Residence Time Distribution for a Class of Gaussian Markov Processes

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We study the distribution of residence time or equivalently that of "mean magnetization" for a family of Gaussian Markov processes indexed by a positive parameter α. The persistence exponent for these processes is simply given by $\theta = \alpha$ but the residence time distribution is nontrivial. The shape of this distribution undergoes a qualitative change as θ increases, indicating a sharp change in the ergodic properties of the process. We develop two alternate methods to calculate exactly but recursively the moments of the distribution for arbitrary α. For some special values of α, we obtain closed form expressions of the distribution function.

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I. INTRODUCTION

The problem of *persistence* in spatially extended nonequilibrium systems has recently generated a lot of interest both theoretically [1–7] and experimentally [8–10]. These systems include the Ising or Potts model undergoing phase ordering dynamics [1–4,11–16], simple diffusion equation with random initial conditions [5,6], several reaction diffusion systems in both pure [17] and disordered [18] environments, fluctuating interfaces [19–21], Lotka-Volterra models [22] and inelastic collapse of a randomly forced particle [23]. In many of these systems the spatial degrees of freedom of the original many body problem can be integrated out and the problem of persistence effectively reduces to the calculation of the probability $P_0(t)$ of no zero crossing upto some time t of an effective single site stochastic process $y(t)$.

In most cases of interests, this probability decays as a power law for large time, $P_0(t) \sim t^{-\theta}$, where the persistence exponent θ is nontrivial. This nontriviality can be traced back to the fact that once the spatial degrees of freedom are integrated out, the effective single site process $y(t)$ becomes non-Markovian. For a non-Markovian process, it is well known that the calculation of any history dependent quantity such as persistence (no zero crossing probability) is extremely hard [29,30]. As an example, for the diffusion equation with random initial condition, the effective single site process $y(t)$ is a Gaussian non-Markovian process characterized by its two-time correlator, $\langle y(t_1)y(t_2)\rangle = [4t_1t_2/(t_1+t_2)^2]^{d/4}$ where d is the spatial dimension [5]. Even for this simple case, the corresponding persistence exponent θ is nontrivial and is known only numerically and approximately by analytical methods [5,25] but not exactly so far. Though recently, an exact series expansion result for the exponent θ has been derived for arbitrary smooth Gaussian processes that includes the diffusion equation [6].

Recently it was argued [24,25] that given this stochastic process $y(t)$, it might be useful to investigate a more general quantity, namely the "residence time distribution", whose limiting behaviour determines the persistence exponent. This is the distribution $f(r,t)$ of the random variable, $r(t) = \frac{1}{t}\int_0^t \theta[y(t')]dt'$ where $\theta(x)$ is the Heaviside theta function. Thus $r(t)$ is simply the fraction of time spent by the process $y(t)$ within time t on one side of zero. It was shown in ref. [25] that for any Gaussian stochastic process, the distribution $f(r)$ is independent of time t. For Gaussian processes with zero mean, the symmetry $r \leftrightarrow (1-r)$ indicates that the function $f(r)$ is symmetric around $r = 1/2$. Also in the limit $r \to 0$ (and symmetrically for $r \to 1$), the function $f(r)$ is clearly the probability that the process remains only on one side of zero and hence is proportional to persistence. This indicates that as $r \to 0$, the function $f(r)$ must behave as $\sim r^{\theta-1}$ (and as $\sim (1-r)^{\theta-1}$ for $r \to 1$), so that $f(r)dr \sim t^{-\theta}$ as $r \to 0$ or 1. A somewhat more convenient variable is the "mean magnetization" [24], $m(t) = 2r(t) - 1$, whose range is $[-1,1]$ and whose distribution function, $P(m) = \frac{1}{2}f[(1+m)/2]$ is symmetric around $m = 0$ and behaves as $P(m) \sim (1 \pm m)^{\theta-1}$ near $m = \pm 1$.

The distribution $P(m)$ is known exactly for the process that represents the position of a one dimensional Brownian walker [26]. Lamperti [27] derived an exact expression of $P(m)$ for a class of renewal processes where successive zero crossing intervals are statistically independent. Recently a special case of Lamperti's results [27], when the successive intervals are distributed according to a Lévy law, was rederived by Baldassari et. al. [28] by a different method. The distribution $P(m)$ has been determined numerically for diffusion equation [25] and for interface growth models [21]. Besides, moments of $P(m)$ have been determined analytically for diffusion equation under the independent interval approximation [24].

