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Relation between Waterman's extended boundary condition and the generalized extinction theorem

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The relation of Waterman's extended boundary condition to the generalized Ewald-Oseen extinction theorem is established. This unifies different formulations of the electromagnetic scattering. Our proof also establishes the validity of Waterman's extended boundary condition for an arbitrary type of material medium.

In a series of papers Waterman[1,2] has presented a new formulation of electromagnetic scattering. This is based on the use of the vectorial form of Huygens's principle[3] generalized in a form so as to include the incident field.[4] This formulation is now known as the extended boundary condition method, and has been used extensively, for example, by Peterson and Ström in the treatment of electromagnetic scattering from different types of material media.[5,6] Ström[7] has also considered how an analog of this principle[8] can be used in quantum-mechanical scattering. In another series of papers[9-15] Bullough, Pattanayak, Wolf, and Agarwal have used the Ewald-Oseen extinction theorem and its generalizations to analyze scattering from a variety of situations. Pattanayak and Wolf[13] also treated the quantum-mechanical scattering from the viewpoint of the extinction theorem. In view of the vast amount of work done on electromagnetic scattering, it is desirable to see how the above two formulations are related to each other. The purpose of this paper is to establish the relation of Waterman's extended boundary condition (EBC) to the Ewald-Oseen extinction theorem.

The EBC states that the incident electric field $\vec{E}^{(i)}(\vec{r})$ at a point $\vec{r}$ inside the medium (assumed to be linear and isotropic) satisfies the relation[1,6]

$$\vec{E}^{(i)}(\vec{r})+\frac{1}{4\pi}\vec{\nabla}\times\int_{S^+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS'+\frac{1}{4\pi}\int_{S^+}\vec{n}\times[\vec{\nabla}'\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS'-\frac{1}{4\pi}\vec{\nabla}\int_{S^+}\vec{n}\cdot\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')dS'=0\,, \tag{1}$$

$$G_0(\vec{r}-\vec{r}')=e^{ik_0|\vec{r}-\vec{r}'|}/|\vec{r}-\vec{r}'|\,,\quad k_0=\omega/c\,.$$

The medium occupies the domain V bounded by the surface S. The fields which appear in the remaining terms of (1) are the fields at a point just outside the medium. $\vec{n}$ is the unit normal to the surface S pointing from inside to outside the medium. The normal component of the field which appears in (1) is usually eliminated by taking curl curl of (1), and hence

$$\vec{E}^{(i)}(\vec{r})+\frac{\vec{\nabla}\times\vec{\nabla}}{4\pi k_0^2}\times\int_{S^+}\{\vec{n}\times[\vec{\nabla}'\times\vec{E}(\vec{r}')]\}\,G_0(\vec{r}-\vec{r}')dS'+\frac{1}{4\pi}\vec{\nabla}\times\int_{S^+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS'=0\,. \tag{2}$$

In this form the EBC involves only the tangential components of the electric field and the magnetic field. For the case of a perfect conductor (2) reduces to

$$\vec{E}^{(i)}(\vec{r})+\frac{i}{\omega}\vec{\nabla}\times\vec{\nabla}\times\int_{S^+}\vec{J}(\vec{r}')G_0(\vec{r}-\vec{r}')dS'=0\,, \tag{3}$$

where $\vec{J}$ is the surface current defined by $\vec{n}\times\vec{H}(+)=(4\pi/c)\vec{J}$. This is the form used by Waterman in his treatment of scattering from perfectly conducting bodies.

