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Spectra of Anderson type models with decaying randomness

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Abstract. In this paper we consider some Anderson type models, with free parts having long range tails and with the random perturbations decaying at different rates in different directions and prove that there is a.c. spectrum in the model which is pure. In addition, we show that there is pure point spectrum outside some interval. Our models include potentials decaying in all directions in which case absence of singular continuous spectrum is also shown.

Keywords. Anderson model; absolutely continuous spectrum; mobility edge; decaying randomness.

1. Introduction

There have been but few models in higher dimensional random operators of the Anderson model type in which presence of absolutely continuous spectrum is exhibited. We present here one family of models with such behaviour.

The results here extend those of Krishna [10] and part of those in Kirsch–Krishna–Obermeit [9], Krishna–Obermeit [12] while making use of wave operators to show the existence of absolutely continuous spectrum, the results of Jaksic–Last [14] to show its purity and those of Aizenman [1] for exhibiting pure point spectrum.

The new results in this paper allow for long range free parts, have models with compact spectrum (in dimensions 2 and more) which contains both absolutely continuous and dense pure point spectrum. Our models include the independent randomness on a surface considered by Jaksic–Molchanov [15, 16] and Jaksic–Last [14, 13], while allowing for the randomness to extend into the bulk of the material.

The literature on the scattering theoretic and commutator methods for discrete Laplacian includes those of Boutet de Monvel–Sahbani [4, 5] who study deterministic operators on the lattice.

The scattering theoretic method that we use is applicable even when the free operator is not the discrete Laplacian but has long range off diagonal parts. We impose conditions on the free part in terms of the structure it has in its spectral representation.

2. Main results

The models we consider in this paper are related to the discrete Laplacian $(\Delta u)(n) = \sum_{|i|=1} u(n+i)$ on $\ell^2(\mathbb{Z}^\nu)$. We denote by $\mathbb{T}^\nu$ the ν dimensional torus $\mathbb{R}^\nu/2\pi\mathbb{Z}^\nu$ and σ the invariant probability measure on it. We use the coordinate chart $\{\vartheta : \vartheta = (\theta_1, \ldots, \theta_\nu), 0 < \theta_i < 2\pi\}$ and the representation $\sigma = \prod_{i=1}^{\nu}(\mathrm{d}\theta_i/2\pi)$ on the torus for calculations below without further explanation. Then Δ is unitarily equivalent to multiplication by $2\sum_{i=1}^{\nu}\cos(\theta_i)$ acting on $L^2(\mathbb{T}^\nu, \sigma)$, written in the above coordinates. We consider a bounded self adjoint operator H_0 which commutes with Δ and which is given by, on $L^2(\mathbb{T}^\nu, \mathrm{d}\sigma)$, an operator of multiplication by a function $h(\vartheta)$ there with h satisfying the assumptions below.

Hypothesis 2.1. Let h be a real valued $C^{3\nu+3}(\mathbb{T}^\nu)$ function satisfying

1. h is separable, i.e. $h(\vartheta) = \sum_{j=1}^{\nu} h_j(\theta_j)$.
2. The sets

$$\mathcal{C}(h_j) = \left\{x : \frac{\mathrm{d}h_j}{\mathrm{d}\theta}(x) = 0\right\}$$

are finite for each $j = 1, \ldots, \nu$. Let

$$\tilde{\mathcal{C}}(h_j) = \mathbb{T} \times \ldots \times \mathbb{T} \times \mathcal{C}(h_j) \times \mathbb{T} \ldots \times \mathbb{T},$$

where the set $\mathcal{C}(h_j)$ occurs in the jth position. We denote by

$$\mathcal{C} = \cup_{j=1}^{\nu} \tilde{\mathcal{C}}(h_j)$$

and note that this is a closed set of measure zero in $\mathbb{T}^\nu$.

We consider random perturbations of bounded self adjoint operators coming from functions as in the above hypothesis. We assume the following on the distribution of the randomness.

Hypothesis 2.2. Let μ be a positive probability measure on $\mathbb{R}$ satisfying:

1. μ has finite variance $\sigma^2 = \int x^2 \mathrm{d}\mu(x)$.
2. μ is absolutely continuous.

Finally we consider some sequences of numbers a_n indexed by the lattice $\mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1} = \mathbb{Z}^+ \times \mathbb{Z}^\nu$ and assume the following on them.

Hypothesis 2.3. (1) a_n is a bounded sequence of non-negative numbers indexed by $\mathbb{Z}^\nu$ which is non-zero on an infinite subset of $\mathbb{Z}^\nu$.
(2) Let $g(R) = a_n \chi_{\{n \in \mathbb{Z}^\nu : |n_i| > R,\ \forall 1 \le i \le \nu\}}$. Then $g \in L^1((1, \infty))$.
(1′) a_n is a bounded sequence of non-negative numbers which are non-zero on an infinite subset of $\mathbb{Z}_+^{\nu+1}$.
(2′) Let $g(R) = a_n \chi_{\{n \in \mathbb{Z}_+^{\nu+1} : |n_i| > R,\ \forall 1 \le i \le \nu\}}$. Then $g \in L^1((1, \infty))$.

Remark 1. In the case of $\mathbb{Z}^\nu$ our hypothesis on the sequence a_n allows for the following type of sequences

- $a_n = (1 + |n|)^\alpha, \ \ \alpha < -1$.
- $a_n = (1 + |n_i|)^\alpha$, for some $i, \ \ \alpha < -1$.
- $a_n = \prod_{i=1}^{\nu}(1 + |n_i|)^{\alpha_i}, \ \ \alpha_i \le 0$ with $\sum_{i=1}^{\nu}\alpha_i < -1$.

Therefore in the theorems, on the existence of absolutely continuous spectrum, we can allow the potentials to be stationary along all but one direction in dimensions $\nu \geq 2$.

2. In the case of $\mathbb{Z}_+^{\nu+1}$, we can allow the sequence to be of the type

- $a_n = 0,\ \ n_1 > N$ and $a_n = 1,\ \ $ for $n_1 \leq N$, for some $0 < N < \infty$.
- $a_n = (1 + |n_1|)^{\alpha},\ \ \alpha < -1$.
- $a_n = \prod_{i=1}^{\nu}(1 + |n_i|)^{\alpha_i},\ \ \alpha_i \leq 0$ with $\sum_{i=1}^{\nu} \alpha_i < -1$.

Thus allowing for models with randomness on a the boundary of a half space.

For the purposes of determining the spectra of the models we are going to consider here in this paper we recall a definition given in Kirsch–Krishna–Obermeit [9], namely,

DEFINITION 2.4

Let a_n be a non-negative sequence, indexed by $\mathbb{Z}^{\nu}$ or $\mathbb{Z}_+^{\nu+1}$. Let μ be a positive probability measure on $\mathbb{R}$. Then the a-supp(μ) is defined as

1. In the case of $\mathbb{Z}^{\nu}$,

$$\text{a-supp}(\mu) = \bigcap_{\substack{k \in \mathbb{Z}+ \\ k \neq 0}} \left\{ x : \sum_{n \in k\mathbb{Z}^{\nu}} \mu(a_n^{-1}(x - \epsilon, x + \epsilon)) = \infty,\ \ \forall\ \epsilon > 0 \right\}.$$

2. In the case of $\mathbb{Z}_+^{\nu+1}$,

$$\text{a-supp}(\mu) = \bigcap_{\substack{k \in \mathbb{Z}+ \\ k \neq 0}} \left\{ x : \sum_{n \in k\mathbb{Z}_+^{\nu+1}} \mu(a_n^{-1}(x - \epsilon, x + \epsilon)) = \infty,\ \ \forall\ \epsilon > 0 \right\}.$$

Remark. 1. In the sums occurring in the above definition we set $\mu(a_n^{-1}(x - \epsilon, x + \epsilon)) \equiv 0$, for those n for which $a_n = 0$. This notation is to allow for sequences a_n that are everywhere zero except on an axis for example.

2. We note that when a_n is a constant sequence $a_n = \lambda \neq 0$,

$$\text{a-supp}(\mu) = \lambda \cdot \text{supp}(\mu).$$

3. When a_n converge to zero as $|n|$ goes to ∞, the a-supp(μ) is trivial if μ has compact support. It could be trivial even for some class of μ of infinite support depending upon the sequence a_n.

4. If a_n is bounded below by a positive number on an infinite subset along the directions of the axes in $\mathbb{Z}^{\nu}$ (respectively $\mathbb{Z}_+^{\nu+1}$), then the a-supp(μ) could be non-trivial even for compactly supported μ.

We consider the operator (for $u \in \ell^2(\mathbb{Z}^+)$),

$$(\Delta_+ u)(n) = \begin{cases} u(n+1) + u(n-1), & n > 0, \\ u(1), & n = 0. \end{cases}$$

Below we use either Δ_+ or its extension by $\Delta_+ \otimes I$ to $\ell^2(\mathbb{Z}_+^{\nu+1})$ by the same symbol, the correct operator is understood from the context. Given a real valued continuous function on the torus $\mathbb{T}^{\nu}$, we consider the bounded self adjoint operators H_0 on $\ell^2(\mathbb{Z}^{\nu})$ which is unitarily equivalent to multiplication by h on $\ell^2(\mathbb{T}^\nu, \sigma)$. We also denote the extension $I \otimes H_0$ of H_0 to $\ell^2(\mathbb{Z}_+^{\nu+1})$ by the symbol H_0 and $L^2(\mathbb{T}^\nu, \sigma)$ as simply $L^2(\mathbb{T}^\nu)$ in the sequel.

We then consider the random operators

$$H^\omega = H_0 + V^\omega, \quad V^\omega = \sum_{n \in I} a_n q^\omega(n) P_n, \quad \text{on} \quad \ell^2(\mathbb{Z}^\nu),$$

$$H_+^\omega = H_{0+} + V^\omega, \quad V^\omega = \sum_{n \in I} a_n q^\omega(n) P_n, H_{0+} = \Delta_+ + H_0, \quad \text{on} \quad \ell^2(\mathbb{Z}_+^{\nu+1}), \tag{1}$$

where P_n is the orthogonal projection onto the one dimensional subspace generated by δ_n when $\{\delta_n\}$ is the standard basis for $\ell^2(I)$ ($I = \mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1}$). $\{q^\omega(n)\}$ are independent and identically distributed real valued random variables with distribution μ. The operator H_0 is some bounded self adjoint operator to be specified in the theorems later.