The distribution function $P(m)$ provides a somewhat more detailed information on the statistical nature of the stochastic process $y(t)$. For example, in the context of diffusion equation it was pointed out by Newman and Toroczkai [25] that an interesting information can be extracted from the shape of the function $P(m)$. For diffusion equation, the exponent $\theta(d)$ (which controls the shape of the function $P(m)$ near $m = \pm 1$) increases monotonically with space dimension d. There exists a critical dimension d_c where $\theta(d_c) = 1$ such that for $d < d_c$, $\theta < 1$ and the function $P(m)$ diverges as $m \to \pm 1$, has a minimum at $m = 0$ and is concave upwards in the range $[-1, 1]$. On the other hand, for $d > d_c$, $\theta > 1$, the function $P(m)$ goes to zero as $m \to \pm 1$, has a maximum at $m = 0$ and is convex upwards in $[-1, 1]$. The peak of the distribution shifts from the edges $m = \pm 1$ to the center $m = 0$ as d increases through d_c. Thus for $d < d_c$, the most probable configurations of the process $y(t)$ are the ones which do not cross zero whereas such configurations are least probable for $d > d_c$, signalling the existence of a sharp change in the ergodic properties of the diffusion field. Such a detailed information is not contained in the persistence exponent θ. In ref. [25], d_c for diffusion equation was approximately determined, $d_c \approx 36$.

These useful informations contained in $P(m)$ of the diffusion equation have so far not been possible to derive exactly mainly due to the non-Markovian nature of the single site Gaussian process. It would therefore be useful to find and study some simpler Markovian Gaussian processes with some tunable parameter (which would play the similar role as the spatial dimension d does in diffusion equation) where exact calculations can be performed. In this paper we study the magnetization distribution $P(m)$ of a family of such Gaussian Markov processes parametrized by an index α. By varying this parameter α, the persistence exponent θ for this process can be varied continuously. The Markovian nature of the process also makes many exact calculations possible.

The Markov process $y(t)$ that we study in this paper, satisfies the following stochastic Langevin equation,

$$\frac{dy}{dt} = \sqrt{2\alpha} t^{\alpha - 1/2} \eta(t) \tag{1}$$

where $\eta(t)$ is a Gaussian white noise with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t_1)\eta(t_2) \rangle = \delta(t_1 - t_2)$ and α is a positive parameter. There are various physical processes that are described by the above Langevin equation. For example, for $\alpha = 1/2$, $y(t)$ represents the position of a one dimensional Brownian random walker. For $\alpha = 1/4$, $y(t)$ can be interpreted [11] to be the "total magnetization" of a Glauber chain undergoing zero temperature coarsening dynamics after being quenched rapidly from infinite temperature.

The persistence of the process $y(t)$ is simply the probability for this process not to cross zero upto time t and decays as $t^{-\theta}$ for large t. The exponent θ for this process can be trivially computed, $\theta = \alpha$. The simplest way to derive this is to define a new time variable $t' = t^{2\alpha}$ such that the equation of motion becomes $dy/dt' = \zeta(t')$ where the new noise $\zeta(t')$ has zero mean and $\langle \zeta(t'_1)\zeta(t'_2) \rangle = \delta(t'_1 - t'_2)$. But this is simply the equation of motion of a one dimensional Brownian walker whose probability of no return to zero upto time t' decays as $\sim 1/\sqrt{t'} \sim t^{-\alpha}$. Thus the persistence exponent for $y(t)$ is simply $\theta = \alpha$.

However we show in the rest of the paper that even though the persistence exponent $\theta = \alpha$ trivially for this process, the magnetization distribution $P(m)$ is nontrivial. In fact, as α and hence θ is increased, the shape of $P(m)$ changes from concave upwards to convex upwards. For $\alpha = 1/2$ (i.e., for ordinary Brownian walker), the distribution $P(m)$ was already known exactly, $P(m) = 1/(\pi\sqrt{1 - m^2})$ [26]. For general α, while we have not been able to determine the full distribution $P(m)$ in closed form, we demonstrate below by two completely different methods that the moments of $P(m)$ can be calculated exactly. In the first method we generalize the formalism developed by Kac [31] for $\alpha = 1/2$ case to arbitrary α. In the second method, we use the formalism recently developed in the context of diffusion equation by Dornic and Godrèche [24] using independent interval approximation (IIA). We point out however that while this latter method yields only approximate results for diffusion equation [24], it gives exact results for the Markov processes that we study in this paper.

