THROUGH A RAILWAY WINDOW.

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The object of this note is to describe in the language of mathematics, the apparent twisting and writhing of a landscape as seen by a moving observer.

1. *Introduction.*

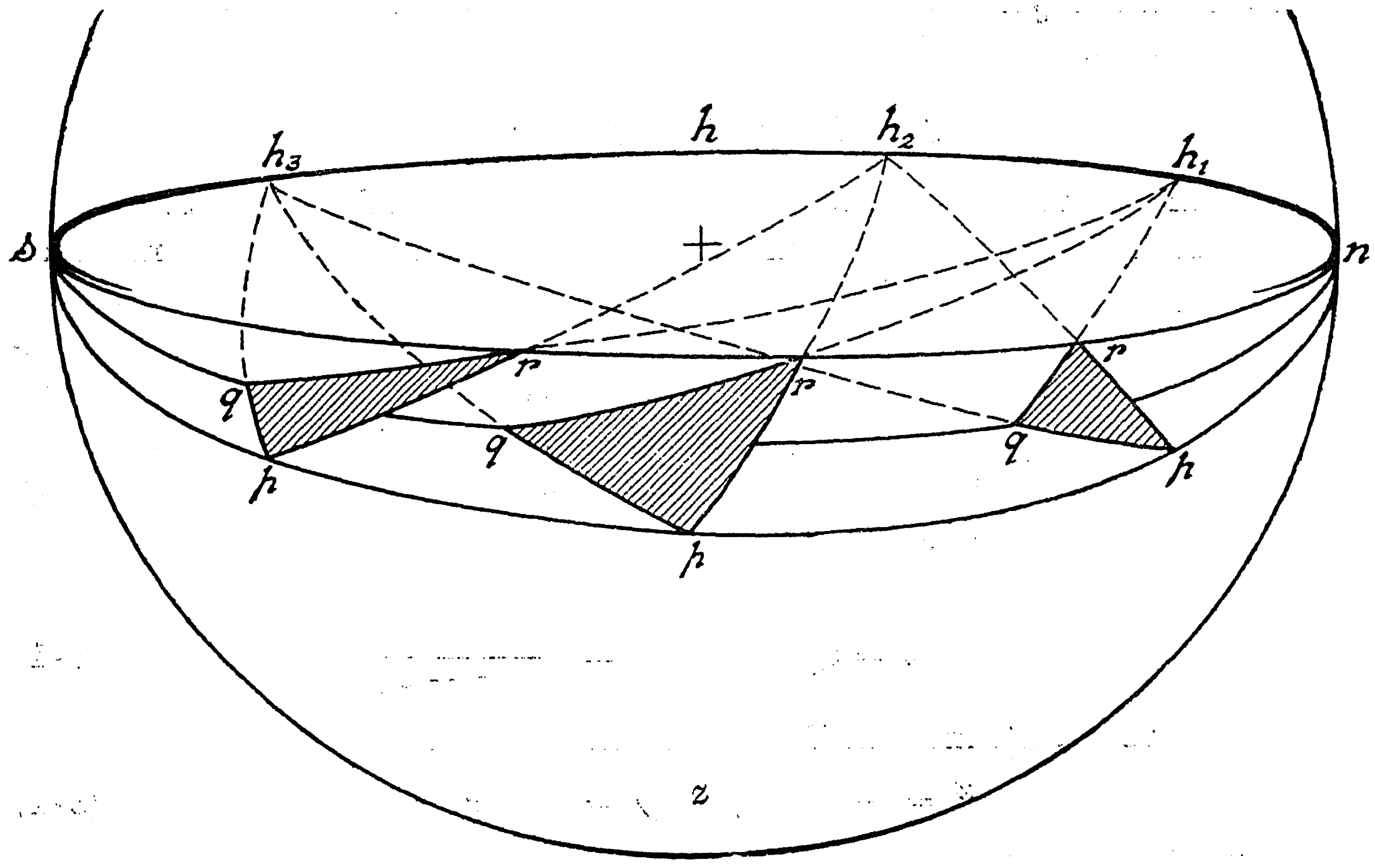
Our experience of the geometry of the external world is derived mainly from two sources—the sensations of touch (including the kinesthetic sense) and of sight. Let us call these experiences " tactile geometry " and " visual geometry ". The former is limited to our immediate neighbourhood and is responsible for our ideas of coincidence and separation and to our concepts of displacement and superposition—concepts which lie at the foundations of Euclid's Geometry. Conversely, Euclidean geometry may be considered as tactile in the sense that whatever the metrical structure of the world its properties *im kleinen* would be those consistent with a flat manifold.

