

# Isospectrality in Chaotic Billiards

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We consider a modification of isospectral cavities whereby the classical dynamics changes from pseudointegrable to chaotic. We construct an example where we can prove that isospectrality is retained. We then demonstrate this explicitly in microwave resonators.

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Recently it has been shown that it is possible to construct two drums which have different shapes but sound exactly the same [1]. This answers the famous question asked by Kac [2] in 1966 “Can you hear the shape of a drum?” the answer being “no”. Gordon, *et. al.* constructed an example of a pair of two-dimensional domains which had different shapes but had identical eigenvalue spectra for the Laplace operator. Since then, a large number of such isospectral pairs have been obtained.

One common feature of all shapes constructed so far is that they are mostly polygonal. Hence the classical dynamics of a particle in billiards of these shapes is pseudointegrable. A question of interest then is whether isospectrality can be achieved even for cavities with chaotic dynamics, which is typical of domains that have convex pieces, and hence are non-polygonal. We address this question, viz. are there sound-alike chaotic drums, both theoretically and through experiments using microwave resonators.

Isospectrality is fundamentally a consequence of topology. The essential aspects of isospectrality can be proved using the example of the two isospectral domains  $\mathcal{C}1$  and  $\mathcal{C}2$  shown in Fig. 1. The proof consists in showing that given any eigenfunction in one domain we can construct a corresponding one in the other domain, with the same eigenvalue, and vice-versa. Each domain consists of seven distinct sub-domains each in the form of a triangle. We label these sub-domains in an arbitrary fashion, using numbers 1, 2...7 for domain  $\mathcal{C}1$  and the alphabets  $A, B, \dots, G$  for domain  $\mathcal{C}2$ . Note that edges of the triangles are marked differently (By dotted, dashed and solid lines) and this allows us to make a unique correspondence between any pairs of triangles. Consider any wavefunction,  $\psi$ , in domain  $\mathcal{C}1$ , which satisfies the eigenvalue equation  $-\nabla^2\psi = k^2\psi$  with Dirichlet boundary conditions ( $\psi$  vanishes on the boundary of the domain). Let us denote by  $\psi_i$  the restriction of the wavefunction  $\psi$  in sub-domain  $i$  (i.e.  $\psi_i(\vec{r}) = \psi(\vec{r})$  if  $\vec{r}$  is a point in the  $i$ th sub-domain, else  $\psi_i(\vec{r}) = 0$ ). Similarly we can define the restricted wavefunctions  $\{\psi_A, \psi_B \dots \psi_G\}$  from any wavefunction in domain  $\mathcal{C}2$ . Starting from the wavefunction  $\psi$  in  $\mathcal{C}1$  let us construct the following restricted wavefunctions in domain  $\mathcal{C}2$ :

$$\psi_A = \psi_2 - \tilde{\psi}_1 + \psi_7$$

$$\begin{aligned} \psi_B &= \psi_3 + \psi_1 + \psi_5 \\ \psi_C &= -\tilde{\psi}_3 + \psi_2 + \psi_4 \\ \psi_D &= \psi_4 - \tilde{\psi}_1 + \psi_6 \\ \psi_E &= \psi_5 - \tilde{\psi}_2 - \tilde{\psi}_6 \\ \psi_F &= \psi_7 - \tilde{\psi}_3 + \psi_6 \\ \psi_G &= -\tilde{\psi}_7 + \psi_5 - \tilde{\psi}_4 \end{aligned} \quad (1)$$

The notation used requires some explanation: to construct  $\psi_A = \psi_2 - \tilde{\psi}_1 + \psi_7$ , we first move the three domains 1, 2 and 7 so that they are on top of each other and all similarly marked edges coincide. This may require us to flip domains about one of the bases and in such cases we have denoted the wavefunction with a tilde (*e.g.*  $\tilde{\psi}_1$ ). The wavefunction  $\psi_A$  is then obtained by adding (or subtracting) the values of the three functions at each point. It is easy to see that  $\psi' = \psi_A + \psi_B + \psi_C + \psi_D + \psi_E + \psi_F + \psi_G$  is an eigenfunction for domain  $\mathcal{C}2$  with the same eigenvalue. For this we notice that:

