ORDERING THEOREMS AND GENERALIZED PHASE SPACE DISTRIBUTIONS ‡

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General theorems are presented, relating to the ordering of functions of operators. Certain new ordering operators are introduced and integral representations for them are given. Applications to phase space descriptions of quantum mechanical systems are noted.

The problem of ordering of operators has recently attracted a good deal of attention in quantum optics in connection with generalized phase space descriptions of optical fields [1-3] and the theory of the laser [4,5] Various results relating to different rules of ordering and to phase space distributions ‡‡ have been derived by ad hoc methods. We have found a general technique for the treatment of such problems in a unified manner and we present in this note our main results.

Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and the creation operators obeying the commutation rule

$$[\hat{a},\hat{a}^\dagger] = 1.$$

We wish to order a given operator function $F(\hat{a},\hat{a}^\dagger)$ according to some prescribed rule, denoted by an ordering operator $\hat{\Omega}$. We denote by $F_\Omega(\hat{a},\hat{a}^\dagger)$ the Ω ordered form of F, i.e. the form of F obtained by arranging F according to the prescribed rule, with the help of the above commutation relation. Further we denote by $\hat{\Omega}\, F(a,a^\dagger)$ the operator obtained by ordering F according to the same rule but whithout making use of the commutation relation, i.e. by treating $\hat{a}$ and $\hat{a}^\dagger$ as c-numbers. For example, if $F = aa^\dagger$ and $\hat{\Omega}$ denotes normal ordering then $F_\Omega(\hat{a},\hat{a}^\dagger) = \hat{a}^\dagger\hat{a}+1$, and $\hat{\Omega} F = \hat{a}^\dagger\hat{a}$.

Consider an ordering such that

$$\hat{\Omega}\exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}] = \Omega(\alpha,\alpha^*)\exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}],$$

where $\Omega(u,v)$ is an entire function of two complex variables u and v which has no zeros and $\Omega(-u,-v) = \Omega(u,v)$, $\Omega(0,0) = 1$. It may easily be shown that the $\Omega(\alpha,\alpha^*)$ functions associated with the usual rules of ordering are of this form. For example, for Weyl's rule of ordering, $\Omega(\alpha,\alpha^*) \equiv 1$, for normal and for standard rules of ordering $\Omega(\alpha,\alpha^*)$ is equal to $\exp[\frac{1}{2}\alpha\alpha^*)$ and to $\exp[\frac{1}{4}(\alpha^2 - \alpha^{*2})]$ respectively.

We now introduce an ordering operator $\Delta^{(\Omega)}$ defined by the formula

$$\Delta^{(\Omega)}(z-\hat{a}, z^*-\hat{a}^\dagger) = \frac{1}{\pi^2}\int d^2\alpha\,\Omega(\alpha,\alpha^*)\exp[\alpha(z^*-\hat{a}^\dagger) - \alpha^*(z-\hat{a})],$$

where the integration extends over the whole complex α plane. $\Delta^{(\Omega)}$ is related via the ordering rule to an operator Δ, essentially identical with one previously introduced by Kubo [6]:

$$\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger) = \hat{\Omega}\,\Delta(z-\hat{a}, z^*-\hat{a}^\dagger) = \hat{\Omega}\,\frac{1}{\pi^2}\int d^2\alpha\exp[\alpha(z^*-\hat{a}^\dagger)-\alpha^*(z-\hat{a})].$$

One of the most important properties of $\Delta^{(\Omega)}$ is expressed by *Theorem* 1: For an arbitrary function $\psi(z,z^*)$,

$$\int\psi(z,z^*)\,\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)\,d^2z = \hat{\Omega}\,\psi(\hat{a},\hat{a}^\dagger).$$

Using this result one may readily establish the following two theorems:

Theorem 2: The Ω-ordered form of an arbitrary operator function $F(\hat{a},\hat{a}^\dagger)$ is given by

$$F_\Omega(\hat{a},\hat{a}^\dagger) = \pi\,\hat{\Omega} f^{(\bar{\Omega})}(\hat{a},\hat{a}^\dagger),$$

where

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$$f^{(\bar{\Omega})}(z,z^*) = \mathrm{Tr}\,[F(\hat{a},\hat{a}^\dagger)\,\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger)]\,.$$

Here $\bar{\Omega}$ is written in place of Ω^{-1}.