Pattanayak and Wolf[10-13] derived a generalization of the Ewald-Oseen extinction theorem. Their extinction theorem is valid for an arbitrary material medium. The generalized extinction theorem reads as

$$\vec{E}^{(i)}(\vec{r})+\frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_{S^-}dS'\left[\vec{E}(\vec{r}')\frac{\partial G_0}{\partial n}(\vec{r}-\vec{r}')-G_0(\vec{r}-\vec{r}')\frac{\partial\vec{E}(\vec{r}')}{\partial n}\right.$$
$$\left.-4\pi\left(\vec{n}\vec{\nabla}'\cdot\vec{P}(\vec{r}')+ik_0\vec{n}\times\vec{M}(\vec{r}')-\frac{ik_0}{c}\vec{J}\right)G_0(\vec{r}-\vec{r}')\right]=0\,, \tag{4}$$

where $\vec{P}$ and $\vec{M}$ represent, respectively, the polarization and magnetization of the medium and $\vec{J}$ is the surface current. The fields which appear in (4) are at a point just inside the medium. It has been assumed that no external charges and currents are present. The magnetic field also satisfies a relation analogous to the EBC and the extinction theorem. The results are similar to (2) and (4). Once the surface fields are known from Maxwell's equations and (2) or (4), the scattered fields can be obtained from

$$\vec{E}^{sc}(\vec{r}) = \frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_{S^+}\vec{n}\times[\vec{\nabla}'\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS' + \frac{1}{4\pi}\vec{\nabla}\times\int_{S^+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS', \tag{5}$$

or from

$$\vec{E}^{sc}(\vec{r}) = \frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_{S^-}dS'\Big[\vec{E}(\vec{r}')\frac{\partial G_0}{\partial n}(\vec{r}-\vec{r}') - G_0(\vec{r}-\vec{r}')\frac{\partial\vec{E}}{\partial n}(\vec{r}') - 4\pi\left(\vec{n}\vec{\nabla}'\cdot\vec{P}(\vec{r}') + ik_0\vec{n}\times\vec{M}(\vec{r}') - \frac{ik_0}{c}\vec{J}(\vec{r}')\right)G_0(\vec{r}-\vec{r}')\Big]. \tag{6}$$

The equivalence of the EBC and the Ewald-Oseen extinction theorem is trivial for the case of a perfectly conducting body as $\vec{E}(-)=0$, $\vec{P}=\vec{M}=0$. In this case it is easily seen that (4) and (6) reduce, respectively, to (2) and (5).

In the more general case such an equivalence is expected to hold as both (2) and (4) describe essentially the cancellation of the incident field with the surface fields, i.e. (2) [(4)] are the constraints on the surface fields, which in turn determine the scattered fields via (5) [(6)]. To establish the equivalence, we proceed as follows. We recall that the extinction theorem has been derived by a suitable application of the vectorial form of the Green's theorem and by using the conventional Maxwell boundary conditions. This has led to the equations[10–12]

$$\vec{E}^{(i)}(\vec{r}) + \frac{1}{4\pi}\int_{S^+}dS'\{\vec{n}\times[\vec{\nabla}'\times\vec{E}(\vec{r}')]\cdot\vec{G}(\vec{r}',\vec{r}) + [\vec{n}\times\vec{E}(\vec{r}')]\cdot\vec{\nabla}'\times\vec{G}(\vec{r}',\vec{r})\} = 0, \tag{7}$$

$$\vec{E}^{sc}(\vec{r}) = \frac{1}{4\pi}\int_{S^+}dS'\{\vec{n}\times[\vec{\nabla}'\times\vec{E}(\vec{r}')]\cdot\vec{G}(\vec{r}',\vec{r}) + [\vec{n}\times\vec{E}(\vec{r}')]\cdot\vec{\nabla}'\times\vec{G}(\vec{r}',\vec{r})\}. \tag{8}$$

Here $\vec{G}$ is the free-space dyadic Green's function defined by

$$\vec{G}(\vec{r},\vec{r}') = \left(\vec{I} + \frac{1}{k_0^2}\vec{\nabla}\vec{\nabla}\right)G_0(\vec{r}-\vec{r}'). \tag{9}$$