- (1) Laplace’s equation is satisfied in every domain and
  - (2) It can be verified that the wavefunction vanishes on the boundary and matches smoothly across sub-domains.
- For example consider the sub-domains  $A$  and  $B$  which are separated by a dashed line. The wavefunctions are given by  $\psi_2 - \tilde{\psi}_1 + \psi_7$  and  $\psi_B = \psi_3 + \psi_1 + \psi_5$ . The smoothness follows since from the wavefunction in  $\mathcal{C}1$  we see that  $\psi_2$  matches smoothly with  $\psi_3$  across the dashed boundary, similarly  $\psi_7$  matches  $\psi_5$  and  $-\tilde{\psi}_1$  matches  $\psi_1$ .

Similarly one can construct an eigenfunction for  $\mathcal{C}1$  starting from any given eigenfunction in  $\mathcal{C}2$ . Thus we have demonstrated a one-to-one correspondence between the states in the two cavities and hence proved isospectrality.

We now modify the domain geometry so as to make the dynamics chaotic. It is expected that making a part of the boundary convex (inwards into the domain) should make the dynamics chaotic. This is related to the fact that on such boundaries, any two particle trajectories which are close to each other, diverge rapidly after being reflected [5]. A well known example of a chaotic billiard is the Sinai billiard obtained by placing a circular scatterer inside a square. In our case, to obtain the modified geometry we first place a scatterer of arbitrary shape inside one of the triangular sub-domains of any one domain

and then place one in a similar position in every other triangle. An example with disc-shaped scatterers is shown in Fig. 2. The identification of edges on the two domains makes this construction unique. Thus notice that, in every triangle, the scatterer is placed close to a vertex where a solid and dotted edge meet. The wavefunction in each domain now changes since it has to vanish on and inside the boundary of the scatterers. Our construction of the modified geometry with scatterers is such that the proof for isospectrality given above can be repeated, since the wavefunctions still satisfy the relations given by eq.(1).

A direct physical proof of isospectrality can be obtained by experiments utilizing microwave cavities [6,7] which provide a simple and powerful method of simulating single particle time-independent quantum mechanics in two dimensions. This follows from the fact that under appropriate geometrical constraints, Maxwell's equations in a cavity reduces to the Schrödinger equation of a free particle inside a two dimensional domain of arbitrary shape and topology. In fact one can show that, for a cavity with thickness (in the  $z$  direction, say) small compared to the dimensions in the other transverse directions (in the  $xy$ -plane), the  $z$ -component of the microwave electric field  $\Psi(x, y) = E_z$  satisfies the time-independent Schrödinger wave equation  $-(\partial_x^2 + \partial_y^2)\Psi = k^2\Psi$  (with the identification  $k = 2\pi f/c$ ,  $f$  being the frequency and  $c$  the speed of light) and  $\psi$  vanishes on the boundary of the domain. This correspondence is exact for all frequencies  $f < c/2d$  where  $d$  is the thickness of the cavity. We note that this is also the Helmholtz equation which describes, for example, vibrations of a drum. Using this equivalence, various phenomena (such as quantum chaos) have been studied. Isospectrality has earlier been demonstrated by Sridhar and Kudrolli [8] using microwave cavities shaped as in Fig. 1. In the experiments one obtains the resonance modes of the cavities. Thus the microwave transmission spectra directly yields the eigenvalues of the cavity being measured. The advantage of this approach is that it can be easily applied to arbitrary 2-D domains, for which numerical simulations are very hard [9,10] and may sometimes be practically impossible.