Then our main theorems are the following. First we state a general theorem on the spectrum of H_0 in such models. For this we consider the operator H_0 to denote a bounded self adjoint operator on $\ell^2(\mathbb{Z}^\nu)$ coming from a function h satisfying the Hypothesis 2.1 and Δ_+ defined as before.

Theorem 2.5. *Let H_0 and H_{0+} be the operators defined as in eq.* (1), *coming from functions h satisfying the hypothesis* 2.1(1)(2)*. Let*

$$E_+ = \sum_{j=1}^{\nu} \sup_{\theta \in [0, 2\pi]} h_i(\theta), \quad E_- = \sum_{j=1}^{\nu} \inf_{\theta \in [0, 2\pi]} h_i(\theta).$$

Then, the spectra of both H_0 and H_{0+} are purely absolutely continuous and

$$\sigma(H_0) = [E_-, E_+], \quad \textit{and} \quad \sigma(H_{0+}) = [-2 + E_-, 2 + E_+].$$

Part of the essential spectra of the operators h^ω and H_+^ω are determined via Weyl sequences constructed from rank one perturbations of the free operators H_0 and H_{0+} respectively. The proof of this theorem is done essentially on the line of the proof of Theorem 2.4 in [9].

Theorem 2.6. *Let the indexing set I be $\mathbb{Z}^\nu$ or $\mathbb{Z}_+^{\nu+1}$ and consider the operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1(1) *in the case of $I = \mathbb{Z}^\nu$ and consider the associated H_{0+} in the case of $I = \mathbb{Z}_+^{\nu+1}$. Suppose $q^\omega(n)$, $n \in I$ are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1)*. Let a_n be a sequence indexed by I satisfying the hypothesis* 2.3(1) (*or* (1′) *as the case may be*)*. Assume also that* $0 \in$ a-supp(μ)*, then*

$$\bigcup_{\lambda \in \text{a-supp}(\mu)} \sigma(H_0 + \lambda P_0) \subset \sigma_{\text{ess}}(H^\omega) \textit{ almost every } \omega$$

and

$$\bigcup_{\lambda \in \text{a-supp}(\mu)} \sigma(H_{0+} + \lambda P_0) \subset \sigma_{\text{ess}}(H_+^\omega) \textit{ almost every } \omega.$$

Remark 1. When μ has compact support and a_n goes to zero at infinity, or when μ has infinite support but a_n has appropriate decay at infinity, there is no essential spectrum outside that of H_0 for H^ω almost every ω. So the point of this theorem is to show that there is essential spectrum outside that of H_0 based on the properties of the pairs $(\{a_n\}, \mu)$.

2. In Kirsch–Krishna–Obermeit [9] some examples of random potentials which have essential spectrum outside $\sigma(H_0)$ even when a_n goes to zero at ∞ were given. The examples presented there had a-supp(μ) as a half axis or the whole axis, this is because of the decay of the sequences a_n. Here however, since we allow for a_n to be constant along some directions, our examples include cases where the spectra of H^ω are compact with some essential spectrum outside $\sigma(H_0)$.

We let $E_\pm$ be as in Theorem 2.5.. We also set $\mathcal{H}_{\omega,n}$ to be the cyclic subspace generated by δ_n and H^ω.

Theorem 2.7. *Consider a bounded self adjoint operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1(1), (2). *Suppose q^ω are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1).

1. *Let $I = \mathbb{Z}^\nu$ and a_n be a sequence satisfying the hypothesis* 2.3(1), (2). *Then,*

$$\sigma_{ac}(H^\omega) \supset [E_-,\ E_+] \ \textit{almost every}\ \ \omega.$$

Further when μ satisfies the hypothesis 2.3(2), *$a_n \neq 0$ on $\mathbb{Z}^\nu$, $\mathcal{H}_{\omega,n}, \mathcal{H}_{\omega,m}$ not mutually orthogonal for any n, m in $\mathbb{Z}^\nu$ for almost all ω and $E_\pm$ as in theorem* 2.5., *we also have*

$$\sigma_s(H^\omega) \subset \mathbb{R} \setminus (E_-,\ E_+) \ \textit{almost every}\ \ \omega.$$

2. *Let $I = \mathbb{Z}_+^{\nu+1}$ and a_n be a sequence satisfying the hypothesis* 2.3$(1')$, $(2')$. *Then,*

$$\sigma_{ac}(H_+^\omega) \supset [-2 + E_-,\ 2 + E_+] \ \textit{almost every}\ \ \omega.$$

Further when μ satisfies the hypothesis 2.3(2), *$a_n \neq 0$ on a subset of $\mathbb{Z}_+^{\nu+1}$ that contains the surface $\{(0, n) : n \in \mathbb{Z}^\nu\}$, the subspaces $\mathcal{H}_{\omega,n}, \mathcal{H}_{\omega,m}$ are not mutually orthogonal almost every ω for m, n in $\{(0, k) : k \in \mathbb{Z}^\nu\}$, we also have*

$$\sigma_s(H^\omega) \subset \mathbb{R} \setminus (-2, +E_-,\ 2 + E_+) \ \textit{almost every}\ \ \omega.$$

Remark 1. When μ is absolutely continuous the theorem says that the spectrum of H^ω in (E_-, E_+) (respectively in $(-2 + E_-, 2 + E_+)$ for the $\mathbb{Z}_+^{\nu+1}$ case) is purely absolutely continuous, this is a consequence of a remarkable theorem of Jaksic–Last [14] who showed that in such models with independent randomness, with the randomness non-zero a.e. on a sufficiently big set (H_0 can be any bounded self adjoint operator in their theorem, provided the set of points where the randomness lives gives a cyclic family for the operators H^ω), whenever there is an interval of a.c. spectrum it is pure almost every ω. Their proof is based on considering spectral measures associated with rank one perturbations and comparing the spectral measures of different vectors (which give rise to the rank one perturbations).

2. Our theorem extends the models of surface randomness considered by Jaksic–Last [13], to allow for thick surfaces where the randomness is located in a strip beyond the surface into the bulk of the material. Such models (which are obtained by taking $a_n = 0,\ n_1 > N,\ \ a_n = 1,\ n_1 \leq N$ for some finite N) have purely absolutely continuous spectrum in $(-2\nu - 2, 2\nu + 2)$. The purity of the a.c. spectrum is again a consequence of a theorem of Jaksic–Last [14].

Finally we have the following theorem on the purity of a part of the pure point spectrum. We denote

$$e_+ = \sup \sigma(H_{0+}),\quad e_- = \inf \sigma(H_{0+})\ \text{ and }\ e_0 = \max(|e_-|, |e_+|). \tag{2}$$

Theorem 2.8. *Consider a bounded self adjoint operator H_0 coming from a function h satisfying the conditions of hypothesis* 2.1. *Let I be the indexing set and suppose $q^{\omega}(n),\ n \in I$ are i.i.d random variables with the distribution μ satisfying the hypothesis* 2.2(1), (2). *Assume further that the density $f(x) = \mathrm{d}\mu(x)/\mathrm{d}x$ is bounded. Set $\sigma_1 = \int \mathrm{d}\mu(x)|x|$. Then,*

1. *Let $I = \mathbb{Z}^{\nu}$ and let a_n be a sequence satisfying the hypothesis* 2.3(1), (2). *Then there is a critical energy $E(\mu) > E_0$ depending upon the measure μ such that*

$$\sigma_c(H^{\omega}) \subset (-E(\mu), E(\mu)) \ \text{almost every} \ \omega.$$

2. *Let $I = \mathbb{Z}_{+}^{\nu+1}$ and let a_n be a sequence satisfying the hypothesis* 2.3(1′), (2′). *Then there is a critical energy $e(\mu) > e_0$ such that*

$$\sigma_c(H_{+}^{\omega}) \subset (-e(\mu), e(\mu)) \ \text{almost every} \ \omega.$$

Remark 1. The $E(\mu)$ and $e(\mu)$, while finite may fall outside the spectra of the operators H^{ω} and H_{+}^{ω}, for some pairs (a_n, μ) when μ is of compact support, so for such pairs this theorem is vacuous. However since the numbers $E(\mu)$ (respectively $e(\mu)$) depend only on the operators H_0 (respectively H_{0+}) and the measure μ we can still choose sequences a_n and μ of large support such that the theorem is non-trivial for such cases. Of course for μ of infinite support, the theorem says that there is always a region where pure point spectrum is present.

2. Since we allow for potentials with a_n not vanishing at ∞ in all directions, we could not make use of the technique of Aizenman–Molchanov [3], for exhibiting pure point spectrum.

3. When μ has compact support, comparing the smallness of a moment near the edges of support one exhibits pure point spectrum there by using the Lemma 5.1 proved by Aizenman [1], comparing the decay rate in energy of the sums of low powers of the integral kernels of the free operators with some uniform bounds of low moments of the measure μ weighted with singular but integrable factors occurring to the same power.

As in Kirsch–Krishna–Obermeit [9], Jaksic–Last [14] we also have examples of cases when there is pure a.c. spectrum in an interval and pure point spectrum outside. The part about a.c. spectrum follows as a corollary of theorem 2.6., while the pure point part is proven as in [9] (following the proof of their theorem 2.3, where Δ can be replaced by any bounded self adjoint operator on $\ell^2(\mathbb{Z}^d)$ and work through the details, as is done in Krishna–Obermeit [12], Lemma 2.1). Further when $H_0 = \Delta$, the Jaksic–Last condition on the mutual non-orthogonality of the subspaces $\mathcal{H}_{\omega,n}$, $\mathcal{H}_{\omega,m}$ is valid since given any n, m we can find a k so that $\langle \delta_n, \Delta^k \delta_m \rangle > 0$ (reason, take $k = |n - m| = \sum_{i=1}^{\nu} |n_i - m_i|$, then

$$\Delta^k = \left(\sum_{i=1}^{\nu} T_i + T_i^{-1} \right)^k = c \prod_{i=1}^{\nu} T_i^{|n_i - m_i|} + c \prod_{i=1}^{\nu} T_i^{-|n_i - m_i|} + \text{lower order}$$

with T_i denoting the bilateral shift in the *i*th direction and c a strictly positive constant coming from the multinomial expansion). We see that we can add any operator diagonal in the basis $\{\delta_n\}$ to Δ without altering the conclusion.

COROLLARY 2.9

Let a_n be a sequence as in Hypothesis 2.3 *and μ as in Hypothesis* 2.2. *Let $H_0 = \Delta$. Assume further that $a_n \neq 0,\ n \in \mathbb{Z}^{\nu}$ goes to zero at ∞ and* a-supp$(\mu) = \mathbb{R}$. *Then we have, for almost all ω,*

1. $\sigma_{ac}(H^\omega) = [-2\nu, 2\nu]$.
2. $\sigma_{pp}(H^\omega) = \mathbb{R} \setminus (-2\nu, 2\nu)$.
3. $\sigma_{sc}(H^\omega) = \emptyset$.