In Eqs. (7) and (8) $\vec{E}$ and $\vec{\nabla}\times\vec{E}$ are fields at points just outside the medium. On simplification (7) and (8) lead to (4) and (6). Equation (7) is in a form which is convenient to establish its equivalence with the EBC (1). Note that the term $(\vec{n}\times\vec{E})\cdot\vec{\nabla}\times\vec{G}(\vec{r},\vec{r}')$ can be written as

$$\begin{aligned}\{[\vec{n}\times\vec{E}(\vec{r}')]\cdot\vec{\nabla}'\times\vec{G}(\vec{r}',\vec{r})\}_l &= \sum[\vec{n}\times\vec{E}(\vec{r}')]_i\epsilon_{ijk}\frac{\partial}{\partial x_j'}\delta_{kl}G_0(\vec{r}-\vec{r}')\\ &= \{[\vec{n}\times\vec{E}(\vec{r}')]\times\vec{\nabla}'\}_l G_0(\vec{r}-\vec{r}')\\ &= -\{[\vec{n}\times\vec{E}(\vec{r}')]\times\vec{\nabla}\}_l G_0(\vec{r}-\vec{r}')\\ &= \{\vec{\nabla}\times[\vec{n}\times\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')]\}_l,\end{aligned} \tag{10}$$

since $\vec{\nabla}$ does not operate on $[\vec{n}\times\vec{E}(\vec{r}')]$. Next consider the term

$$[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]\cdot\vec{G}(\vec{r}',\vec{r}) = [\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]G_0(\vec{r}-\vec{r}') + \{[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]\cdot\vec{\nabla}'\}\vec{\nabla}'G_0(\vec{r}-\vec{r}')k_0^{-2}. \tag{11}$$

On substituting (10) and (11) in (7) we get (note that $\vec{\nabla}G_0 = -\vec{\nabla}'G_0$)

$$\vec{E}^{(i)}(\vec{r}) + \frac{1}{4\pi}\int_{S^+}[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]G_0(\vec{r}-\vec{r}')dS' + \frac{1}{4\pi}\vec{\nabla}\times\int_{S^+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS' - \frac{1}{4\pi}\vec{\nabla}Q = 0, \tag{12}$$

where

$$Q = k_0^{-2}\int_{S^+}[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]\cdot\vec{\nabla}'G_0(\vec{r}-\vec{r}')dS'. \tag{13}$$

Equation (13) can be simplified by the application of Gauss's theorem. Since it is not clear what the value of Q is, if the surface is at infinity, we use the Maxwell boundary conditions to express Q in terms of the inside fields

$$[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]_+ = ik_0[\vec{n}\times\vec{H}(\vec{r}')]_+$$
$$= ik_0[\vec{n}\times\vec{H}(\vec{r}')]_- + \frac{4\pi i k_0}{c}\vec{J}(\vec{r}')$$
$$= [\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]_- - 4\pi i k_0[\vec{n}\times\vec{M}(\vec{r}')]_- + \frac{4\pi i k_0}{c}\vec{J}(\vec{r}')\,,$$

and hence Q can be written as

$$k_0^2 Q = \int_{S-}\vec{n}\cdot\{[\vec{\nabla}'\times\vec{E}(\vec{r}')]\times\vec{\nabla}'G_0(\vec{r}-\vec{r}')\}dS' + 4\pi i k_0\int_{S-}\vec{n}\cdot[\vec{\nabla}'G_0(\vec{r}-\vec{r}')\times\vec{M}(\vec{r}')]dS'$$
$$+\frac{4\pi i k_0}{c}\int\vec{J}(\vec{r}')\cdot\vec{\nabla}'G_0(\vec{r}-\vec{r}')dS'$$
$$= \int_{S-}\vec{n}\cdot[\vec{\nabla}'\times\vec{\nabla}'\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS' - \int_{S-}\vec{n}\cdot\vec{\nabla}'\times[G_0(\vec{r}-\vec{r}')\vec{\nabla}'\times\vec{E}(\vec{r}')]dS'$$
$$+4\pi i k_0\int_{S-}\vec{n}\cdot[\vec{\nabla}'G_0(\vec{r}-\vec{r}')\times\vec{M}(\vec{r}')]dS' + \frac{4\pi i k_0}{c}\int\vec{J}\cdot\vec{\nabla}'G_0(\vec{r}-\vec{r}')dS'\,,$$