In our experiments we consider the same set of cavities (Fig. 1) as the ones considered by Sridhar and Kudrolli [8] and investigate the question of isospectrality in the presence of scatterers placed in the specified way inside the cavities. A schematic of the experimental cavity is shown in Fig. 3. The desired domain is cut out from a brass plate of thickness  $d = 6mm$ . Two other brass plates are placed on top and below the hole to form a closed cavity. As shown in the figure microwaves were coupled in and out using loops terminating coaxial lines that enter through the sides of the cavity. The length of bases of the triangular sub-domains was taken to be  $a = 8cm$  and the thickness of the cavity was  $d = 6mm$ . The small thickness of the cavity makes it essentially 2-dimensional and the correspondence between Maxwell's equations and Schrödinger's equation is good for frequencies  $f < c/2d = 25GHz$ . For all metallic objects in the

2-D space between the plates, Dirichlet boundary conditions apply inside the metal.

All measurements were carried out using an HP8510B vector network analyzer which measured the transmission ( $S_{21}$ ) parameters. The typical values of quality factor obtained range from a maximum of 850, at the lower end of the spectrum, to a minimum of 250.

Results: We first attempt to reproduce the results in [8] for the cavities shown in Fig. 4. We show in Fig 4 the traces of the spectrum for the two cavities in the frequency range 1 – 5 GHz. The first 30 resonances of the two cavities are listed in Table I and one sees that the eigenvalues match to better than 1%. One sees that *each resonance present in one is present in the other*. A few lines are missing and this is attributed to the fact that the particular coupling positions we used may not excite some modes. The remaining inaccuracies are due to imperfections in the machining, and in the clamping together of various parts of the cavity. Note, that the amplitudes themselves may be different, as that depends on the location of the coupling and hence to the way the modes are excited.

TABLE I. The table lists the first 33 resonances in the two cavities.

| Resonant frequency<br>in C1(in MHz) | Resonant frequency<br>in C2(in MHz) | percentage<br>discrepancy |
|-------------------------------------|-------------------------------------|---------------------------|
| 1902.500                            | 1903.750                            | 0.0657                    |
| 2271.250                            | 2274.375                            | 0.1374                    |
| 2700.625                            | 2719.375                            | 0.6895                    |
| 3045.625                            | 3062.500                            | 0.5510                    |
| 3217.500                            | 3215.000                            | -0.0778                   |
| 3612.500                            | 3631.250                            | 0.5163                    |
| —                                   | 3892.500                            | —                         |
| 4054.375                            | —                                   | —                         |
| 4184.375                            | 4200.625                            | 0.3868                    |
| 4303.125                            | 4328.125                            | 0.5776                    |
| 4488.125                            | 4528.125                            | 0.8834                    |
| 4743.750                            | 4756.250                            | 0.2628                    |
| 4898.750                            | —                                   | —                         |
| 5026.875                            | 5031.875                            | 0.0994                    |
| 5171.875                            | 5194.325                            | 0.4322                    |
| 5426.250                            | 5474.325                            | 0.8782                    |
| 5488.750                            | —                                   | —                         |
| 5625.625                            | 5627.500                            | 0.0333                    |
| 5793.750                            | 5808.750                            | 0.2582                    |
| —                                   | 5903.125                            | —                         |
| 5928.750                            | 5940.625                            | 0.1999                    |
| 6085                                | —                                   | —                         |
| 6222.500                            | 6242.500                            | 0.3204                    |
| 6312.500                            | 6352.500                            | 0.6297                    |
| 6497.500                            | 6492.500                            | -0.0770                   |
| 6680.000                            | 6707.500                            | 0.4100                    |
| 6750.000                            | 6775.000                            | 0.3690                    |
| —                                   | 6790.000                            | —                         |
| 6855.000                            | 6877.500                            | 0.3272                    |
| 6930.000                            | 6992.500                            | 0.8938                    |

Thus we have obtained the energy spectrum for the given set of isospectral cavities and verified that each eigenvalue in one is present in the other at the same resonance value.

As a check on the quality of the spectral data we compare the cumulative number of resonance levels as a function of frequency, obtained experimentally, with the Weyl formula for the integrated density of states in a two-dimensional domain [11]:

$$N(k) = \frac{Ak^2}{4\pi} - \frac{Sk}{4\pi} + K, \quad (2)$$

where  $A$  and  $S$  are the area and perimeter of the domain, and  $K$  is a correction term associated with its topology. For a polygonal billiard with inner angles  $\alpha_i$  this is given by  $K = \sum_i \frac{1}{24} (\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi})$ . In the present case we find  $K = 0.42$ . We show in Table II a comparison between the experimental results with the above formula. The agreement is quite good.