The paper is organized as follows. In section (II), we generalize Kac's formalism for $\alpha = 1/2$ to arbitrary α and derive an exact recursion relations satisfied by the moments of $P(m)$. In section (III), we rederive the same results by using an alternate IIA formalism. In section (IV) we use the formalism developed in sections-II and III to obtain explicit results for the distribution of mean magnetization for some special values of the parameter α. Finally we conclude with a summary and discuss the relative merits of the two formalisms and some applications.

II. METHOD I: GENERALIZATION OF KAC'S FORMALISM

We consider the Gaussian process $y(t)$ evolving stochastically via Eq. 1 and define the "mean magnetization", $m(t) = \frac{1}{t}\int_0^t V[y(t')]dt'$, where the functional $V(y)$ in our case is simply, $V(y) = sgn(y)$. Let $G(y, t \mid y', t')$ denote the propagator of the process, i.e., the probability that the process takes the value y at time t given that it was at y' at time $t' < t$. This can be easily computed for our process and is given by,

$$G(y, t \mid y', t') = \frac{1}{\sqrt{2\pi(t^{2\alpha} - t'^{2\alpha})}} e^{-(y-y')^2/2(t^{2\alpha} - t'^{2\alpha})}. \tag{2}$$

Following Kac [31], we define the moment generating function,

$$\langle e^{-utm}\rangle = \sum_{n=0}^{\infty} \frac{(-u)^n}{n!}\nu_n, \tag{3}$$

where ν_n are the moments defined by:

$$\nu_n = \langle (\int_0^t V[y(t')]dt')^n \rangle. \tag{4}$$

To compute the moments ν_n, it is useful to first define a set of functions $Q_n(y,t)$ via the recursion relation,

$$\begin{aligned} Q_{n+1}(y,t) &= \int_0^t dt' \int_{-\infty}^{\infty} dy' G(y,t \mid y',t')V(y')Q_n(y',t') \\ Q_0(y,t) &= G(y,t \mid 0,0). \end{aligned} \tag{5}$$

It can then easily be checked that,

$$\nu_n = n! \int_{-\infty}^{\infty} Q_n(y,t)dy. \tag{6}$$

Using Eq. 3 and Eq. 6, we finally get,

$$\langle e^{-utm}\rangle = \int_{-\infty}^{\infty} dy Q_u(y,t) \tag{7}$$

where $Q_u(y,t)$ is the generating function,

$$Q_u(y,t) = \sum_0^{\infty} Q_n(y,t)(-u)^n. \tag{8}$$

Thus the moments of the mean magnetization m can be computed exactly from Eq. 7 provided we can evaluate the function $Q_u(y,t)$. By using the recursion relation Eq. 5, it can be checked that $Q_u(y,t)$ satisfies the following integral equation,

$$\begin{aligned} & Q_u(y,t) = G(y,t \mid 0,0) \\ & -u\int_0^t dt' \int_{-\infty}^{\infty} dy' G(y,t \mid y',t')V(y')Q_u(y',t'). \end{aligned} \tag{9}$$

Using the definition of the propagator G, this integral equation can then be converted to a partial differential equation,

$$\frac{\partial Q_u(y,t)}{\partial t} = \alpha t^{2\alpha-1}\frac{\partial^2 Q_u(y,t)}{\partial y^2} - uV(y)Q_u(y,t) \tag{10}$$

with $Q_u(y,t=0) = \delta(y)$ and $V(y) = sgn(y)$.