The h given in the corollary below is a smooth $2\pi\mathbb{Z}^\nu$ periodic function, so it satisfies the conditions of the Hypothesis 2.1. It is also not hard to verify that, because of the term $\sum_{i=1}^{\nu} \cos(\theta_i)$ occurring in its expression, the cyclic subspaces generated by the associated H_0 on any pair of $\{\delta_n, \delta_m\}$ are mutually non-orthogonal.

COROLLARY 2.10

Let a_n be a sequence as in Hypothesis 2.3 *and μ as in Hypothesis* 2.2*. Let H_0 be a bounded self adjoint operator coming from the function h given by* $h(\vartheta) = \sum_{j=1}^{\nu} \sum_{k=1}^{N} \cos(k\theta_j)$. *Assume that $a_n \neq 0,\ n \in \mathbb{Z}^\nu$ goes to zero at ∞ and* a-supp$(\mu) = \mathbb{R}$. Then we have, for almost all ω,

1. $\sigma_{ac}(H^\omega) = [E_-, E_+]$.
2. $\sigma_{pp}(H^\omega) = \mathbb{R} \setminus (E_-, E_+)$.
3. $\sigma_{sc}(H^\omega) = \emptyset$.

3. Proofs

In this section we present the proofs of the theorems stated in the previous section.

Proof of Theorem 2.5. The statement about the spectrum of H_0 follows from the Hypothesis 2.1(1) on the function h. Each of the functions h_i is a real valued continuous 2π periodic function, hence has compact range. By the intermediate value theorem, we see that the range of $(0, 2\pi)$ under h_i is also an interval. Since the spectrum of H_0 is the algebraic sum of the intervals I_i, – if H_{0j} denotes the operator associated with h_j on $\ell^2(\mathbb{T})$, then $H_0 = H_{01} \otimes I + I \otimes H_{02} \otimes I + \cdots + I \otimes H_{0\nu}$ hence this fact – the statement follows.

We note that $\ell^2(\mathbb{Z}^+)$ is unitarily equivalent to the Hardy space $\mathbb{H}^2(\mathbb{T})$ of functions on $\mathbb{T}$ whose negative Fourier coefficients vanish. Under this unitary transformation, the operator Δ_+ is unitarily equivalent to the operator of multiplication by the function $2\cos(\theta)$ acting on $\mathbb{H}^2(\mathbb{T})$, which can be seen by the definitions of Δ_+, $\mathbb{H}^2(\mathbb{T})$ and the unitary isomorphism U that takes $\mathbb{H}^2(\mathbb{T})$ to $\ell^2(\mathbb{Z}^+)$ (explicitly this is $2\pi(Uf)(n) = \int_0^{2\pi} \mathrm{d}\theta\ \mathrm{e}^{-in\theta} f(\theta)$). Therefore the spectrum of Δ_+ is $[-2, 2]$ and is purely absolutely continuous (there are no eigenvalues). Therefore the spectrum of H_{0+} is also purely a.c. and equals $\sigma(\Delta_+) + [E_-, E_+]$, with $E_\pm$ as above. Hence the theorem follows.

Proof of Theorem 2.6. We prove the theorem for the case H^ω the proof for the case H_+^ω proceeds along essentially the same lines and we give a sketch of the proof for that case. We consider any $\lambda \in$ a-supp(μ), which means that we have

$$\sum_{n \in k\mathbb{Z}_+^{\nu+1}} \mu(a_n^{-1}(\lambda - \epsilon, \lambda + \epsilon)) = \infty, \quad \forall k \in \mathbb{Z}^+, k \neq 0, \quad \text{and all } \epsilon > 0.$$

We consider the distance function $|n| = \max|n_i|, i = 1, \ldots, \nu$ on $\mathbb{Z}^\nu$. We consider the events, with $\epsilon > 0$, $m \in k\mathbb{Z}^\nu$,

$$A_{k,m,\epsilon} = \{\omega : a_m q^\omega(m) \in (\lambda - \epsilon,\ \lambda + \epsilon),\ \ |a_n\, q^\omega(n)| < \epsilon,\ \forall 0 < |n - m| < k - 1\}$$

and

$$B_{k,m,\epsilon} = \{\omega : \ |a_n \, q^{\omega}(n)| < \epsilon, \ \forall 0 \leq |n-m| < k-1\},$$

where the index n in the definition of the above sets varies in $\mathbb{Z}^{\nu}$. Then each of the events $A_{k,m,\epsilon}$ are mutually independent for fixed k and ϵ as m varies in $k\mathbb{Z}^{\nu}$, since the random variable defining them live in disjoint regions in $\mathbb{Z}^{\nu}$. Similarly $B_{k,m,\epsilon}$ is a collection of mutually independent events for fixed k and ϵ as m varies in $k\mathbb{Z}^{\nu}$. Further these events have a positive probability of occurrence, the probability having a lower bound given by

$$\text{Prob}(A_{k,m,\epsilon}) \geq \mu(a_m^{-1}(\lambda - \epsilon, \lambda + \epsilon))(\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}}$$

and

$$\text{Prob}(B_{k,m,\epsilon}) \geq (\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}},$$

where we have taken $c = \inf_{n \in \mathbb{Z}^{\nu}} a_n^{-1} > 0$. The definition of c implies that

$$(-c\,\epsilon, c\,\epsilon) \subset a_m^{-1}(-\epsilon, \epsilon), \ \forall \ m \in \mathbb{Z}^{\nu}.$$

Therefore the assumption that $\lambda \in$ a-supp(μ) implies that $\forall k \in \mathbb{Z}^{+} \setminus \{0\}$,

$$\sum_{m \in k\mathbb{Z}^{\nu}} \text{Prob}(A_{k,m,\epsilon}) \geq (\mu(-c\,\epsilon, c\,\epsilon))^{(k-1)^{\nu+1}} \sum_{m \in k\mathbb{Z}^{\nu}} \mu(a_m^{-1}(\lambda - \epsilon, \lambda + \epsilon)) = \infty$$

and similarly

$$\sum_{m \in k\mathbb{Z}^{\nu}} \text{Prob}(B_{k,m,\epsilon}) = \infty, \ \forall k \in \mathbb{Z}^{+} \setminus \{0\}.$$

Then Borel–Cantelli lemma implies that for all $\epsilon > 0$, (setting $R_{\epsilon} = (\lambda - \epsilon, \lambda + \epsilon)$ and $S_{\epsilon} = (-\epsilon, \epsilon)$ and $\Lambda_k(m) = \{n \in \mathbb{Z}^{\nu} : 0 \leq |n-m| < k-1\}$), the events

$$\Omega(\epsilon, k) = \bigcap_{\substack{m \in I \subset \mathbb{Z}^{\nu} \\ \# I = \infty}} \{\omega : a_m q^{\omega}(m) \in R_{\epsilon}, \ a_n \, q^{\omega}(n) \in S_{\epsilon}, \ \forall n \in \Lambda_k(m) \setminus \{m\}\}$$

have full measure. Therefore the event

$$\Omega_1 = \bigcap_{l,k \in \mathbb{Z}^{+} \setminus \{0\}} \Omega\left(\frac{1}{l}, k\right)$$

has full measure, being a countable intersection of sets of full measure. Similarly the sets

$$\Omega_2(\epsilon, k) = \bigcap_{\substack{m \in I \subset \mathbb{Z}^{\nu} \\ \# I = \infty}} \{\omega : a_n \, q^{\omega}(n) \in S_{\epsilon}, \ \forall n \in \Lambda_k(m)\}$$

have full measure. Therefore the events

$$\Omega_2 = \bigcap_{l,k \in \mathbb{Z}^{+} \setminus \{0\}} \Omega_2\left(\frac{1}{l}, k\right)$$

have full measure.

We take

$$\Omega_0 = \Omega_1 \cap \Omega_2$$

and note that it has full measure. We use this set for further analysis. We denote $H(\lambda) = H_0 + \lambda P_0$. Then suppose $E \in \sigma(H(\lambda))$, then there is a Weyl sequence ψ_l of compact support, $\psi_l \in \ell^2(\mathbb{Z}^\nu)$ such that $\|\psi_l\| = 1$ and

$$\|(H(\lambda) - E)\psi_l\| < \frac{1}{l}.$$

Suppose the support of ψ_l is contained in a cube of side $r(l)$, centered at 0. Denote by $\Lambda_k(x)$ a cube of side k centered at x in $\mathbb{Z}^\nu$. We denote $V^\omega(n) = a_n q^\omega(n)$, for ease of writing. We then find cubes $\Lambda_{r(l)}(\alpha_l)$ centered at the points α_l such that

$$|V^\omega(\alpha_l) - \lambda| < \frac{1}{l}, \ \ |V^\omega(x)| < \frac{1}{l}, \ \ \forall x \in \Lambda_{r(l)}(\alpha_l) \setminus \{\alpha_l\}.$$

Now consider $\phi_l(x) = \psi_l(x - \alpha_l)$. Then by the translation invariance of H_0 we have for any $\omega \in \Omega_0$,

$$\begin{aligned}\|(H^\omega - E)\phi_l\| &\leq \|(H_0 + V^\omega(\cdot + \alpha_l)) - E)\psi_l\| \\ &\leq \|(H_0 + \lambda P_0 - E)\psi_l\| + \|V^\omega(\cdot + \alpha_l) - \lambda P_0)\phi_l\| \\ &\leq \frac{1}{l} + \frac{1}{l}.\end{aligned} \tag{3}$$

Clearly since ϕ_l is just a translate of ψ_l, $\|\phi_l\| = 1$ for each l. We now have to show that the sequence ϕ_l goes to zero weakly. This is ensured by taking successively α_k large so that

$$\cup_{j=1}^{k-1}\text{supp}(\phi_j) \cap \Lambda_{r(k)}(\alpha_k) = \emptyset, \ \text{ and } \ \text{supp}(\phi_k) \subset \Lambda_{r(k)}(\alpha_k).$$

This is always possible for each ω in Ω_0 by its definition, thus showing that the point E is in the spectrum of H^ω, concluding the proof of the theorem.