which by Maxwell's equations $\nabla\times\nabla\times\vec{E} = k_0^2\vec{D} + 4\pi i k_0\vec{\nabla}\times\vec{M}$ becomes

$$k_0^2 Q = k_0^2\int_{S-}\vec{n}\cdot\vec{D}(\vec{r}')G_0(\vec{r}-\vec{r}')dS' + \left(\frac{4\pi i k_0}{c}\right)\int_{S-}\vec{J}\cdot\vec{\nabla}'G_0(\vec{r}-\vec{r}')dS' - \int_{S-}\vec{n}\cdot\vec{\nabla}'\times[G_0(\vec{r}-\vec{r}')\vec{\nabla}'\times\vec{E}(\vec{r}')]dS'$$
$$+4\pi i k_0\int_{S-}\vec{n}\cdot\vec{\nabla}'\times[\vec{M}(\vec{r}')G_0(\vec{r}-\vec{r}')]dS'\,. \tag{14}$$

The last two integrals in (14) are identically zero since div curl $\vec{A}=0$. On using the boundary conditions again (14) becomes

$$Q = \int_{S+}\vec{n}\cdot\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')dS' + \frac{4\pi i}{k_0 c}\int\vec{J}\cdot\vec{\nabla}'G_0 dS' - 4\pi\int\rho\, G_0(\vec{r}-\vec{r}')dS',$$

which in view of the continuity equation $\nabla\cdot\vec{J} = -i\omega\rho$ reduces to

$$Q = \int_{S+}\vec{n}\cdot\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')dS'\,. \tag{15}$$

On substituting (15) in (12) we get

$$\vec{E}^{(i)}(\vec{r}) + \frac{1}{4\pi}\vec{\nabla}\times\int_{S+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS' + \frac{1}{4\pi}\int_{S+}[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]G_0(\vec{r}-\vec{r}')dS' - \frac{1}{4\pi}\vec{\nabla}\int_{S+}\vec{n}\cdot\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')dS' = 0\,, \tag{16}$$

which is the EBC. Similar analysis enables us to show that the expression (8) for the scattered field leads to

$$\vec{E}^{sc}(\vec{r}) = \frac{1}{4\pi}\vec{\nabla}\times\int_{S+}[\vec{n}\times\vec{E}(\vec{r}')]G_0(\vec{r}-\vec{r}')dS' + \frac{1}{4\pi}\int_{S+}[\vec{n}\times(\vec{\nabla}'\times\vec{E}(\vec{r}'))]G_0(\vec{r}-\vec{r}')dS'$$
$$-\frac{1}{4\pi}\vec{\nabla}\int_{S+}\vec{n}\cdot\vec{E}(\vec{r}')G_0(\vec{r}-\vec{r}')dS'\,. \tag{17}$$

It could be seen that (16) has been derived without any assumption on the properties of the material medium and hence it is applicable to a material medium characterized by *arbitrary constitutive relations*. We have thus shown that the EBC and the extinction theorem are just different facets of the same relation (7).

The T matrix for the scattering can be formulated in the usual way and one obtains results equivalent to those of Waterman, Ström, and coworkers. It is perhaps interesting to note that the results of the above authors are directly obtained from (7) and (8) without ever going to the explicit form of the EBC. This is easily seen by expanding $\vec{G}, \vec{E}, \vec{E}^{(i)}$ in terms of a complete set of vector eigenfunctions [(2.3) of Ref. 5b] and by expressing the surface fields at a point just outside the medium in terms of the surface fields inside the medium. The latter is done via the Maxwell boundary conditions.

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