We first make the scale transforms:

$$z = \frac{y}{t^\alpha}; \qquad a = ut; \qquad Q_u(y,t) = \frac{1}{t^\alpha}F(z,a).$$

Substituting in Eq. 10 we get the following equation for F:

$$\begin{aligned} a\frac{\partial F}{\partial a} &= \alpha\frac{\partial^2 F}{\partial z^2} + \alpha z\frac{\partial F}{\partial z} + [\alpha - aV(z)]F \\ F(z,a=0) &= \frac{e^{-z^2/2}}{\sqrt{2\pi}} \end{aligned} \tag{11}$$

where $V(z) = sgn(z)$. This equation has the following series solution:

$$\begin{aligned} F(z,a) &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} b_n a^n e^a D_{-n/\alpha}(-z)e^{-z^2/4} \qquad z<0 \\ F(z,a) &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty} c_n a^n e^{-a} D_{-n/\alpha}(z)e^{-z^2/4} \qquad z>0, \end{aligned} \tag{12}$$

where $D_p(z)$ are parabolic cylinder functions. The coefficients b_n and c_n are to be determined from the boundary conditions, namely the continuity of both F and $\partial F/\partial z$ at $z=0$. The initial conditions determine $b_0 = c_0 = 1$. Using the boundary conditions we get:

$$\begin{aligned} \sum_{n=0}^{\infty} b_n a^n e^a D_{-n/\alpha}(0) &= \sum_{n=0}^{\infty} c_n a^n e^{-a} D_{-n/\alpha}(0) \\ \sum_{n=0}^{\infty} b_n a^n e^a D_{-n/\alpha+1}(0) &= -\sum_{n=0}^{\infty} c_n a^n e^{-a} D_{-n/\alpha+1}(0). \end{aligned} \tag{13}$$

By expanding in powers of a and equating coefficients of all powers of a we finally obtain the following recursions for the coefficients b_n and c_n

$$\begin{aligned} c_n &= \sum_{m=0}^{n-1} \frac{(-1)^m c_m}{(n-m)!}\frac{D_{-m/\alpha}(0)}{D_{-n/\alpha}(0)} && (n \text{ odd}) \\ &= -\sum_{m=0}^{n-1} \frac{(-1)^m c_m}{(n-m)!}\frac{D_{-m/\alpha+1}(0)}{D_{-n/\alpha+1}(0)}c_{n-m} && (n \text{ even}) \\ b_n &= (-1)^n c_n \end{aligned} \tag{14}$$

where we have used few identities satisfied by the parabolic cylinder functions [34]. Now from Eq. 7 it follows that the moments $\mu_k = \langle m^k\rangle$ satisfy,

$$\int_{-\infty}^{\infty} F(z,a)dz = \sum_{k=0}^{\infty}\frac{(-a)^k}{k!}\mu_k. \tag{15}$$

Finally, substituting the series solution for $F(z,a)$ (Eq. 12) in the above equation we obtain:

$$\mu_n = n!\sqrt{\frac{2}{\pi}}\sum_{m=0}^{n}\frac{(-1)^m c_m}{(n-m)!}D_{-m/\alpha-1}(0) \tag{16}$$

for the even moments, while the odd ones vanish. The coefficents c_m are determined from Eq. 14. This thus gives an iterative scheme to generate all moments of the required distribution.

III. AN ALTERNATE DERIVATION OF THE MOMENTS

There is an alternate scheme to calculate the moments of the distribution $P(m)$. This scheme assumes statistical independence of the successive zero crossing intervals of the process $y(t)$ and was first used by Dornic and Godrèche in the context of diffusion equation [24]. We however stress that while this assumption is only approximate for non-Markov processes such as diffusion equation [24], it is however exact for Markov processes such as the one we study in this paper. An additional complication in our case arises due to the fact that the average distance between zero-crossings vanishes. This is a standard result which is true for any Gaussian Markov process [32]. In our calculations we introduce this average distance between two consequtive zeros $\langle l\rangle$ as a cut-off parameter and then take the limit $\langle l\rangle \to 0$ in the end.