Proof of Theorem 2.7. We first consider the part (1) of the theorem and address the proof of (2) later. The set $\mathcal{C}$ below is as in Hypothesis 2.1. We consider the set

$$\mathcal{D} = \{\phi \in \ell^2(\mathbb{Z}^\nu) : \text{supp}(\widehat{\phi}) \subset \mathbb{T}^\nu \setminus \mathcal{C} \ \text{ and } \ \widehat{\phi} \ \text{ smooth}\}, \tag{4}$$

where we denote by $\widehat{\phi}$ the function in $\ell^2(\mathbb{T}^\nu)$ obtained by taking the Fourier series of ϕ. Since the set $\mathcal{C}$ is of measure zero, such functions form a dense subset of $\ell^2(\mathbb{Z}^\nu)$. We also note that the set $\mathcal{C}$ is closed in $\mathbb{T}^\nu$, thus its complement is open (in fact it is a finite union of open rectangles) and each ϕ in $\mathcal{D}$ has compact support in $\mathbb{T}^\nu \setminus \mathcal{C}$.

We first consider the case when μ has compact support. The general case is addressed at the end of the proof.

If we show that the sequence $W(t, \omega) = \mathrm{e}^{itH^\omega}\mathrm{e}^{-itH_0}$ is strongly Cauchy for any ω, then standard scattering theory implies that $\sigma_{ac}(H^\omega) \supset \sigma_{ac}(H_0)$ for that ω. We will show below this Cauchy property for a set ω of full measure.

To this end we consider the quantity

$$\mathbb{E}\{\|(W(t, \omega) - W(r, \omega))\phi\|\}, \ \ \phi \in \mathcal{D} \tag{5}$$

and show that this quantity goes to zero as t and r go to $+\infty$. Then the integrand being uniformly bounded by an integrable function $\|\phi\|$ and since ϕ comes from a dense set, Lebesgue dominated convergence theorem implies that $W(t, \omega)$ is strongly Cauchy for every ω in a set of full measure $\Omega(f)$ that depends on f in $\ell^2(\mathbb{Z}^\nu)$. Since $\ell^2(\mathbb{Z}^\nu)$ is separable, we take the countable dense set $\mathcal{D}_1$ and consider

$$\Omega_3 = \bigcap_{f \in \mathcal{D}_1} \Omega(f)$$

which also has full measure being a countable intersection of sets of full measure. For each $\omega \in \Omega_3$, $W(t,\omega)$ is a family of isometries such that $W(t,\omega)f$ is a strongly Cauchy sequence for each $f \in \mathcal{D}_1$, therefore this property also extends by density of $\mathcal{D}_1$ to all of $\ell^2(\mathbb{Z}^\nu)$ point wise in Ω_3. Thus it is enough to show that the quantity in (5) goes to zero as t and r go to $+\infty$.

We have the following inequality coming out of Cauchy–Schwarz and Fubini, for an arbitrary but fixed $\phi \in \mathcal{D}$. In the inequality below we denote, for convenience the operator of multiplication by the sequence a_n as A and in the first step we write the left hand side as the integral of the derivative to obtain the right hand side

$$\begin{aligned}\mathbb{E}\{\|W(t,\omega)\phi - W(r,\omega)\phi\|\} &\leq \mathbb{E}\left\{\| \int_r^t \mathrm{d}s \ \ \mathrm{e}^{isH^\omega} V^\omega \mathrm{e}^{-isH_0}\phi\|\right\} \\ &\leq \int_r^t \mathrm{d}s \ \ \mathbb{E}\{\|V^\omega \mathrm{e}^{-isH_0}\phi\|\} \\ &\leq \int_r^t \mathrm{d}s \ \ \|\sigma A \mathrm{e}^{-isH_0}\phi\|. \end{aligned} \tag{6}$$

The required statement on the limit follows if we now show that the quantity in the integrand of the last line is integrable in s. To do this we define the number

$$v_\phi = \inf_j \inf\{|h_j'(\theta_j)| : \vartheta \in \operatorname{supp} \widehat{\phi}\}, \quad \vartheta = (\theta_1, \ldots, \theta_\nu). \tag{7}$$

We note that since the support of $\widehat{\phi}$ is compact in $\mathbb{T}^\nu \setminus \mathcal{C}$, $h_j{}'$, $j = 1, \ldots, \nu$ (which are continuous by assumption), have non-zero infima there, so v_ϕ is strictly positive. Then consider the inequalities

$$\begin{aligned}\|\sigma A \mathrm{e}^{-isH_0}\phi\| &\leq \|\sigma A F(|n_j| > v_\phi\ s/4 \ \ \forall j\ \ )\mathrm{e}^{-isH_0}\phi\| \\ &\quad + \|\sigma A F(|n_j| \leq v_\phi\ s/4 \ \text{ for some } \ \ j)\mathrm{e}^{-isH_0}\phi\| \\ &\leq \sigma|g(s)|\|\phi\| + \sigma\|A\|\|F(|n_j| \leq v_\phi\ s/4, \ \text{ for some } \ j)\mathrm{e}^{-isH_0}\phi\|, \end{aligned} \tag{8}$$

where we have used the notation that $F(S)$ denotes the orthogonal projection (in $\ell^2(\mathbb{Z}^\nu)$) given by the indicator function of the set S and used the function g as in the Hypothesis 2.3(2) which is integrable in s, so the first term is integrable in s. We concentrate on the remaining term.

$$\|F(|n_j| \leq v_\phi\ s/4, \ \text{ for some } \ j)\mathrm{e}^{-isH_0}\phi\|. \tag{9}$$

To estimate the term we go to the spectral representation of H_0 and do the computation there as follows. Since $|n_j| \leq v_\phi\ s/4$ for some j, we may without loss of generality set $j = 1$ and proceed with the calculation. Let us denote the set $S_1(s) = \{n : |n_1| \leq v_\phi s/4, \ \ n_j \in \mathbb{Z}, \ \ j \neq 1\}$. In the steps below we pass to $L^2(\mathbb{T}^\nu)$ via the Fourier series, (where the normalized measure on $\mathbb{T}^\nu$ is denoted by $\mathrm{d}\sigma(\vartheta)$).

$$\begin{aligned} T &= \|F(|n_1| \leq v_\phi\ s/4)\mathrm{e}^{-isH_0}\phi\| \\ &= \left\{\sum_{n \in S_1(s)} \left|\langle \delta_n, \mathrm{e}^{-isH_0}\phi\rangle\right|^2\right\}^{1/2} \\ &= \left\{\sum_{n \in S_1(s)} \left|\int_{\mathbb{T}^\nu} \mathrm{d}\vartheta \ \ \mathrm{e}^{-in\cdot\vartheta - is\sum_{j=1}^\nu h_j(\theta_j)}\widehat{\phi}(\vartheta)\right|^2\right\}^{1/2} \end{aligned}$$

$$= \left\{ \sum_{n\in\mathbb{Z}^{\nu-1}} \sum_{|n_1|\le\frac{v_\phi s}{4}} \left| \int_{\mathbb{T}^{\nu-1}} \prod_{j=2}^{\nu} \mathrm{d}\sigma(\theta_j)\ \mathrm{e}^{-i\sum_{j=2}^{\nu}(n_j\theta_j+sh_j(\theta_j))} \int_{\mathbb{T}} \mathrm{d}\sigma(\theta_1) \right.\right.$$

$$\left.\left. \mathrm{e}^{-i(n_1\theta_1+sh_1(\theta_1))}\widehat{\phi}(\vartheta)\mathrm{d}\sigma(\theta_1)\right|^2\right\}^{1/2}. \tag{10}$$

We define the function $J(\theta, s, n_1) = n_1\theta + sh_1(\theta)$. When ϑ is in the support of $\widehat{\phi}$, we have that $|h_1'(\theta_1)| \ge v_\phi$, by eq. (7). This in turn implies that when $\vartheta = (\theta_1, \ldots, \theta_\nu) \in \mathrm{supp}\widehat{\phi}$,

$$\left|\frac{\partial}{\partial\theta} J(\theta_1, s, n_1)\right| = |n_1 + s\ h_1'(\theta_1)| \ge 3v_\phi\ s/4 \ \text{ when }\ n_1 \le v_\phi\ s/4.$$

We use this fact and do integration by parts twice with respect to the variable θ_1 to obtain

$$T = \left\{ \sum_{|n_1|\le\frac{v_\phi s}{4}} \sum_{n\in\mathbb{Z}^{\nu-1}} \left| \int_{\mathbb{T}^{\nu-1}} \prod_{j=2}^{\nu} \mathrm{d}\sigma(\theta_j)\ \mathrm{e}^{-i\sum_{j=2}^{\nu} n_j\theta_j+sh_j(\theta_j)} \int_{\mathbb{T}} \mathrm{d}\sigma(\theta_1) \right.\right.$$

$$\left.\left. \mathrm{e}^{-i(n_1\theta_1+sh_1(\theta_1))} \left\{ \left(\frac{\partial}{\partial\theta_1}\frac{1}{J'(\theta_1, n_1, s)}\right)^2 \widehat{\phi}(\vartheta)\right\} \mathrm{d}\sigma(\theta_1)\right|^2\right\}^{1/2}. \tag{11}$$

We note that the quantity

$$I_1 = \left(\frac{\partial}{\partial\theta_1}\frac{1}{J'(\theta_1, n_1, s)}\right)^2 \widehat{\phi}(\vartheta)$$

$$= \left(\frac{-J^{(3)}}{(J')^3} + \frac{3J^{(2)}(J')^2}{(J')^6}\right)\widehat{\phi} + \frac{1}{(J')^2}\frac{\partial^2}{\partial\theta_1^2}\widehat{\phi} + \frac{-J^{(2)}}{(J')^3}\frac{\partial}{\partial\theta_1}\widehat{\phi} \tag{12}$$

is in $L^2(\mathbb{T}^\nu)$.

The assumptions on the lower bound on J' (when $|n_1| \le v_\phi s/4$) and the boundedness of its higher derivatives by Cs (which is straightforward to verify by the assumption on h_j) together now yield the bound

$$T \le \frac{C}{s^2}\left\{ \|\phi\| + \|\widehat{\frac{\partial}{\partial\theta_1}\phi}\|_{L^2(\mathbb{T}^\nu)} + \|\widehat{\frac{\partial^2}{\partial\theta_1^2}\phi}\|_{L^2(\mathbb{T}^\nu)} \right\}$$

which gives the required integrability.