TABLE II. The table gives a comparison of the cumulative frequency in the cavities with the Weyl estimate

| frequency<br>(in GHz) | cumulative<br>resonant frequency | Weyl's estimate |
|-----------------------|----------------------------------|-----------------|
| 1                     | 0                                | 0               |
| 2                     | 1                                | 0.8             |
| 3                     | 3                                | 3.4             |
| 4                     | 7                                | 7.5             |
| 5                     | 13                               | 13.1            |
| 6                     | 21                               | 20.4            |
| 7                     | 30                               | 29.2            |
| 8                     | 38                               | 39.5            |

Results for the chaotic geometry: We place the scatterers inside the cavity following the prescription outlined above. The scatterers are taken to be metallic cylinders of diameter 1.0 cm and height equal to the thickness of the cavities. The modified spectrum from the two cavities is shown in Fig. 5 and we list the first 22 resonances in Table III. We again find that the eigenvalues in the two cavities match to within 1%. Thus there is clear evidence that isospectrality is retained in the modified chaotic geometry.

It may be noted that the introduction of the scatterers changes the topology of the domain from being simply-connected to now being multiply-connected. We now have a polygonal box with  $p = 7$  circular holes. In this case the topology term in Weyl's formula for the integrated density of states is given by:  $K = \sum_i \frac{1}{24} (\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}) - \frac{p}{6}$ . The comparison with Weyl's formula is given in Table IV. The agreement is not very good at low frequencies and the number of levels seems to be somewhat *higher* than that given by the Weyl estimate.

TABLE III. The first 22 resonant frequencies in the cavities with scatterers

| Resonant frequency<br>in C1(in MHz) | Resonant frequency<br>in C2(in MHz) | percentage<br>discrepancy |
|-------------------------------------|-------------------------------------|---------------------------|
| 2181.875                            | 2175.000                            | -0.3161                   |
| 2330.000                            | 2338.875                            | 0.3795                    |
| 2906.250                            | 2933.125                            | 0.9163                    |
| 3303.750                            | 3311.250                            | 0.2265                    |
| 3346.875                            | 3361.875                            | 0.4462                    |
| 3746.875                            | 3766.250                            | 0.5144                    |
| 4135.625                            | —                                   | —                         |
| 4175.625                            | 4175.625                            | 0.000                     |
| 4207.500                            | 4222.500                            | 0.3552                    |
| 4645.000                            | 4666.250                            | 0.4554                    |
| 4966.875                            | 4956.250                            | -0.2144                   |
| 5056.250                            | —                                   | —                         |
| 5106.250                            | 5101.250                            | -0.0980                   |
| 5148.750                            | 5124.375                            | -0.4757                   |
| 5540.000                            | 5576.875                            | 0.6612                    |
| 5603.125                            | 5633.125                            | 0.5326                    |
| —                                   | 5889.375                            | —                         |
| 6045.000                            | 6002.500                            | -0.7080                   |
| 6435.000                            | 6450.000                            | 0.2325                    |
| 6490.000                            | 6505.000                            | 0.2306                    |
| 6850.000                            | 6872.500                            | 0.3274                    |
| 6955.000                            | 6995.000                            | 0.5718                    |

TABLE IV. Comparison of Weyl's estimate and cumulative frequency obtained experimentally after placing the scatterers

| frequency<br>(in GHz) | cumulative<br>resonant frequency | Weyl's estimate |
|-----------------------|----------------------------------|-----------------|
| 1                     | 0                                | 0               |
| 2                     | 0                                | 0               |
| 3                     | 3                                | 0.9             |
| 4                     | 6                                | 4.5             |
| 5                     | 11                               | 9.7             |
| 6                     | 17                               | 16.3            |
| 7                     | 22                               | 24.5            |