Consider a particular realization of the process $y(t)$ ending at time t. Let at time t the process y have a positive sign. Let t_n denote the time instant at which the nth zero-crossing takes place. Then the mean magnetization is given by:

$$
\begin{aligned}
m &= \frac{1}{t}((t-t_n)-(t_n-t_{n-1})+...) = 1-2\xi, \quad \text{where} \\
\xi &= \frac{t_n}{t}-\frac{t_{n-1}}{t}+\frac{t_{n-2}}{t}+...
\end{aligned}
$$

Similarly if $y(t)<0$ then we get $x = 2\xi - 1$. We note that at any t, the sign of y can be positive and negative with equal probability. Hence if we can find the distribution of ξ, that of m can be computed easily. Now in the logarithmic time variable $T = \log(t)$, we can write ξ in the form:

$$
\begin{aligned}
\xi &= e^{-(T-T_n)} - e^{-(T-T_{n-1})} + e^{-(T-T_{n-2})} + ... \\
&= e^{-\lambda}(1-e^{-l_n}+e^{-l_n-l_{n-1}}+...) \\
&= e^{-\lambda}X_n \qquad \text{where} \qquad (17) \\
X_n &= (1-e^{-l_n}+e^{-l_n-l_{n-1}}+...), \qquad (18)
\end{aligned}
$$

$l_n = T_n - T_{n-1}$ and λ is the time from the last zero-crossing to time t. The variables X_n satisfy Kesten recursion relations,

$$
X_n = 1 - e^{-l_n}X_{n-1}. \tag{19}
$$

One then assumes that the successive zero crossing intervals are statistically independent. In the long time limit the distribution of X is determined by the following set of equations:

$$
\begin{aligned}
X &= \eta(1-2\xi)+(1-\eta)(2\xi-1) \\
\xi &= e^{-\lambda}X \\
X &= 1-e^{-l}X,
\end{aligned}
$$

where η is an independent random variable that can take values 0 and 1 with equal probabilities. Since one can compute the distributions of l and λ, it is then straightforward though tedious to compute all the moments of the mean magnetization, $\mu_k = \langle m^k\rangle$ recursively [24].

As noted above, the mean distance between zero crossings, $\langle l\rangle$ vanishes and we introduce this as a cut-off parameter. We now show that the μ_n are actually independent of $\langle l\rangle$.

We first note that the Laplace transforms of the distributions of l and λ, which we denote by $\hat{f}(s)$ and $\hat{q}(s)$ respectively, are given by [24]:

$$
\begin{aligned}
\hat{f}(s) &= \frac{1-\langle l\rangle g(s)}{1+\langle l\rangle g(s)} \\
\hat{q}(s) &= \frac{2g(s)}{s(1+\langle l\rangle g(s))},
\end{aligned} \tag{20}
$$

where $g(s) = s(1-s\hat{A}(s))/2$, and $\hat{A}(s)$ is defined as follows. Consider the normalized process $Y = y(t)/\sqrt{\langle y^2(t)\rangle}$. In the logarithmic time, $T = \log(t)$, this has a stationary autocorrelator, $C(T = \mid T_1 - T_2 \mid) = e^{-\alpha|T|}$. Now consider the autocorrelation function $A(T)$ of the "signed" process, $A(T) = \langle sgn(Y(0))sgn(Y(T))\rangle$. The quantity $\hat{A}(s)$ is just the Laplace transforms of $A(T)$. Using the fact that $y(T)$ is Gaussian with a correlator $C(T)$, the function $A(T)$ can be easily computed, $A(T) = (2/\pi)\sin^{-1}[C(T)]$. In our case, $C(T) = e^{-\alpha|T|}$ which finally gives,

$$
\hat{A}_s = \frac{1}{s} - \frac{1}{s\pi}B\left[\frac{s+\alpha}{2\alpha},\frac{1}{2}\right] \tag{21}
$$

where $B[a,b]$ is the standard Beta function.

It is now convenient to define the moments $r_n =< X^n > /(1+\langle l\rangle g_n)$. Then taking the nth power of the equation $X = 1-e^{-l}X$ and using the expression for $\hat{f}(s)$ given in Eq. 20, it can be shown that r_n's are recursively generated through the following sets of equations:

$$
\begin{aligned}
2r_{2n+1} &= \sum_{k=0}^{2n}\binom{2n+1}{k}(-1)^k r_k \\
2g_{2n}r_{2n} &= -\sum_{k=1}^{2n-1}\binom{2n}{k}(-1)^k g_k r_k
\end{aligned} \tag{22}
$$

with $r_0 = 1$. Note that all r_n's are *independent of* $\langle l\rangle$. The moments of ξ are then given by:

$$
<\xi^n>=\hat{q}(n)<X^n>=\frac{2g_n r_n}{n} \tag{23}
$$

and are also independent of the cut-off $\langle l\rangle$. Finally the non-vanishing even moments of m can be obtained through:

$$
\mu_{2n} = \langle(2\xi-1)^{2n}\rangle \tag{24}
$$

and clearly do not depend on $\langle l\rangle$.