We proved the case (1) of the theorem assuming that μ has compact support. The case when μ has infinite support requires only a comment on the function $\mathrm{e}^{-isH_0}\phi$ being in the domain on V^ω almost everywhere, when s is finite and for fixed $\phi \in \mathcal{D}$. Once this is ensured the remaining calculations are the same. To see the stated domain condition we first note that for each fixed s, the sequence $(\mathrm{e}^{-isH_0}\phi)(n)$ decays faster than any polynomial, (in $|n|$). The reason being that, by assumption, $\widehat{\phi}$ is smooth and of compact support in $\mathbb{T}^\nu \setminus \mathcal{C}$, $|\phi(n)| \le |n|^{-N}$ for any $N > 0$, as $|n| \to \infty$. On the other hand for $|n - m| > s\|H_0\|$, we have

$$|\mathrm{e}^{-isH_0}(n, m)| \le \frac{1}{|n - m|^N}, \ \text{ for any } N > 0.$$

These two estimates together imply that

$$\|(1 + |m|)^{2\nu+2}\mathrm{e}^{-isH_0}\phi\| < \infty, \ \ \forall\phi \in \mathcal{D}. \tag{13}$$

We now consider the events

$$A_n = \{\omega : |q^\omega(n)| > |n|^{2\nu+1}\}$$

and they satisfy the condition

$$\sum_{n\in\mathbb{Z}^\nu} \text{Prob}(A_n) < \infty,$$

by a simple application of Cauchy–Schwarz and the finiteness of the second moment of μ. Hence, by an application of Borel–Cantelli lemma, only finitely many events A_n occur with full measure. Therefore on a set of full measure all but finitely many $q^\omega(n)$ satisfy, $|q^\omega(n)| \leq |n|^{2\nu+1}$. Let the set of full measure be denoted by Ω_1. Then for each $\omega \in \Omega_1$ we have a finite set $S(\omega)$ such that $\mathrm{e}^{-isH_0}\phi$ is in the domain of the operator $V_1^\omega = V^\omega(I - P_{S(\omega)})$, where $P_{S(\omega)}$ is the orthogonal projection onto the subspace $\ell^2(S(\omega))$, in view of the eq. (13). Then the proof that the a.c. spectrum of the operator

$$H_1^\omega = H_0 + V_1^\omega, \forall \omega \in \Omega_1 \cap \Omega_0$$

goes through as before. Since for each $\omega \in \Omega_1 \cap \Omega_0$, H_1^ω differs from H^ω by a finite rank operator, its absolutely continuous spectrum is unaffected (by trace class theory of scattering) and the theorem is proved.

The statement on the singular part of the spectrum of H^ω, is a direct corollary of the Theorem 5.2. We note firstly that since $\{\delta_n, n \in \mathbb{Z}^\nu\}$ is an orthonormal basis for $\ell^2(\mathbb{Z}^\nu)$ it is automatically a cyclic family for H^ω for every ω.

Secondly, by assumption, the subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not mutually orthogonal, so the conditions of Theorem 5.2 are satisfied. Therefore, since the a.c. spectrum of H^ω contains the interval (E_-, E_+) almost every ω the result follows.

(2) We now turn to the proof of part 2 of the theorem. The essential case to consider again as in (1) is when μ has compact support, the general case goes through as before. The proof is again similar to the one in (1), but we need to choose a dense set $\mathcal{D}_1$ in the place of $\mathcal{D}$ properly.

The operator Δ_+ is self adjoint on $\ell^2(\mathbb{Z}^+)$ and its restriction Δ_{+1} to $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ is unitarily equivalent to multiplication by $2\cos(\theta)$ acting on the image of $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ under the Fourier series map. We now consider the operator

$$H_{0+1} = \Delta_{+1} + H_0$$

in the place of H_{0+} and show the existence of the Wave operators

$$W_+ = \text{slim}_{t\to\infty} \mathrm{e}^{itH_+^\omega} \mathrm{e}^{-itH_{0+1}}$$

almost every ω.

We take the set $\mathcal{D}$ as in eq. (4), $\mathcal{D}_2$ as in Lemma 3.1 and define

$$\mathcal{D}_+ = \left\{ \phi : \phi = \sum_{i,j \text{ finite}} \alpha_{ij}\phi_i \psi_j, \quad \psi_j \in \mathcal{D}, \quad \phi_i \in \mathcal{D}_2, \quad \alpha_{ij} \in \mathbb{C} \right\}. \tag{14}$$

Then $\mathcal{D}_+$ is dense in

$$\mathcal{H}_0 = \{f \in \ell^2(\mathbb{Z}_+^{\nu+1}) : f(0, n) = 0\}.$$

We then define the minimal velocities for $\phi \in \mathcal{D}_+$ with w_{ϕ_1} defined as in Lemma 3.1 for $\phi_1 \in \mathcal{D}_2$.

$$w_{1,\phi} = \inf_k w_{\phi_k}$$
$$w_{2,\phi} = \inf_l \inf_j \inf\{|h'_j(\theta_j)| : \vartheta \in \mathrm{supp}\widehat{\psi_l}\}$$
$$v_\phi = \min\{w_{1,\phi}, w_{2,\phi}\}. \tag{15}$$

Calculating the limits, as in eq. (5)

$$\|(\mathrm{e}^{itH^{\omega+}}\mathrm{e}^{-itH_{0+1}} - \mathrm{e}^{irH^{\omega+}}\mathrm{e}^{-irH_{0+1}})\phi\|$$
$$= \int_r^t \mathrm{d}s\ \|(\mathrm{e}^{isH^{\omega+}}(V^\omega - P_0\Delta_+ + -\Delta_+ P_0 + P_0\Delta_+ P_0)\mathrm{e}^{-itH_{0+1}}\phi\|, \tag{16}$$

where P_0 is the operator $p_0 \otimes I$, with p_0 being the orthogonal projection onto the one dimensional subspace spanned by the vector δ_0 in $\ell^2(\mathbb{Z}^+)$. We note that by the definition of Δ_+, the term $P_0\Delta_+ P_0$ is zero. The estimates proceed as in the proof of (1), after taking averages over the randomness and taking $\phi \in \mathcal{D}_+$. As in that proof it is sufficient to show the integrability in s of the functions

$$\|\sigma A\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_1\rangle\langle\delta_0| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_0\rangle\langle\delta_1| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|,$$

respectively. By the definition of $\mathcal{D}_+$, any ϕ there is a finite sum of terms of the form $\phi_j(\theta_1)\psi_j(\theta_2, \ldots, \theta_{\nu+1})$, so it is enough to show the integrability when ϕ is just one such product, say $\phi = \phi_1\psi_1$. Therefore we show the integrability in s of the functions

$$\|\sigma A\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_1\rangle\langle\delta_0| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|, \quad \||\delta_0\rangle\langle\delta_1| \otimes I\mathrm{e}^{-isH_{0+1}}\phi\|,$$

for s large we are done. We have

$$F(|n_1| > v_\phi s/4)\delta_i = 0, \quad i = 0, 1 \text{ and } \|\sigma A F(|n_j| > v_\phi s/4, \forall j)\| \in L^1(1, \infty),$$

by the Hypothesis 2.3(2′) on the sequence a_n. Therefore it is enough to show the integrability of the norms

$$\|F(|n_j| < v_\phi s/4)\mathrm{e}^{-is\Delta_{0+1}}\phi_1\psi_1\|, \quad \forall \phi_1 \in \mathcal{D}_2, \quad \psi_1 \in \mathcal{D},$$

for each $j = 1, \ldots, \nu+1$. When $j = 2, \ldots, \nu+1$, the proof is as in the previous theorem, while for $j = 1$, the proof is given in the Lemma 3.1 below.

The statement on the absence of singular part of the spectrum of $H^\omega+$ in $(E_- - 2, E_+ + 2)$, is as before a direct corollary of the Theorem 5.2, since the set of vectors $\{\delta_n, n = (0, m), m \in \mathbb{Z}^\nu\}$ is a cyclic family for H^ω_+, for almost all ω and $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$ are not mutually orthogonal for almost all ω when m, n are in $\{(0, n) : n \in \mathbb{Z}^\nu\}$, and the fact that the a.c. spectrum of H^ω contains the interval $(-2 + E_-, 2 + E_+)$ almost every ω.

The lemma below is as in Jaksic–Last [13](Lemma 3.11) and the enlarging of the space in the proof is necessary since there are no non-trivial functions in $\ell^2(\mathbb{Z}^+)$ whose Fourier series has compact support in $(0, 2\pi)$ (all of them being boundary values of functions analytic in the disk).

Lemma 3.1. *Consider the operator Δ_{+1} on $\ell^2(\mathbb{Z}^+)$. Then there is a set $\mathcal{D}_2$ dense in $\ell^2(\mathbb{Z}^+)$ and a number w_ϕ such that for $s \geq 1$,*

$$\|F(|n| < w_\phi s/4)\mathrm{e}^{-is\Delta_{1+}}\phi\| \leq C|s|^{-2}, \quad \forall \phi \in \mathcal{D}_2.$$

with the constant C independent of s.

Proof. We first consider the unitary map $\mathcal{W}$ from $\mathcal{H}_0$ to a subspace $\mathcal{S}$ of $\{f \in \ell^2(\mathbb{Z}) : f(0) = 0\}$, given by

$$(\mathcal{W}f)(n) = \begin{cases} \frac{1}{\sqrt{2}} f(n), & n > 0 \\ -\frac{1}{\sqrt{2}} f(-n), & n < 0. \end{cases} \tag{17}$$

Then the range of $\mathcal{W}$ is a closed subspace of $\ell^2(\mathbb{Z})$ and consists of functions

$$\mathcal{S} = \{f \in \ell^2(\mathbb{Z}) : f(n) = -f(-n)\}.$$

Under the Fourier series map this subspace goes to

$$\widehat{\mathcal{S}} = \{\phi \in L^2(\mathbb{T}) : \phi(\theta) = -\phi(-\theta)\}$$

so that the functions here have mean zero. Then under the map from $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ to $\widehat{\mathcal{S}}$ obtained by composing $\mathcal{W}$ and the Fourier series map, the operator Δ_{1+} goes to multiplication by $2\cos(\theta)$. We now choose a set

$$\mathcal{D}_1 = \{\phi \in \widehat{\mathcal{S}} : \mathrm{supp}(\phi) \subset \mathbb{T} \setminus \{0, \pi\}\},$$

and define the number

$$w_\phi = \inf\{|2\sin(\theta)| : \theta \in \mathrm{supp}(\phi)\},$$

for each $\phi \in \mathcal{D}_1$. We denote by $\mathcal{D}_2$ all those functions whose images under the composition of $\mathcal{W}$ and the Fourier series lies in $\mathcal{D}_1$. The density of $\mathcal{D}_2$ in $\ell^2(\mathbb{Z}^+ \setminus \{0\})$ is then clear. We shall simply denote by f_ϕ elements in $\mathcal{D}_2$ whose images in $\mathcal{D}_1$ is ϕ. Given a $\phi \in \mathcal{D}_1$ and a w_ϕ we see that

$$\|F(|n| \leq w_\phi s/4)\mathrm{e}^{-is\Delta_{1+}} f_\phi\|^2 = \sum_{|n| < w_\phi s/4} \left| \int_{\mathbb{T}} \mathrm{d}\sigma(\theta) \mathrm{e}^{-in\theta - i2s\cos(\theta)} \phi(\theta) \right|^2 \leq C|s|^{-4},$$

by a simple integration by parts, done twice, using the condition that $||n| + 2s\sin(\theta)| > w_\phi s/4$ in the support of ϕ.