To make the demonstration more convincing and illustrate the non-triviality of the isospectral construction with scatterers, we consider another geometry (Fig. 6) where the scatterers in the second cavity are placed in a somewhat different manner. The arrangement still seems to follow the folding construction and naively one would expect isospectrality. However on closer inspection one finds that the correct correspondence between the edges of the sub-domains has not been satisfied and the wavefunction matching condition in fact no longer holds and so we *should not* get isospectrality. We plot the spectrum for this case in Fig. 7. We see a marked difference from that in Fig. 5, namely we find that there is no correspondence between the spectral lines from the two cavities. This shows clearly that isospectrality is indeed obtained

only for the special arrangement of scatterers in Fig. 2.

In conclusion we have demonstrated that isospectrality is unrelated to the underlying classical dynamics of a particle. We have shown a simple way of introducing scatterers of arbitrary shape into polygonal cavities in such a way that isospectrality is retained. This leads us to a new class of isospectral scatterers and also a better understanding of the essential features necessary for isospectrality.

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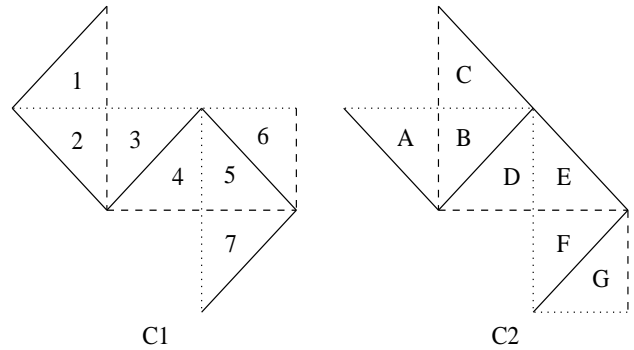


FIG. 1. Isospectral cavities  $C1$  and  $C2$ . The outer edges of the polygonal structure constitute the boundary of the cavity. The inner edges have been marked to show the seven triangular sub-domains within each cavity.

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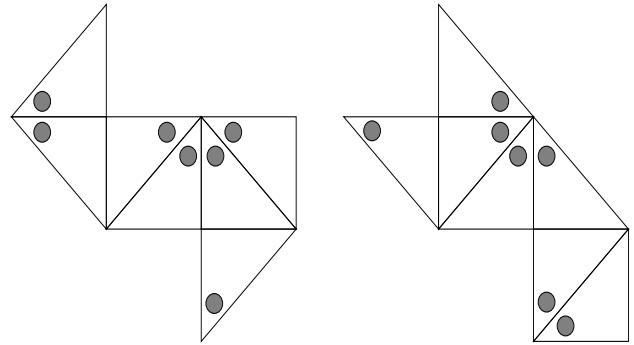


FIG. 2. Isospectral cavities with scatterers in the shape of discs. The wavefunction vanishes on the boundary and inside of every scatterer.

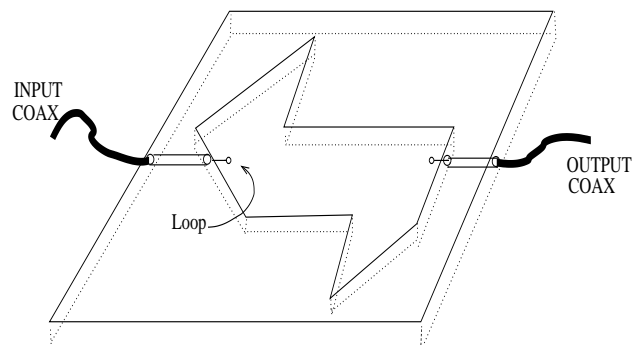


FIG. 3. A schematic of the experimental cavity. This shows a brass plate on which a hole of the desired cavity shape has been cut. This plate is sandwiched between two other brass plates to form a closed cavity.