Visual Geometry has an enormously greater range and brings the distant nebulæ within the ambit of our perception. Except in our own immediate neighbourhood where the physiological sensation of directing and focussing both eyes on the same object provides some correlate of the concept of distance, it associates with each point in the universe only a direction. Visual geometry is thus a geometry of directions and two points in the line of sight are visually equivalent. The apparent separation of two objects may be measured by the angle between their directions of view, or if we prefer it, by the arc of the unit sphere with the observer's eye for centre connecting the two points on it corresponding to the two directions. Visual geometry is thus two dimensional and non-Euclidean while tactile geometry is three-dimensional and Euclidean and the incongruity between the two is the reason why children (including many old children) find it difficult to draw intelligible figures of solid objects.

The building up of the geometry and the optics of the world *im Grossen* out of small scale tactile experiences may be done in many ways giving us different cosmological models. In this note I assume that the portion of 3-space which is visible to the observer is Euclidean and shall deduce such apparent changes as he observes as due to the peculiar character of visual geometry.

2. *Descriptive Treatment.*

Let us suppose that the observer is travelling along an infinite plain from South to North along a straight track, and let h be the height of his eye above the ground-level. We take rectangular co-ordinates with the observer's eye O for origin, the x-axis in the direction of motion and the z-axis vertically downward. Let n, h, z, s be the points where the observer's phenosphere[1] (a sphere with centre O and radius unity) is cut by the positive x, y, z axes and the negative x-axis. The circle nhs is then the observer's *horizon circle* on his phenosphere and every point P below his eye level is represented by a point p lying in the quadrant bounded by nhs and nzs. A fixed point (fixed relatively to the ground) appears to move parallel to the x-axis and its apparent path on the phenosphere is a great circle nps described from n to s. A straight line PQ is represented by a great circle pq on the sphere and since PQ moves so as to take up a series of parallel



Parallel Projection of the Phenosphere of a Moving Observer.

$h_1\ h_2\ h_3$ is the horizon circle. The sides pqr of the moving triangle always pass through fixed points $h_1\ h_2\ h_3$ on the horizon circle while the vertices trace out great circles through n and s.

[1] From Gr. $\phi\alpha\iota\nu\epsilon\iota\nu$ to show. The phenosphere is the sphere on which all phenomena appear to take place. In Kantian philosophy a phenomenon is a thing as it appears, as distinguished from the thing in itself.

positions, pq will pass through a fixed point which, when PQ is horizontal, will lie on the horizon circle. Thus if P, Q, R be three fixed points in space the representative triangle pqr on the phenosphere will so move that each vertex describes a great circle through n and s while each side passes through a fixed point, these fixed points lying on a great circle whose plane is parallel to PQR. These properties suffice to determine the one parameter family of spherical triangles into which pqr is deformed, and since any configuration in space may be completely defined by means of triangular facets, we see that

The continuous deformation of the apparent shape of fixed objects as seen by a moving observer is fully characterised by the property that every triangle pqr *formed by three points on the phenosphere moves so as always to be in perspective with itself, the centres of perspective being* n *and* s *and the axis of perspective the great circle whose plane is parallel to PQR. Thus when PQR is a horizontal plane, each side of* pqr *cuts the horizon circle in a fixed point.*

3. *Mathematical Treatment.*

The mathematical treatment is not essentially different from that of the streaming of a deformable medium in a two-dimensional Riemannian manifold (the phenosphere) with a source at n and a sink at s. Let us represent a point on the sphere by two Gaussian co-ordinates u, v corresponding to longitude and latitude with n and s for North and South poles. Thus u is the angle which the plane Opn makes with the horizon circle (measured positively downwards) and v is the complement of the angle which Op makes with On. If P (x, y, z) be a point in space and p (u, v) the representative point on the sphere, it is readily seen that

$$\tan u = z/y\,; \quad \tan v = \frac{x}{\sqrt{(y^2+z^2)}}. \tag{3·1}$$

Due to the observer's motion, P has a relative velocity

$$\dot{x} = -c \text{ (say)}\,; \quad \dot{y} = 0\,; \quad \dot{z} = 0. \tag{3·2}$$

An easy calculation gives

$$\dot{u} = 0\,; \quad \dot{v} = \frac{-c \sin u \cos^2 v}{z}$$

$$= \frac{-c \sin u \cos^2 v}{h} \tag{3·3}$$

in the case of points on the ground for which z is constant and equal to h, the height of the observer's eye.