Proof of Theorem 2.8. The proof of this theorem is based on a technique of Aizenman [1]. We break up the proof into a few lemmas. First we show that the free operators H_0 and H_{0+} have resolvent kernels with some summability properties, for energies in their resolvent set.

Lemma 3.2. *Consider a function h satisfying the Hypothesis* 2.1 *and consider the associated operators H_0 or H_{0+}. Then for all $s \geq \nu/(3\nu + 3)$,*

$$\sup_{n \in \mathbb{Z}^\nu} \sum_{n \in \mathbb{Z}^\nu} |\langle \delta_n, (H_0 - E)^{-1} \delta_m \rangle|^s < C(E),$$

and $C(E) \to 0, |E| \to \infty$. Similarly we also have for all $s > \nu/(3\nu + 3)$,

$$\sup_{n \in \mathbb{Z}_+^{\nu+1}} \sum_{n \in \mathbb{Z}^\nu} |\langle \delta_n, (H_{0s} - E)^{-1} \delta_m \rangle|^s < C(E).$$

Proof. We will prove the statement for H_0, the proof for H_{0+} is similar. We write the expression for the resolvent kernel in the Fourier transformed representation (we write the Fourier series of an $\ell^2(\mathbb{Z}^\nu)$ function as $\widehat{u}(\vartheta) = \sum_{n\in\mathbb{Z}^\nu} \mathrm{e}^{in\cdot\vartheta} u(n))$, use the Hypothesis 2.1(1), and integrate by parts $3\nu + 3$ times with respect to the variable θ_j (recall that $\vartheta = (\theta_1, \ldots, \theta_\nu))$, to get the inequalities

$$\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle \;= \int_{\mathbb{T}^\nu} \mathrm{d}\sigma(\vartheta)\;\; \mathrm{e}^{i(m-n)\cdot\vartheta}\;\, (h(\vartheta) - E)^{-1} = \frac{(i)^{3\nu+3}}{((m-n)_j)^{3\nu+3}} \times \int_{\mathbb{T}^\nu} \mathrm{d}\sigma(\vartheta)\;\, \mathrm{e}^{i(m-n)\cdot\vartheta}\;\, \frac{\partial^{3\nu+3}}{\partial\theta_j^{3\nu+3}}(h(\vartheta) - E)^{-1}, \tag{18}$$

where we have chosen the index j such that $|(m-n)_j| \geq |m-n|/\nu$ and assumed that $m \neq n$ (when $m = n$ the quantity is just bounded). Let us set

$$C_0(E) = \max\;\left\{ \sup_{\vartheta\in\mathbb{T}^\nu} \left| \frac{\partial^{3\nu+3}}{\partial\theta_j^{3\nu+3}}(h(\vartheta) - E)^{-1} \right|,\; |(h(\vartheta) - E)^{-1}| \right\}.$$

It is easy to see that since the function h is of compact range and all its $3\nu + 3$ partial derivatives are bounded, by hypothesis $C_0(E)$ goes to zero as $|E|$ goes to ∞. We then get the bound for any $s > \nu/(3\nu + 3)$,

$$|\,\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle\,| \leq \frac{\nu^{3\nu+3}}{|m-n|^{3\nu+3}} C_0(E).$$

Given this estimate we have

$$\begin{aligned} \sup_{n\in\mathbb{Z}^\nu} \sum_{n\in\mathbb{Z}^\nu} |\,\langle\, \delta_n, (H_0 - E)^{-1}\delta_m \,\rangle\,|^s &\leq C_0(E)^s \left(\sup_{n\in\mathbb{Z}^\nu} \left(1 + \sum_{\substack{n\in\mathbb{Z}^\nu\\ m\neq n}} \left| \frac{\nu^{s(3\nu+3)}}{|m-n|^{s(3\nu+3)}} \right| \right)\right) \\ &\leq C_0(E)^s \left(1 + \sum_{n\in\mathbb{Z}^\nu, m\neq 0} \left| \frac{\nu^{s(3\nu+3)}}{|m|^{s(3\nu+3)}} \right| \right), \\ &\leq C_0(E)^s C(s), \end{aligned} \tag{19}$$

where $C(s)$ is finite since $|m|^{-s(3\nu+3|)},\; m \neq 0$ is a summable function in $\mathbb{Z}^\nu$ when $s(3\nu + 3) > \nu$.

Proof of Corollary 2.9. We prove the theorem only for the case H^ω the proof of the other case is similar.

By the Hypothesis 2.3(2) on the finiteness of the second moment of μ we see that $\int \mathrm{d}\mu(x)\, |x| < \infty$, so that we can set $\tau = 1$ in the Lemma 5.1. Since the assumption in the theorem ensures the boundedness of the density of μ we can also set $q = \infty$ in the Lemma 5.1 with then $Q^{1/1+q} = \|\mathrm{d}\mu/\mathrm{d}x\|_\infty$. Then in the Lemma 5.1 the constant C is given by

$$C\left(Q, \frac{\kappa}{1-2\kappa}, \infty\right) = 1 + \frac{2\kappa Q}{1-\kappa}.$$

The condition on the constant κ becomes

$$\kappa < 1/3.$$

Below we choose a s satisfying $\frac{\nu}{(3\nu+3)} < s < 1/3$, and consider the expression

$$G(\omega, z, n, m) = \;\langle\, \delta_n, (H^\omega - z)^{-1}\delta_m \,\rangle,\;\; G(0, z, n, m) = \;\langle\, \delta_n, (H_0 - z)^{-1}\delta_m \,\rangle,$$

where we take $z = E + i\epsilon$ with $\epsilon > 0$. Then by the resolvent equation we have

$$G(\omega, z, n, m) = G(0, z, n, m) - \sum_{l \in Znu} G(\omega, z, n, l) V^{\omega}(l) G(0, z, l, m). \quad (20)$$

We denote by

$$G_l(\omega, z, n, m) = \langle \delta_n, (H^{\omega} - V^{\omega}(l) P_l - z)^{-1} \delta_m \rangle,$$

where P_l is the orthogonal projection onto the subspace generated by δ_l. Then using the rank one formula

$$G(\omega, z, n, l) = \frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + G_l(\omega, z, l, l)^{-1}}$$

whose proof is again by resolvent equation, we see that eq. (20) can be rewritten as

$$\begin{aligned} G(\omega, z, n, m) &= G(0, z, n, m) \\ &\quad + \sum_{l \in \mathbb{Z}^{\nu}} \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + G_l(\omega, z, l, l)^{-1}} \right) V^{\omega}(l) G(0, z, l, m). \end{aligned} \quad (21)$$

Raising both the sides to power s (noting that $s < 1$ so the inequalities are valid), we get

$$\begin{aligned} |G(\omega, z, n, m)|^s &= |G(0, z, n, m)|^s \\ &\quad + \sum_{l \in \mathbb{Z}^{\nu}} \left| \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + (G_l(\omega, z, l, l)^{-1}} \right) \right|^s |V^{\omega}(l)|^s |G(0, z, l, m)|^s. \end{aligned} \quad (22)$$

Now observing that G_l is independent of the random variable $V^{\omega}(l)$, we see that

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &= |G(0, z, n, m)|^s \\ &\quad + \sum_{l \in \mathbb{Z}^{\nu}} \mathbb{E} \left(\left| \left(\frac{\frac{G_l(\omega,z,n,l)}{G_l(\omega,z,l,l)}}{V^{\omega}(l) + (G_l(\omega, z, l, l)^{-1}} \right) \right|^s |V^{\omega}(l)|^s \right) |G(0, z, l, m)|^s. \end{aligned} \quad (23)$$

This then becomes, integrating with respect to the variable $q^{\omega}(l)$, remembering that $V^{\omega}(l) = a_l q^{\omega}(l)$,

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &= |G(0, z, n, m)|^s \\ &\quad + \sum_{l \in \mathbb{Z}^{\nu}} \mathbb{E} \left(\left| \frac{G_l(\omega, z, n, l)}{G_l(\omega, z, l, l)} \right|^s \right) \\ &\quad \times \int \left(\mathrm{d}\mu(x) \, \frac{|x|^s}{|x + a_l^{-1} G_l(\omega, z, l, l)^{-1}|^s} \right) |G(0, z, l, m)|^s \end{aligned} \quad (24)$$

which when estimated using the Lemma 5.1 yields

$$\begin{aligned} \mathbb{E}(|G(\omega, z, n, m)|^s) &\leq |G(0, z, n, m)|^s \\ &\quad + \sum_{l \in \mathbb{Z}^{\nu}} K_s \mathbb{E} \left(\left| \frac{G(\omega, z, n, l)}{G_l(\omega, z, l, l)} \right|^s \right) \\ &\quad \times \int \left(\mathrm{d}\mu(x) \, \frac{1}{|x + a_l^{-1} G_l(\omega, z, l, l)^{-1}|^s} \right) |G(0, z, l, m)|^s, \end{aligned} \quad (25)$$

where K_s is the constant appearing in Lemma 5.1 with κ set equal to s. We take $K = (\sup_n |a_n|^s)K_s$, and rewrite the above equation to obtain

$$\mathbb{E}(|G(\omega, z, n, m)|^s) = |G(0, z, n, m)|^s + \sum_{l \in \mathbb{Z}^\nu} K\mathbb{E}(|(G(\omega, z, n, l)|^s |G(0, z, l, m)|^s. \qquad (26)$$

We now sum both the sides over m, set

$$I = \sum_{m \in \mathbb{Z}^\nu} \mathbb{E}(|G(\omega, z, n, m)|^s)$$

and obtain the inequality

$$I \leq \sum_{m \in \mathbb{Z}^\nu} |G(0, z, n, m)|^s + \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in \mathbb{Z}^\nu} KI|G(0, z, l, m)|^s.$$

Therefore when there is an interval (a, b) in which

$$K \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in Znu} |G(0, z, l, m)|^s < 1, \quad E \in (a, b), \qquad (27)$$

we obtain that

$$\int_a^b \mathrm{d}E \sum_{m \in \mathbb{Z}^\nu} \mathbb{E}(|G(\omega, E + i0, n, m)|^s) < \infty,$$

by an application of Fatou's lemma implying that for almost all $E \in (a, b)$ and almost all ω, we have the finiteness of

$$\sum_{m \in \mathbb{Z}^\nu} |G(\omega, E + i0, n, m)|^2 < \infty,$$

satisfying the Simon–Wolff [19] criterion. This shows that (the proof follows as in Theorems II.5, II.6 [18]) the measures

$$\nu_n^\omega(\cdot) = \langle \delta_n, E_{H^\omega}(\cdot)\delta_n \rangle$$

are pure point in (a, b) almost every ω. This happens for all n, hence the total spectral measure of H^ω itself is pure point in (a, b) for almost all ω.