The final expressions of the first few even moments are as follows (see Eq. (B.4) in the appendix of ref. [24]),

$$\mu_2 = \hat{A}_1$$
$$\mu_4 = 1 - \frac{(1-3\hat{A}_1+4\hat{A}_2)(1-3\hat{A}_3)}{1-2\hat{A}_2}$$
$$\ldots \qquad (25)$$

We have checked that the moments μ_n's calculated recursively by this method are identical to those obtained by the first method in section-II.

IV. MOMENTS FOR SOME SPECIAL VALUES OF α

In this section, we use the formalisms developed in the previous two sections to derive some explicit results for the moments of the distribution $P(m)$. While the iterative schemes developed in the previous sections are exact, it seems that for general α it is quite hard to obtain an exact closed form expression of μ_n for aribitrary n. They have to be determined only recursively. However the equations simplify for some special values of the parameter α, for which not just the moments but the full distribution $P(m)$ can be obtained explicitly.

In order to see that the peak of the distribution shifts from $m = \pm 1$ for small α to $m = 0$ for large α, it is natural to examine the two extreme limits $\alpha = 0$ and $\alpha = \infty$ for which fortunately we can obtain exact form of the distribution. Consider first $\alpha = 0$. In this case it is somewhat easier to consider the second method used in section (III). It can be easily seen then that all the g_n's vanish while the moments r_n's remain finite. Thus from Eq. 23 all moments of ξ vanish. Hence from Eq. 24 we get $\mu_n = 1$ for all even n. The same result can also be derived via the first method of section (II) by taking carefully the $\alpha \to 0$ limit in Eqns. 14 and 16. Immediate inspection then determines,

$$P(m) = \frac{1}{2}\left[\delta(m-1) + \delta(m+1)\right] \qquad (26)$$

for $\alpha = 0$. Now consider the other extreme limit, $\alpha = \infty$. In this case, one finds from Eq. 14 that $c_m = 1/m!$. Then the series in Eq. 16 just reduces to the expansion of $(1-1)^n$. Hence $\mu_n = 0$ for all n. This indicates that for $\alpha = \infty$

$$P(m) = \delta(m). \qquad (27)$$

Another case where exact form of $P(m)$ can be obtained is for $\alpha = 1/2$. In this case, using the known values of the parabolic cylinder functions, it is easy to compute the first few terms of the series $\{c_n, n = 0, 1, 2, \cdots\} = \{1, 1, 2, 5, 14, 42, 132, 429, \cdots\}$ from Eq. 14. We then make an ansatz, $c_n = (2n)!/[n!(n+1)!]$ and verify from Eq. 14 that it is indeed the solution for arbitrary n. Substituting this in Eq. 16, we get

$$\mu_{2n}(\alpha = 1/2) = \frac{(2n)!}{(n!)^2 2^{2n}}, \qquad (28)$$

and the odd moments are identically zero. A little inspection then shows that these are the moments of the distribution function,

$$P(m) = \frac{1}{\pi\sqrt{1-m^2}} \qquad (29)$$

with m varying in $[-1, 1]$. We thus reproduce the well known [26] magnetization distribution for the ordinary random walk ($\alpha = 1/2$).

Unfortunately we were unable to get a closed form expression of μ_n for other values of α. For example, for $\alpha = 1/4$, we get by solving Eq. 14 the first few terms of the sequence, $\{c_n, n = 0, 1, 2, \cdots\} = \{1, 3, 72, 3663, 292824, 32227002, \cdots\}$. We found however that this is not listed in the catalogue of known integer sequences [35] and we could not guess any formula for this sequence.

Thus as expected, the distributions of mean magnetization show a qualitative change in shape as α changes. As we go from small α to large α, the peak of the distribution shifts from the edges to the center. This can be understood physically since for small α the noise becomes small as time increases and the probability of zero crossing becomes negligible. On the other hand, for large α, the noise increases with time and the magnetization keeps changing sign and thus the most probable value gets peaked at $m = 0$.