If $\mathbf{R}$ be the position vector of a point on the sphere so that

$$\mathbf{R} = (\sin v,\ \cos u \cos v,\ \sin u \cos v)$$

and

$$\mathbf{R}_1 = \frac{\partial \mathbf{R}}{\partial u},\ \mathbf{R}_2 = \frac{\partial \mathbf{R}}{\partial v},\ \mathbf{R}_{11} = \frac{\partial^2 \mathbf{R}}{\partial u^2},\ \text{etc., and } \mathbf{R}^i \cdot \mathbf{R}_j = \delta^i_j$$

we have for the sphere

$$\begin{aligned} ds^2 &= g_{11}\,(du)^2 + 2g_{12}\,du\,dv + g_{22}\,(dv)^2 \\ &= \cos^2 v\,(du)^2 + (dv)^2 \end{aligned}$$

and

$$\dot{\mathbf{R}} = \mathbf{R}_1 \dot{u} + \mathbf{R}_2 \dot{v} = -\frac{c \sin u \cos^2 v}{h}\,\mathbf{R}_2 = \mathbf{R}_2 f \text{ (say) from } (3\cdot 3).$$

We now calculate the divergence and curl of this vector field $\dot{\mathbf{R}}$.

$$\begin{aligned} \operatorname{div} \dot{\mathbf{R}} &= \nabla \cdot \mathbf{R}_2 f = \left(\mathbf{R}^1 \frac{\partial}{\partial u} + \mathbf{R}^2 \frac{\partial}{\partial v}\right) \cdot \mathbf{R}_2 f \\ &= \mathbf{R}^1 \cdot \mathbf{R}_{21} f + \mathbf{R}^2 \cdot \mathbf{R}_2 \frac{\partial f}{\partial v} \text{ since } \mathbf{R}^1 \cdot \mathbf{R}_2 = 0 \text{ and } \mathbf{R}^2 \cdot \mathbf{R}_{22} = 0 \\ &= \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} f + \frac{\partial f}{\partial v} \qquad (3\cdot 4) \\ &= \frac{3c \sin u \sin 2v}{2h} \text{ since } \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} \text{ is found[2] to be } -\tan v \end{aligned}$$

on calculation.

Lastly,

$$\begin{aligned} \operatorname{curl} \dot{\mathbf{R}} &= \nabla \times \mathbf{R}_2 f = \left(\mathbf{R}^1 \frac{\partial}{\partial u} + \mathbf{R}^2 \frac{\partial}{\partial v}\right) \times \mathbf{R}_2 f \\ &= \mathbf{R}^1 \times \mathbf{R}_{21} f + \mathbf{R}^1 \times \mathbf{R}_2 \frac{\partial f}{\partial u} + \mathbf{R}^2 \times \mathbf{R}_{22} f + \mathbf{R}^2 \times \mathbf{R}_2 \frac{\partial f}{\partial v}. \end{aligned}$$

But

$$\begin{aligned} \mathbf{R}^1 &= g^{1i}\,\mathbf{R}_i = \sec^2 v\,\mathbf{R}_1\,; \\ \mathbf{R}^2 &= g^{2i}\,\mathbf{R}_i = \mathbf{R}_2. \quad \text{Hence } \mathbf{R}^2 \times \mathbf{R}_2 = 0 \end{aligned}$$

and $\mathbf{R}_1 \times \mathbf{R}_2 = \cos v\,\mathbf{N}$ where $\mathbf{N}$ is the unit outward drawn normal since our axes are left handed.

We have

$$\begin{aligned} \mathbf{R}^1 \times \mathbf{R}_{21} &= \sec v\,[\mathbf{R}_2 \times \mathbf{N}] \times \mathbf{R}_{21} \\ &= \sec v\,[21,\,2]\,\mathbf{N} - \sec v\,(\mathbf{N} \cdot \mathbf{R}_{21})\,\mathbf{R}_2 = 0. \\ \mathbf{R}^1 \times \mathbf{R}_2 &= \sec^2 v\,\mathbf{R}_1 \times \mathbf{R}_2 = \sec v\,\mathbf{N} \\ \mathbf{R}^2 \times \mathbf{R}_{22} &= \sec v\,[\mathbf{N} \times \mathbf{R}_1] \times \mathbf{R}_{22} \\ &= \sec v\,(\mathbf{N} \cdot \mathbf{R}_{22})\,\mathbf{R}_1 - [22,\,1]\sec v\,\mathbf{N} \\ &= -\mathbf{R}_1 \sec v. \end{aligned}$$

[2] The Christoffel 3-index symbols are those derived from the metrical tensor g_{ij}. We use in this para the summation convention for repeated indices, the indices being summed up for the values 1, 2.

