n)$ as follows.

$$B_j = \lambda_j b^{(1)}(e_j) \otimes I^{(2)} + \mu_j I^{(1)} \otimes b^{(2)}(e_j),$$

where

$$\lambda_j^2 = 1 + \mu_j^2,$$

and $b^{(i)}(e_j)$ $(i = 1, 2)$ are two copies of the canonical (zero temperature) irreducible Fock representation of CCR in $r(\mathbb{C}^n)$, the Fock space over $\mathbb{C}^n$. The cyclic vector is $\Omega \otimes \Omega$, where Ω is the vacuum vector in $r(\mathbb{C}^n)$ so that

$$[B_j, B_k] = \delta_{jk},$$

$$\langle \Omega \otimes \Omega,\ B_j^\dagger B_k\, \Omega \otimes \Omega \rangle = \mu_j^2 \delta_{jk}\,. \tag{4}$$

The B_j's are the finite temperature (reducible) representation of the CCR of finite degrees of freedom.

The states of the total system is described by vectors in $\mathbb{C}^2 \otimes H_n$ and the interaction Hamiltonian by

$$H_I = \sum_{j=1}^{n} g_j(\varepsilon a + \eta a^\dagger) B_j + \bar{g}_j(\varepsilon a^\dagger + \eta a) B_j^\dagger, \tag{5}$$

with ε, $\eta \geq 0$, and $g_j \in \mathbb{C}$. The total Hamiltonian $H = H_S + H_E + H_I = H_0 + H_I$ is clearly self-adjoint since B_j is bounded relative to $B_j^\dagger B_j$.

In the interaction representation in $\mathbb{C} \otimes H_n$, set $W(t) = \exp(iH_0 t)\exp(-iHt)$ so that a simple calculation leads to

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -ik(t)\, W(t),$$

where $K(t) = [\varepsilon a \exp(-iw_0 t) + \eta a^\dagger\, exp\,(iw_0 t)]\, F(t)$

$+ [\varepsilon a^\dagger \exp(iw_0 t) + \eta a \exp(-iw_0 t)]\, F^\dagger(t)$, with

$$F(t) = \sum_{j=1}^{n} g_j \exp(-iw_j t)\, B_j\,. \tag{6}$$

It then follows easily from (4) and (6) that

$$[F(s), F(t)] = 0,\ [F(s),\ F(t)^\dagger] = \sum_{j=1}^{n} |g_j|^2 \exp[-iw_j(s-t)]\,,$$

and $\langle\, \Omega \otimes \Omega,\ \ F(s)^\dagger\ \ F(t)\, \Omega \otimes \Omega\,\rangle$

$$= \sum_{j,k} \bar{g}_j\, g_k \exp[i(w_j s - w_k t)]\ \mathrm{tr}\ \rho_E\, a_j^\dagger a_k$$

$$= \sum_j |g_j|^2\, \mu_j^2 \exp[-iw_j\,(t-s)]\,. \tag{7}$$

Formally the solution of (5) can be written down as either a time-ordered exponential series of whose convergence we cannot say much or as a continuous product integral. For a partition $0 = t_0 < t_1 < t_2 \ldots t_n = t$ with $\Delta_j = t_{j+1} - t_j$, we can formally write

$$W(t) = \lim_{|\Delta_j| \to 0} \prod_{j=1}^{n} \exp[-ik(t_j)\,\Delta_j]. \tag{8}$$

3. Singular coupling limit or Wigner-Weisskopf approximation and quantum stochastic differential equation

Now we suppose that in the limit of the degree of freedom $n \to \infty$ the frequencies w_j are uniformly distributed over IR and $|g_i|^2$ are all close to 1. Then formally (still denoting the formal limit of F by the same symbol F)

$$[F(t)] = 0,\ [F(s),\ F(t)^\dagger] = \delta(s-t)$$

and $\langle\, \Omega \otimes \Omega,\ F(s)^\dagger F(t)\, \Omega \otimes \Omega \,\rangle = \int \mathrm{d}w\, \mu(w)^2 \exp[-iw(t-s)]$,

where $\mu(w)^2 = [\exp(\beta w) - 1]^{-1}$. With a further simplifying assumption that $\mu(w)$ is a constant $= \mu_0^2 = [\exp(\beta w_0) - 1]^{-1}$, the above looks like

$$\langle\, \Omega \otimes \Omega,\ F(s)^\dagger\ F(t)\, \Omega \otimes \Omega \,\rangle = \mu_0^2\, \delta(s-t). \tag{9}$$