There are two different ways to fix the critical energy $E(\mu)$ now. Firstly if K is large, then in view of the Lemma 3.2 (by which $C_0(E) \to 0,\ |E| \to \infty$) and the fact that K is finite (by Lemma 5.1)

$$K \sup_{l \in \mathbb{Z}^\nu} \sum_{m \in Znu} |G(0, z, l, m)|^s \leq KC_0(E)^s C(s) < 1, \quad |E| \to \infty. \qquad (28)$$

Therefore there is a large enough $E(\mu)$ such that for all intervals (a, b) in $(-\infty, -E(\mu)) \cup (E(\mu), \infty)$, the condition in eq. (25) is satisfied.

On the other hand if the moment $B = \int |x|\, \mathrm{d}\mu(x)$ is very small, then we can choose $E(\mu)$ by the condition,

$$KC_0(E)C_s < 1,$$

even when $C_0(E) > 1$, since it is finite for E in the resolvent set of H_0 by Lemma 3.2.

4. Examples

In this section we present some examples of the operators H_0 considered in the theorems. We only give the functions h stated in the Hypothesis 2.1.

- *Examples of operators H_0*

1. $h(\vartheta) = \sum_{i=1}^{\nu} 2\cos(\theta_i)$, corresponds to the usual discrete Schrödinger operator and it is obvious that the Hypothesis 2.1 are satisfied. The Jaksic–Last condition 5.2 on mutual non-orthogonality of the subspaces generated by H_0 and δ_n for different n in $\mathbb{Z}^\nu$ are also satisfied, by an elementary calculation taking powers of H_0 depending upon a pair of vectors δ_n and δ_m, since the operator H_0 is given by $T + T^{-1}$, with T being the bilateral shift on $\ell^2(\mathbb{Z})$.

2. $h(\vartheta) = \sum_{i=1}^{\nu} h_i(\theta_i), h_i(\theta_i) = \sum_{k=1}^{N(i)} \cos(k\theta_i), \quad N(i) < \infty$. Clearly each h_i is a smooth function in $\mathbb{R}^\nu$ and each h_i and all its derivatives are 2π periodic. Hence the Hypothesis 2.3 is satisfied. Further each of h_i is a trigonometric polynomial, and its derivative is also a trigonometric polynomial and hence has only finitely many zeros on the circle.

   The condition in Jaksic–Last condition Theorem 5.2 on mutual non-orthogonality is again elementary to verify in this case.

3. Consider the functions

$$h_i(\theta_i) = \theta_i^{3\nu+4}(2\pi - \theta_i)^{3\nu+4}, \ \ 0 \le \theta_i \le 2\pi, \ \ i = 1, \ldots, \nu$$

   and take $h = \sum_{i=1}^{\nu} h_i(\theta)$ extended to the whole of $\mathbb{R}^\nu$ periodically. Clearly these are in $C^{3\nu+3}(\mathbb{T}^\nu)$, by construction.

- *Examples of pairs (a_n, μ)*

We give next some examples of sequences a_n satisfying the Hypothesis 2.2 such that

$$\text{supp}(\mu) = \text{a-supp}(\mu).$$

We consider $\nu \ge 2$ and the sequence $a_n = (1 + |n_1|)^\alpha, \ \alpha < -1$. Then we have that

$$k\mathbb{Z}^\nu \cap \{(0, n) : n \in \mathbb{Z}^{\nu-1}\} = \{(0, n) : n \in k\mathbb{Z}^{\nu-1}\}$$

and $a_{(0,n)}^{-1}(a,\ b) =$ (a, b) for any interval (a, b) and any $n \in \mathbb{Z}^{\nu-1}$. Therefore for any positive integer k, we have

$$\sum_{m \in k\mathbb{Z}^\nu} \mu(a_m^{-1}(a,\ b)) \ge \sum_{m \in k\mathbb{Z}^{\nu-1}} \mu((a,\ b)) = \infty$$

whenever $\mu((a,\ b)) > 0$.

- *Examples of measures μ with small moment*

We next give an example of an absolutely continuous measure of compact support such that the Aizenman condition (in Lemma 5.1 is satisfied. We use the notation used in that lemma for the example.

We consider numbers $0 < \epsilon, \delta < 1$, R and let μ be given by

$$\mathrm{d}\mu(x)/\mathrm{d}x = \begin{cases} \frac{1-\epsilon}{\delta}, & 0 \le x \le \delta, \\ \frac{\epsilon}{R-\delta}, & \delta < x \le R, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Then μ is an absolutely continuous probability measure and

$$Q \le \frac{1}{\delta} + \frac{1}{R-\delta}.$$

We take $\tau = 1$, then the moment B is bounded by

$$B \le (1-\epsilon)\delta + (R+\delta)\epsilon/2.$$

Now if we fix R large and choose $\epsilon = 1/R^3$ and $\delta = 1/R^2$, we obtain an estimate

$$B^\kappa \le \frac{2^\kappa}{R^{2\kappa}} \text{ and } B^\kappa Q^{1-2\kappa} \le 8R^{2-6\kappa}.$$

Taking $\kappa = s$ in the lemma and noting that $s < 1/3$ implies $2 - 6s < 0$ so that both the terms above go to zero as R goes to ∞. We see that by taking μ with large support but small moment, we can make the constant K in the Lemma 5.1 as small as we want. This in particular means that in the Theorem 2.8. given a energy E_0 outside the spectrum of H_0 we can find a measure μ which is absolutely continuous of small moment such that K is smaller than $C_0(E_0)^s C_s$ in the proof of Theorem 2.8. and hence $E(\mu) < |E_0|$. We can use such measures to give examples of operators with compact spectrum with both a.c. spectrum and pure point spectrum present but in disjoint regions.

- *Example when Jaksic–Last condition is violated*

We finally give examples where Jaksic–Last condition is violated and yet the conclusion of their theorem is valid.

Consider $\nu = 1$, for simplicity, and let $h(\theta) = 2\cos(2\theta)$. Then the associated H_0 has purely a.c. spectrum in [-2, 2] and we see that the operator $H_0 = T^2 + T^{-2}$ if T is the bilateral shift acting on $\ell^2(\mathbb{Z})$. Then if we consider the operators $H^\omega = H_0 + V^\omega$, and the cyclic subspaces $\mathcal{H}_{\omega,1}, \mathcal{H}_{\omega,2}$ generated by the H^ω and the vectors δ_1, δ_2 respectively, such an operator satisfies

$$\mathcal{H}_{\omega,1} \subset \ell^2(\{1\} + 2\mathbb{Z}), \quad \mathcal{H}_{\omega,2} \subset \ell^2(\{1+1\} + 2\mathbb{Z}), \quad \text{almost every } \omega.$$

We then have

$$\mathcal{H}_{\omega,1} \subset \ell^2(\{2n+1,\ n \in \mathbb{Z}\}), \quad \mathcal{H}_{\omega,2} \subset \ell^2(2\mathbb{Z}), \quad \text{almost every } \omega.$$

The subspaces $\ell^2(\{n :\ n \text{ odd}\})$ and $\ell^2(2\mathbb{Z})$ are generated by the families $\{\delta_k,\ k \text{ odd}\}$ and $\{\delta_k,\ k \text{ even}\}$ respectively. (We could have taken any odd integer k in the place of 1 to do the above)

These two are invariant subspaces of H^ω which are mutually orthogonal, a.e. ω. Therefore the Jaksic–Last theorem is not directly valid. However, by considering the restrictions of H^ω to these two subspaces, one can go through their proof in these subspaces to again obtain the purity of a.c. spectrum for such operators when they exist.

We consider two examples to illustrate the point, for which we let $q^\omega(n)$ denote a collection of i.i.d. random variables with an absolutely continuous distribution μ of compact support in $\mathbb{R}$, its support containing 0.

1. If $V^\omega(n) = a_n q^\omega(n)$, with $0 < a_n < (1+|n|)^{-\alpha}$, $\alpha > 0$, we see that there is pure a.c. spectrum in [–2, 2], a.e. ω by applying trace class theory of scattering.
2. On the other hand if, with $0 < a_n < (1+|n|)^{-\alpha}$, $\alpha > 1$,

$$V^\omega(n) = \begin{cases} a_n q^\omega(n), & n \text{ odd} \\ q^\omega(n), & n \text{ even}, \end{cases}$$

then there is dense pure point spectrum embedded in the a.c. spectrum in [–2, 2].

We can give similar, but non trivial, examples in higher dimensions but we leave it to the reader.

5. Appendix

In this appendix we collect two theorems we use in this paper. One is a lemma of Aizenman [1] and another a theorem of Jaksic–Last [14].

The first lemma and its proof are those of Aizenman [1](Lemma A.1) which reproduce below (with some modifications in the form we need), with a slight change in notation (we in particular call the number s in Aizenman's lemma as κ),

Lemma 5.1 (*Aizenman*). *Let μ be an absolutely continuous probability measure whose density f satisfies $\int_{\mathbb{R}} \mathrm{d}x |f(x)|^{1+q} = Q < \infty$ for some $q > 0$. Let $0 < \tau \leq 1$ and suppose $B \equiv \int_{\mathbb{R}} \mathrm{d}\mu(x)\, |x|^\tau < \infty$. Then for any*

$$\kappa < \left[1 + \frac{2}{\tau} + \frac{1}{q}\right]^{-1}$$

we have

$$\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa} < K_\kappa \int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa}, \quad \text{for all} \quad \alpha \in \mathbb{C},$$

with K_κ given by

$$K_\kappa = B^{\frac{\kappa}{\tau}}(2^{1+2\kappa} + 4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q)^{\frac{\tau-2\kappa}{\tau}}\right] < \infty.$$

Remark. We see from the explicit form of the constant K_κ that the moment B can be made sufficiently small by the choice of μ even when its support is large. This will ensure that in some models of random operators, the region where the Simon–Wolff criterion is valid extends to the region in the spectrum. This is the reason for our writing K_κ in this form.