Hence

$$\text{curl } \dot{\mathbf{R}} = \sec v \frac{\partial f}{\partial u} \mathbf{N} - f \sec v \, \mathbf{R}_1$$

$$= -\frac{c}{h} \cos u \cos v \, \mathbf{N} + \frac{c}{h} \sin u \cos v \, \mathbf{R}_1. \qquad (3 \cdot 5)^3$$

4. *Interpretation.*

The component of the curl along $\mathbf{R}_1$ which is the direction conjugate to that of motion arises out of considering the sphere as embedded in a Euclidean 3-space and is, in fact, the rotation involved in the Levi-Civita parallel transport of the surface element along the surface.[4] It is thus irrelevant in relation to the intrinsic geometry of the mainfold.[5] The other part corresponds to a rotation of the surface element in its own plane. A positive rotation about $\mathbf{N}$ is one which carries $\mathbf{R}_1$ into the direction of $\mathbf{R}_2$ through just one right angle and will be counter-clockwise as seen from the centre of the sphere. Since for the field of observation $-\pi/2 < (u, v) < \pi/2$, the rotation $-\frac{c}{h} \cos u \cos v$ has the same sign as $-h$, so that the rotation is clockwise or counter-clockwise as seen from the centre, according as the point under observation is below or above the eye-level.

The deformation of the surface elements on the phenosphere may thus be resolved into

(i) a translatory motion (Levi-Civita parallel displacement) along the geodesics u = const. given by $\dot{u} = 0$, $\dot{v} = -\frac{c}{h} \sin u \cos^2 v$;

(ii) a relative dilatation of the area of each surface element with coefficient

$$\text{div } \dot{\mathbf{R}} = \frac{3c}{2h} \sin u \sin 2v \; ;$$

(iii) a rotation of each surface element (counter-clockwise for points above eye-level, clockwise for points below) given by

$$\text{curl } \dot{\mathbf{R}} = -\frac{c}{h} \cos u \cos v.$$

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[3] Weatherburn, *Differential Geometry of Curves and Surfaces*, Vol. I, p. 229, last line.

[4] Levi-Civita, *The Absolute Differential Calculus*—tr. by Miss Long, p. 105.

[5] If for a vector field in a two-dimensional manifold, λ_1 λ_2 are the covariant components, the Stokes tensor $\lambda_{ij} = \lambda_{i,j} - \lambda_{j,i}$ has only two non-vanishing components $\lambda_{12} = -\lambda_{21}$. In the ambient 3-space we may associate with this tensor the contravariant vector $\mu^k = -\epsilon^{ijk} \lambda_{ij}/2 \sqrt{g}$ which has only one component μ^3 along the normal. The value of this component is precisely $\sec v \frac{\partial f}{\partial u} \mathbf{N}$ found above. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Tract No. 24, p. 64.

These results agree both with observation and commonsense, for they lead to the following corollaries :—

(1) The translatory velocity $\dot{v}$ vanishes for points on the horizon ($u = 0$) and is a maximum just below the eye of the observer ($u = \pi/2$, $v = 0$).

(2) The dilatation coefficient is positive while the objects are approaching ($v > 0$) and negative when they are receding ($v < 0$). It vanishes for points on the horizon.

(3) The rotation is clockwise for objects below eye-level for, the upper part of a surface element—corresponding to the more distant points—will appear to move more slowly than the lower part. For surface elements above the eye-level it is the lower parts (nearer to the horizon circle) which are slower.

Other striking results which may be predicted from our equations are :—

(4) The points near the horizon right in front of the observer ($u = 0$, $v = 0$) and those just below his eye-level ($u = \pi/2$, $v = 0$) are subject to an exceedingly simple type of deformation. In the first case there is only pure rotation (without translation or dilatation), and in the second pure translation only.

(5) The dilatation coefficient is a maximum or minimum at points along the tract at a distance h in front or behind the observer ($u = \pi/2$, $v = \pm \pi/4$).

(6) By leaning down or otherwise reducing the height h of the eye, the magnitude of each of the components (i), (ii), (iii) may be increased.

These results are worth verifying by looking out THROUGH A RAILWAY WINDOW.

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