Mathematically meaningful objects satisfying (9) can be realized by introducing the 'smeared fields' for $f, g \in L^2(IR)$,

$$A(f) \equiv \int F(t)\, \bar{f}(t)\, \mathrm{d}t, \quad A^\dagger(g) = \int F(t)^\dagger\, g(t)\, \mathrm{d}t$$

leading to $[A(f), A^\dagger(g)] = \int \bar{f}(s)\, g(s)\, \mathrm{d}s$. If we write $A(t) \equiv A(\chi_{[0,t]})$ and $A^\dagger(t) = A^\dagger(\chi_{[0,t]})$, then the exponent in the RHS of (8) can be expressed as:

$$\begin{aligned} -ik(t)\,\mathrm{d}t &= G(t)^\dagger\, F(t)\, \mathrm{d}t - G(t)\, F^\dagger(t)\, \mathrm{d}t \\ &= G(t)^\dagger\, \mathrm{d}A(t) - G(t)\, \mathrm{d}A^\dagger(t), \end{aligned}$$

with $G(t) = i[\varepsilon a^\dagger \exp(iw_0 t) + \eta a \exp(-iw_0 t)]$. Thus (8) formally looks like

$$W(t) = \lim \prod_{j=1}^{n} \exp[G(t)^\dagger\, \mathrm{d}A\,(t) - G(t)\, \mathrm{d}A^\dagger\,(t)].$$

This form of $W(t)$ can be given a precise meaning in the context of quantum Ito integral or quantum stochastic differential equation as in Hudson and Parthasarathy (1982, 1984), viz.,

$$\begin{aligned} \mathrm{d}W(t) = W(t)\, \{&G(t)^\dagger\, \mathrm{d}A(t) - G(t)\, \mathrm{d}A^\dagger(t) \\ &-\tfrac{1}{2}[\lambda_0^2\, G(t)^\dagger\, G(t) + \mu_0^2\, G(t)\, G(t)^\dagger]\, \mathrm{d}t\} \end{aligned} \tag{10}$$

with initial value $W(0) = I$, and $\lambda_0^2 = 1 + \mu_0^2$. Here we have used the notations of Hudson and Parthasarathy (1984) (see also Sinha 1986), with the proviso that the Ito product looks like: $\mathrm{d}A\, \mathrm{d}A^\dagger = \lambda_0^2\, \mathrm{d}t$, $\mathrm{d}A^\dagger\, \mathrm{d}A = \mu_0^2\, \mathrm{d}t$.

We rewrite (10) in terms of $U(t) = \exp(-iH_s t)\, W(t)$ as

$$\mathrm{d}U = U\{G^\dagger\, \mathrm{d}A - G\mathrm{d}A^\dagger - [iH_S + \tfrac{1}{2}(\lambda_0^2\, G^\dagger\, G + \mu_0^2\, GG^\dagger)]\, \mathrm{d}t\},$$

where

$$G = G(0) = i(\varepsilon a + \eta a). \tag{11}$$

We can instead look at the effect of this evolution on the system observables in $M_2(\mathbb{C})$ and find that for X ($\in M_2(\mathbb{C})$ if we write

$$T_t(X) = \langle \cdot \otimes(\Omega \otimes \Omega),\ U(t)\, X\, U(t)^\dagger \cdot \otimes (\Omega \otimes \Omega),$$

then T_t is a one-parameter completely positive semigroup with generator L given by

$$L(X) = -i[H_S, X] + \lambda^2(G^\dagger XG - \tfrac{1}{2}[G^\dagger G, X]_+) + \mu^2(GXG^\dagger - \tfrac{1}{2}[GG^\dagger, X]_+). \tag{12}$$

Thus U plays the role of a unitary quantum stochastic dilation of the quantum dynamical semigroup T_t with generator L given by (12) (see Gorrini *et al* 1976).