Proof. The strategy employed in proving the lemma is to consider the ratio

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa}}$$

and obtain upper bounds for the numerator and lower bounds for the denominator.

Note first that B finite and $\kappa < \tau$ implies that $|x-\alpha|^\kappa$ is integrable even if α is purely real and we have

$$\int_a^b f(x)\mathrm{d}x \leq Q^{\frac{1}{1+q}} |b-a|^{\frac{q}{1+q}} \tag{30}$$

by Hölder inequality. Hence

$$\begin{aligned}\int \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa} &\le 1 + \int_1^\infty \mathrm{d}t\ \mu(\{x : \frac{1}{|x-\alpha|^\kappa} \ge t\}) \\ &\le 1 + \frac{\kappa(2^q Q)^{\frac{1}{1+q}}}{\frac{q}{1+q} - \kappa} \\ &\equiv C(Q,\kappa,q), \end{aligned} \tag{31}$$

where the integral is estimated using the estimate in eq. (30).

Consider the region $|\alpha| > (2B)^{\frac{1}{\tau}}$: We then estimate for fixed α the contributions from the regions $|x| \le |\alpha|/2$ and $|x| > |\alpha|/2$ to obtain

$$\begin{aligned}\int \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa} &\le \frac{2^\kappa}{|\alpha|^\kappa}\left(\int \mathrm{d}\mu(x)\ |x|^\kappa + \int \mathrm{d}\mu(x)\ \frac{|x|^{2\kappa}}{|x-\alpha|^\kappa}\right) \\ &\le \frac{2^\kappa}{|\alpha|^\kappa}(B + B^{\frac{2\kappa}{\tau}}\ C\left(Q, \frac{\kappa}{1-2\kappa/\tau}, q)^{\frac{\tau-2\kappa}{\tau}}\right), \end{aligned} \tag{32}$$

with κ chosen so that $\kappa/(1-2\kappa/\tau) < q/(1+q)$. (Here we have explicitly calculated the p occurring in the lemma of Aizenman in terms of κ and τ). For a fixed τ and q this condition is satisfied whenever κ satisfies the inequality stated in the lemma.

The lower bounds on $\int \mathrm{d}\mu(x)\ 1/|x-\alpha|^\kappa$ is obtained first by noting that $B < \infty$ implies

$$\mu(\{x : |x|^\tau > (2B)\}) \le \frac{1}{2}.$$

Since $|\alpha| > (2B)^{\frac{1}{\tau}}$, we have the trivial estimate

$$\begin{aligned}\int \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^\kappa} &\ge \int_{|x|>(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^\kappa} + \int_{|x|\le(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\ge \int_{|x|\le(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\ge \frac{1}{2(|\alpha| + (2B)^{\frac{1}{\tau}})^\kappa}. \end{aligned} \tag{33}$$

Putting the inequalities in (32) and (33) together we obtain, (remembering that $|\alpha| > (2B)^{\frac{1}{\tau}}$),

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^\kappa}} \le 2^{1+2\kappa} B^{\frac{\kappa}{\tau}}\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q)^{\frac{\tau-2\kappa}{\tau}}\right]. \tag{34}$$

We now consider the region $|\alpha| < (2B)^{\frac{1}{\tau}}$: Estimating as in eq. (32) but now splitting the region as $|x| \le (2B)^{\frac{1}{\tau}}$ and $|x| > (2B)^{\frac{1}{\tau}}$, we obtain the analogue of the estimate in eq. (32), in this region of α as

$$\begin{aligned}\int \mathrm{d}\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa} &\le \frac{1}{(2B)^{\frac{1}{\tau}}}\left(\int \mathrm{d}\mu(x)\ |x|^\kappa + \int \mathrm{d}\mu(x)\ \frac{|x|^{2\kappa}}{|x-\alpha|^\kappa}\right) \\ &\le \frac{1}{(2B)^{\frac{1}{\tau}}}\left(B + B^{\frac{2\kappa}{\tau}}\ C\left(Q, \frac{\kappa}{1-2\kappa/\tau}, q\right)^{\frac{\tau-2\kappa}{\tau}}\right). \end{aligned} \tag{35}$$

Similarly the estimate for the denominator term is done as in eq. (33),

$$\begin{aligned}\int \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}} &\geq \int_{|x|>(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}} + \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}}\\ &\geq \int_{|x|\leq(2B)^{\frac{1}{\tau}}} \mathrm{d}\mu(x)\frac{1}{|x-\alpha|^{\kappa}}\\ &\geq \frac{1}{2((2B)^{\frac{1}{\tau}}+(2B)^{\frac{1}{\tau}})}\\ &= \frac{1}{4(2B)^{\frac{1}{\tau}}}. \end{aligned} \tag{36}$$

Using the above two inequalities we obtain the estimate,

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^{\kappa}}} \leq 4\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C\left(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q\right)^{\frac{\tau-2\kappa}{\tau}}\right], \tag{37}$$

when $|\alpha| \leq (2B)^{\frac{1}{\tau}}$. Using the inequalities (34) and (36) obtained for these two regions of values of α we finally get

$$\frac{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{|x|^{\kappa}}{|x-\alpha|^{\kappa}}}{\int_{\mathbb{R}} \mathrm{d}\mu(x)\ \frac{1}{|x-\alpha|^{\kappa}}} \leq B^{\frac{\kappa}{\tau}}(2^{1+2\kappa}+4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C\left(Q, \frac{\kappa}{1-\frac{2\kappa}{\tau}}, q\right)^{\frac{\tau-2\kappa}{\tau}}\right], \tag{38}$$

for any $\alpha \in \mathbb{R}$.

We next state a theorem (Corollary 1.1.3) of Jaksic–Last [14] without proof, its proof is as in Corollary 1.1.3 of Jaksic–Last [14]. We state it in the form we use in this paper.

Theorem 5.2 [*Jaksic–Last*]. *Suppose $\mathcal{H}$ is a separable Hilbert space and A a bounded self adjoint operator. Suppose $\{\phi_n\}$ are normalized vectors and let P_n denote the orthogonal projection on to the one dimensional subspace generated by each ϕ_n. Let $q^{\omega}(n)$ be independent random variables with absolutely continuous distributions μ_n. Consider*

$$A^{\omega} = A + \sum_n q^{\omega}(n) P_n, \quad \textit{almost every}\ \ \omega.$$

Suppose that the following conditions are valid

1. *The family $\{\phi_n\}$ is a cyclic family for A^{ω} a.e. ω.*

2. *Let $\mathcal{H}_{\omega,n}$ denote the cyclic subspace generated by A^{ω} and ϕ_n. Then the cyclic subspaces $\mathcal{H}_{\omega,n}$ and $\mathcal{H}_{\omega,m}$, are not orthogonal.*

Then whenever there is an interval (a,b) in the absolutely continuous spectrum of $A^{\omega} = A + \sum_n q^{\omega}(n) P_n$, almost all ω, we have

$$\sigma_s(A^{\omega}) \cap (a,b) = \emptyset, \quad \textit{almost every}\ \ \omega.$$

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References

[1] Aizenman M, Localization at weak disorder: Some elementary bounds, *Rev. Math. Phys.* **6** (1994) 1163–1182

[2] Aizenman M and Graf S, *Localization bounds for electron gas*, preprint mp_arc 97–540 (1997)

[3] Aizenman M and Molchanov S, Localization at large disorder and at extreme energies: an elementary derivation, *Commun. Math. Phys.* **157** (1993) 245–278

[4] Boutet de Monvel A and Sahbani J, On the spectral properties of discrete Schrödinger operators, *C. R. Acad. Sci. Paris, Series I* **326** (1998) 1145–1150

[5] Boutet de Monvel A and Sahbani J, On the spectral properties of discrete Schrödinger operators: multidimensional case, to appear in *Rev. Math. Phys.*

[6] Carmona R and Lacroix J, *Spectral theory of random Schrödinger operators* (Boston: Birkhäuser Verlag) (1990)

[7] Cycon H, Froese R, Kirsch W and Simon B, *Topics in the Theory of Schrödinger operators* (New York: Springer-Verlag, Berlin, Heidelberg) (1987)

[8] Figotin A and Pastur L, *Spectral properties of disordered systems in the one body approximation* (Berlin, Heidelberg, New York: Springer-Verlag) (1991)

[9] Kirsch W, Krishna M and Obermeit J, Anderson model with decaying randomness-mobility edge. *Math. Zeit.* (2000) DOI 10.1007/s002090000136

[10] Krishna M, Anderson model with decaying randomness – Extended states, *Proc. Indian. Acad. Sci. (Math. Sci.)* **100** (1990) 220–240

[11] Krishna M, Absolutely continuous spectrum for sparse potentials, *Proc. Indian. Acad. Sci. (Math. Sci.)*, **103(3)** (1993) 333–339

[12] Krishna M and Obermeit J, Localization and mobility edge for sparsely random potentials, preprint xxx.lanl.gov/math-ph/9805015

[13] Jaksic V and Last Y, Corrugated surfaces and a.c. spectrum (to appear in *Rev. Math. Phys*)

[14] Jaksic V and Last Y, Spectral properties of Anderson type operators, *Invent. Math.* **141** (2000) 561–577

[15] Jaksic V and Molchanov S, On the surface spectrum in dimension two, *Helvetica Phys. Acta* **71** (1999) 169–183

[16] Jaksic V and Molchanov S, Localization of surface spectra, *Commun. Math. Phys.* **208** (1999) 153–172

[17] Reed M and Simon B, *Methods of modern mathematical physics: Functional analysis* (New York: Academic Press) (1975)

[18] Simon B, Spectral analysis of rank one perturbations and applications in CRM Lecture Notes (eds) J Feldman, R Froese, and L Rosen, *Am. Math. Soc.* **8** (1995) 109–149

[19] Simon B and Wolff T, Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, *Comm. Pure Appl. Math.* **39** (1986) 75–90

[20] Stein E, *Harmonic analysis – real variable methods, orthogonality and oscillatory integrals* (New Jersey: Princeton University Press, Princeton) (1993)

[21] Weidman J, *Linear operators in Hilbert spaces, GTM-68* (Berlin: Springer-Verlag) (1987)