While obtaining an exact form of $P(m)$ is difficult for general α, there is no problem in obtaining the exact values of the moments of $P(m)$ by using the recursion relations and the known values of the parabolic cylinder functions. In Fig. 1 we plot the moments for $\alpha = 1/4$, 1/2 and 3/4.

V. CONCLUSION

In this paper, we have studied the distribution of residence times or equivalently that of mean magnetization of a family of Gaussian markov processes parametrized by an index α which takes values continuously from 0 to ∞. We have shown that the shape of the distribution $P(m)$ undergoes a qualitative change as α is increased from 0 to ∞. For small α, $P(m)$ has peaks at the edges $m = \pm 1$ and has a minimum at $m = 0$ whereas for large α, the peak of the distribution shifts to $m = 0$ with minima at the edges $m = \pm 1$. This change in the ergodicity properties of a stochatic process as one changes a parameter was first noted in ref. [25] in the context of diffusion equation. The advantage of the process studied here, apart from representing various physical situations, is that the Markov nature of the process makes it possible to derive many exact analytical results.

In this paper we have developed two alternate formalisms to compute the moments of the residence times or mean magnetization. While both methods yield exact results for the moments, they do so only recursively. A closed form expression for the moments and hence that of the full distribution is possible only for some special values of the parameter α that characterizes the process. But unfortunately this special set of solvable values of α turn out to be the same for both these methods. Thus so far as the problem studied in this paper is concerned, both these methods are on equal footing. However there are other problems where the former method that generalizes Kac's formalism seems to have an advantage over the second method. We briefly mention below one such application.

The general problem of a random walker in a space with moving boundaries has been well studied and has lot of applications [33]. It would be interesting to study the residence time distribution in such problems. For example, consider a random walker moving in one dimension and ask what is the distribution of the fraction of times spent by the walker in the region bounded by $+\infty$ and a point O that moves deterministically as $x_O(t)$ where $x_O(t)$ is some arbitrary function of t. For the special case when $x_O(t) = c\sqrt{t}$ where c is a constant, this problem can be solved by using the techniques presented in section-II of this paper. The calculations will be similar except that the potential $V(z) = sgn(z)$ as used in Eq. 11 should be replaced by $V(z) = \theta(z - c)$. The corresponding equations can be solved as before except that now the boundary conditions are to be applied at $z = c$. We note however that the second method illustrated in section-III does not seem to be easily generalizable to solve this problem.

We conclude with one last remark. The magnetization distribution $P(m)$ is a useful quantity to study for a generic stochastic process and contain in it many useful informations regarding ergodicity etc. However as is obvious from the efforts of this paper, exact analytical calculation of $P(m)$ seems quite nontrivial even for the simple Gaussian Markov processes studied here. Thus at present, the only hope to compute $P(m)$ for non-Markov processes which are richer and more abundant in nature, seems to be via numerical or approximate methods.

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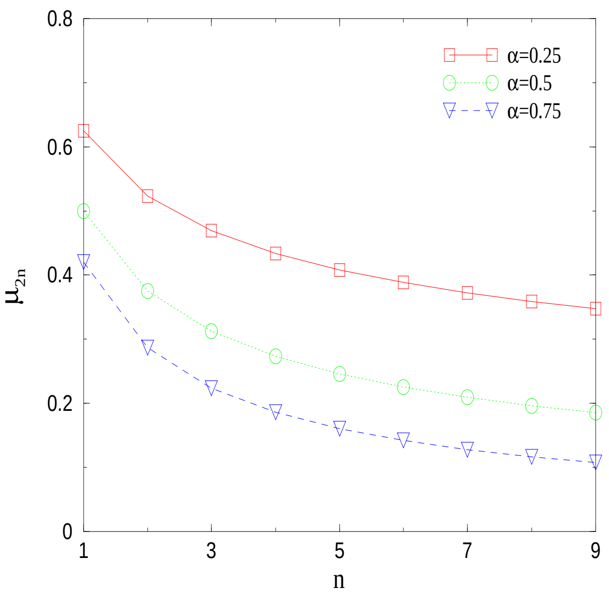

FIG. 1. In this figure the first few non-vanishing moments of $P(m)$ are plotted, for $\alpha = 1/4$, $1/2$ and $3/4$.