In view of this theorem, we shall refer to $f^{(\bar{\Omega})}(z,z^*)$ as *the Ω-ordering function associated with* $F(\hat{a},\hat{a}^\dagger)$.

Theorem 3: The trace of the product of two operator functions $F(\hat{a},\hat{a}^\dagger)$ and $G(\hat{a},\hat{a}^\dagger)$, assuming it exists, may be expressed as follows:

$$\mathrm{Tr}\,[F(\hat{a},\hat{a}^\dagger)G(\hat{a},\hat{a}^\dagger)] = \pi\int f^{(\bar{\Omega})}(z,z^*)g^{(\Omega)}(z,z^*)\mathrm{d}^2z,$$

where $f^{(\bar{\Omega})}(z,z^*)$ is the Ω-ordering function associated with $F(\hat{a},\hat{a}^\dagger)$ and $g^{(\Omega)}(z,z^*)$ is the $\bar{\Omega}$-ordering function associated with $G(\hat{a},\hat{a}^\dagger)$.

Some recently discovered results follow readily from our theorems. If in theorem 2 one chooses Ω to represent antinormal ordering and one makes use of theorem 1 for the integral representation of the associated ordering function $f^{(\bar{\Omega})}(z,z^*)$ and if further one uses the completeness relation on the eigenstates of the annihilation operator one obtains the Sudarshan-Glauber diagonal representation [1,2]. Further if in theorem 2 one specializes to the cases of normal and antinormal ordering, one immediately obtains other recently derived results [5].

If G is the density operator, then theorem 3 expresses the expectation value of F, in the state represented by G as an average over phase space. If in particular F is a Ω-ordered operator function ($F \equiv F_\Omega$) then theorem 2 and 3 give

$$\mathrm{Tr}\,[F_\Omega(\hat{a},\hat{a}^\dagger)G(\hat{a},\hat{a}^\dagger)] = \int F_\Omega(z,z^*)g^{(\Omega)}(z,z^*)\mathrm{d}^2z\,.$$

This result, with Ω representing normal ordering, is the essence of Sudarshan's theorem [1,7] on the equivalence of the quantum and semicalssical descriptions of optical coherence. A closed expression for $g^{(\Omega)}(z,z^*)$ with Ω, representing normal ordering, recently derived by Mehta [8], can also be obtained from our theorems.

For the sake of brevity we have considered here only systems with one degree of freedom and functions of operators $\hat{a}$ and $\hat{a}^\dagger$ which satisfy an equal time commutation relation. However the method can be readily extended to systems with arbitrary number of degrees of freedom and also to problems of multi-time correspondence [9], between quantum and classical stochastic processes.

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References

1. E.C.G.Sudarshan, Phys.Rev.Letters 10 (1963) 277, Proc.Symp.on Optical Masers, ed. J.Fox (Wiley, New York. 1963) p.45.
2. R.J.Glauber, Phys.Rev.131 (1963) 2766.
3. L.Mandel and E.Wolf, Rev.Mod.Phys.37(1965) 231.
4. H.Haken, H.Risken and W.Weidlich, Z.Physik 206 (1967) 355, and references therein.
5. M.Lax and W.H.Louisell, J.Quant.Electron., QE-3 (1967) 47 and references herein.
6. R.Kubo, Journ.Phys.Soc.Japan 19 (1964) 2127.
7. J.R.Klauder, Phys.Rev.Letters 16 (1966) 534.
8. C.L.Mehta, Phys.Rev.Letters 18 (1967) 752.
9. M.Lax, Quantum noise XI, to be published.

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