When in the special case $\varepsilon = \eta = 1$, $G = i(a+a^\dagger) = iq$ and (11) reduces to

$$\mathrm{d}U = U[-iq\, \mathrm{d}Q - \{iH_S + \tfrac{1}{2}(\lambda_0^2 + \mu_0^2)\, q^2\}\, \mathrm{d}t],$$

which with the initial value $U(0) = I$ can be explicitly solved as

$$U(t) = \exp(-iH_S t)\exp[-iq\, Q(t)]. \tag{13}$$

Here $Q(t)$ is a classical Gaussian process with covariance $\exp[Q(s)\, Q(t)] = (\lambda_0^2 + \mu_0^2)\, min(s, t)$. In the limit $\beta \to \infty$ (zero temperature) $\mu_0^2 \to 0$ and $\lambda_0^2 \to 1$ and in this limit $Q(t)$ reduces to the standard Brownian process.

One can instead compute the evolved state $\rho_S(t)$ defined as:

$$\mathrm{tr}\, \rho_S(t)\, X = \mathrm{tr}\, \rho_S\, T_t(X)$$

and find that

$$\rho_S(t) = \rho_S - \varepsilon^2/[\sigma^2(\varepsilon^2+\eta^2)]\,\{1 - \exp[-t\sigma^2(\varepsilon^2+\eta^2]\}\, (aa^\dagger - a^\dagger a),$$

where $\sigma^2 = \lambda_0^2 + \mu_0^2 \geq 1$. Thus $\rho_S(t) = \rho_S$ for all t if and only if $\varepsilon = 0$. One can also show that $\varepsilon = 0$ is the condition needed for satisfying the detailed balance condition, and this case is known in the literature as the rotating wave approximation.

In general, $\rho_S(\infty) = \rho_S - \varepsilon^2/[\sigma^2(\varepsilon^2+\eta^2)]\,(aa^\dagger - a^\dagger a)$ and in the classical case of $\varepsilon = \eta = 1$, $\rho_S(t) = \rho_S - (2\sigma^2)^{-1}[1 - \exp(-2t\sigma^2)]\,(aa^\dagger - a^\dagger a)$. The relaxed state at large time is thus different from the initial state.

4. Non-uniqueness of dilations of dynamical semigroups

Here we look at a specific example of a dynamical semigroup, constructed from a model as in Dattagupta (1984). The system observables $XB(h_S)$ in this model evolve under a dynamical semigroup T_t with generator L given by

$$L(X) = -i[H_S, X] + \lambda(WXW^* - X), \tag{14}$$

where H_S is the Hamiltonian of conservative Hiesenberg-type evolution, W is unitary in h_S, and $\lambda \geq 0$ is a parameter of the model.

The first kind of dilation $U^{(1)}$ is given in terms of the quantum Poisson process with intensity λ viz. $\pi_\lambda(t) = \Lambda(t) + \sqrt{\lambda}[A(t) + A^\dagger(t)] + \lambda t$. Here the gauge or conservation process $\Lambda(t)$ is defined as in Hudson and Parthasarathy (1984). One can easily verify the Ito product rule: $(d\pi_\lambda)^2 = d\pi_\lambda$ and also that Exp $\pi_\lambda \equiv \langle \Omega, \pi_\lambda \Omega \rangle = \lambda t$. We write down the quantum stochastic differential equation (q.s.d.e) satisfied by $U^{(1)}(t)$ as

$$dU^{(1)} = U^{(1)}[(W-1)\,d\pi_\lambda - iH_S\,dt],\ U^{(1)}(0) = I. \qquad (15)$$

One can easily solve such an equation. However, it is more interesting to verify that such an $U^{(1)}$ is indeed a dilation of T_t given by (14)

The second kind of dilation $U^{(2)}$ is given entirely in terms of quantum Brownian motion. Here we write

$$dU^{(2)} = U^{(2)}\left[\sqrt{\lambda}\,W\,dA - \sqrt{\lambda}\,W^*\,dA^\dagger - \left(iH_S + \frac{\lambda}{2}\right)dt\right],\ U^{(2)}(0) = I. \qquad (16)$$

Again one can verify that the semigroup computed as $T_t(X) = \langle \cdot \otimes \Omega, U^{(2)} X U^{(2)^\dagger} \cdot \otimes \Omega \rangle$ has as generator L with the form given by (14). Thus we see that the same physical semigroup of evolution given by (14) has two dilations, one given entirely in terms of the quantum Poison process and the other in terms of the quantum Brownian motion only.

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