RELATIVISTIC EQUATIONS FOR PARTICLES OF ARBITRARY SPIN

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THE problem has often been formulated as to how far the equations for a relativistic particle of any assigned spin can be put into the form

$$(p_k \alpha^k + \chi)\,\psi = 0 \qquad (1)$$

where $p_k = ih \dfrac{\partial}{\partial x^k}$, χ is some arbitrary constant and the α^k's are four matrices which satisfy a different set of commutation relations in each case. It is well known that the equation (1) is invariant for all transformations of the Lorentz group if the α's satisfy the commutation relations

$$[\alpha^k, I^{rs}] \equiv \alpha^k I^{rs} - I^{rs}\alpha^k = g^{kr}\alpha^s - g^{ks}\alpha^r \qquad (2)$$

where the metric tensor g^{kr} is defined by $kg^{00} = -g^{11} = -g^{22} = -g^{33} = 1, g^{kl} = 0, l \neq$ and the $I^{rs} = -I^{sr}$ are the six infinitesmal transformations of a particular representation of the Lorentz group satisfying the commutation rules

$$[I^{kl}, I^{rs}] = -g^{kr} I^{ls} + g^{ks} I^{lr} + g^{lr} I^{ks} - g^{ls} I^{kr} \qquad (3)$$

The further equation

$$[\alpha^k, \alpha^l] = I^{kl} \qquad (4)$$

can be shown to be consistent with (2) and (3) but it cannot in general be deduced from them. It should be noted that a possible numerical constant on the right of (4) can always be removed by absorption into the α's and results merely in a change of the value of χ in (1) which is without any significance.

I have investigated all possible equations of the form (1). It can be shown that these include a set equivalent to the one given by Dirac[1] and the alternative equivalent formulations in the force free case given by Fierz[2] for particles of any assigned spin. The necessary subsidiary conditions are not included. It also includes a set which is a generalisation to higher spins of the type of the scalar wave-equation. There are other more complicated sets. But I have proved that except for the case of spins 0, ½ and 1 equation (4) is not necessarily satisfied.

It can, however, be postulated that equation (4) *shall* hold for all spins. All the irreducible representations of the set of ten operators I^{kl} and α^k satisfying (2) to (4), *i.e.*, all possible irreducible wave equations of the form (1) can then be found by the following artifice. We introduce a new index 4 and define

$$I^{k4} \equiv -I^{4k} \equiv \alpha^k$$
$$g^{44} = -1, g^{k4} = 0 \qquad k \neq 4 \qquad (5)$$

The equations (2) and (4) are then included in the set (3) if we let the indices in the latter run from 0 to 4 instead of from 0 to 3. But the resulting ten matrices I^{kl} then satisfy the commutation rules for the infinitesimal transformations of the Lorentz group in five dimensions, and all irreducible representations of these are known. The problem is, therefore, completely solved. It can also be deduced immediately that the four α^k's and the six original $^kI^l$'s have the same eigenvalues (possibly multiplied by i to allow for the time-like character of the first co-ordinate). For example, by (2) and (3) the three quantities I^{kl}, α^k and α^l for $k,l = 1,2,3$ satisfy the three equations

$$[\alpha^k, I^{kl}] = -\alpha^l, \; [I^{kl}, \alpha^l] = -\alpha^k,$$
$$[\alpha^l, \alpha^k] = -I^{kl} \qquad (6)$$

which are just the commutation rules for the three components of angular momentum, and it follows from this that in any representation, irreducible or otherwise, I^{kl}, α^k and α^l have the same eigenvalues and satisfy the same characteristic equation. It can be proved further that for any irreducible representation the eigenvalues are always $s, s-1, \ldots\ldots\ldots -s+1, -s$ where s is any integer or half odd integer. One can define a particle of spin s as one for which the maximum eigenvalue of the I^{kl} is s. In that case more restricted commutation rules which the α^k's have to satisfy for a given value of s can be deduced from equations (2) to (4), as has been done by Madhava Rao[3] for $s = 3/2$ and $s = 2$.

The imposition of the condition (4) has very far-reaching consequences. It drastically cuts down the number of possible equations. The allowed set includes the Dirac equation and the scalar and vector Kemmer[4] equations, but it *excludes* the equations given by Dirac for particles of higher spin. The allowed equations for higher spins are such that each component of the wave-function in the force-free case does not satisfy the usual second order wave equation but a factorisable equation of higher order. To see this we note that since the α^k are matrices of a finite number of rows and columns, say n, the operator $P \equiv p_k\alpha^k$ must satisfy a characteristic equation of order $\leqslant n^2$ whose coefficients can only contain products of the four quantities p_k multiplied by pure numbers. It can also be seen quite easily that this characteristic equation must be invariant for all transformations of the Lorentz group and hence must contain the p_k only in powers of the combination $p^2 \equiv p_k p^k$. To find the numerical coefficients we consider the special case when $P = p_0\alpha^0$, the other three components of p_k being zero. Since the eigenvalues of α^0 for spin s are $+s, \pm(s-1), \ldots$ it follows that the characteristic equation of P must be

$$\{P^2 - s^2p^2\}\{P^2 - (s-1)^2p^2\} \ldots\ldots = 0 \qquad (7)$$

the last factor being either P or $P^2 - p^2/4$ depending on whether s is an integer or half odd integer. Our derivation shows that this is the lowest order characteristic equation that P can satisfy, for otherwise α^0 would also satisfy one of lower order. Letting this equation act on ψ and replacing every P in it by $-\chi$ through a repeated use of (1) we see that each component of ψ must satisfy the equation

$$\{\chi^2 - s^2p^2\}\{\chi^2 - (s-1)^2p^2\}\cdots$$
$$\{\chi^2 - p^2\}\chi\psi = 0 \qquad (8a)$$

if s is an integer, or

$$\{\chi^2 - s^2p^2\}\{\chi^2 - (s-1)^2p^2\}\cdots \{\chi^2 - {}^9/_4p^2\}\{\chi^2 - {}^1/_4p^2\}\psi = 0 \qquad (8b)$$

if s is half-odd integer. These equations show that a particle of spin s must necessarily appear with $2s$ and $2s+1$ values of the mass respectively, namely, $\pm\chi/s$, $\pm\chi/(s-1)$, ... Thus a particle of spin $^3/_2$ in this theory would necessarily be capable of appearing with two different values of the rest mass, the higher value being three times the lower. These higher values of the rest mass cannot be eliminated by an artifice any more than the states of negative mass (energy) in Dirac's theory of the electron, and we are, therefore, compelled to regard them as different states of the same particle. The above theory has the advantage over the theories of Dirac, Fierz and Pauli[5] that the equation (1) can be deduced naturally from a Lagrange function even in the presence of an electromagnetic field. There are no awkward subsidiary conditions.

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1. Dirac, *Proc. Roy. Soc.*, A, **1936**, **155**, 447–59. 2. Fierz, *Helv. Phys. Acta*, **1939**, **12**, 3–37. 3. Madhava Rao, *Proc. Ind. Acad. Sci.*, A, **1942**, **15**, 139–47. 4. Kemmer, *Proc. Roy. Soc.*, A, 1939, **173**, 91–116. 5. Fierz and Pauli, *Ibid.*, **1939**, **173**, 211–32.