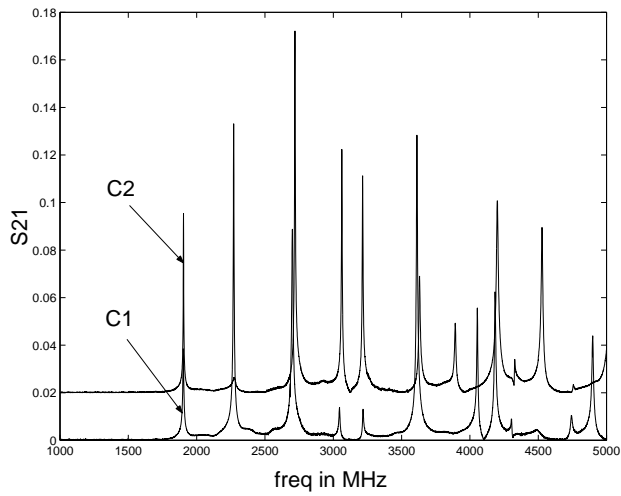


FIG. 4. Comparison of spectrum of isospectral cavities C1 and C2 in the absence of scatterers.  $S_{21}$  is the transmission amplitude.

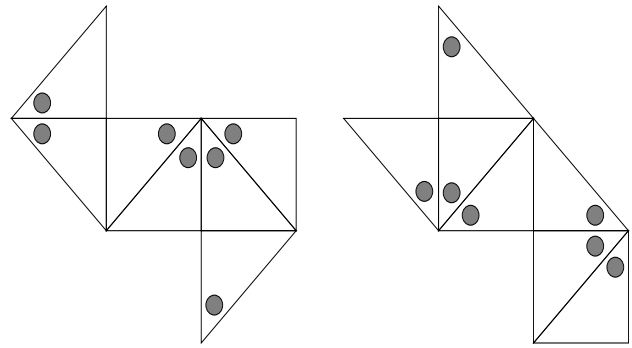


FIG. 6. A *non-isospectral* arrangement of scatterers. The scatterers in cavity C2 are now in a different position.

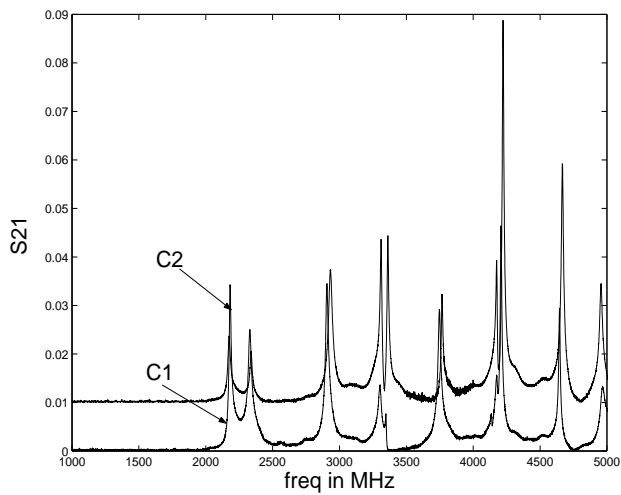


FIG. 5. Comparison of spectrum of the isospectral cavities with scatterers placed as in Fig. 2.

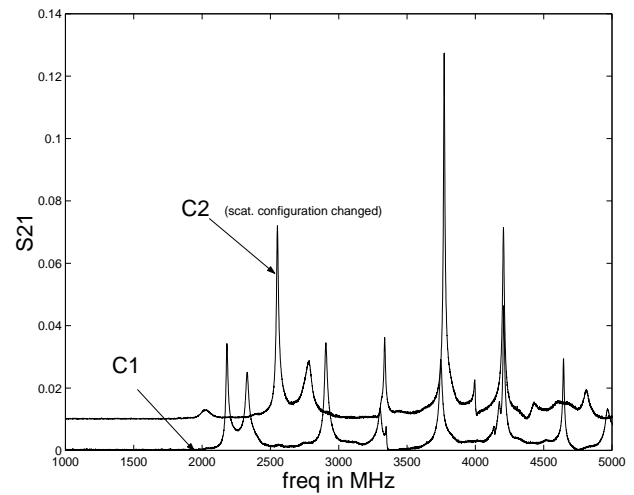


FIG. 7. Spectrum of the cavities in Fig. 6.