Interaction of electromagnetic waves at rough dielectric surfaces

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A systematic perturbation theory is developed to study the interaction of electromagnetic waves at rough dielectric surfaces. The Ewald-Oseen extinction theorem is used as the basis of the perturbation theory. Explicit expressions for both first-order and second-order fields (in terms of the surface roughness parameter) at all points in space are presented. The incident field is treated very generally and thus it can be in any state of polarization. The surface is characterized by a structure function and hence the results are valid both for periodic and statistical surfaces. The first-order fields are used to calculate various types of scattering cross sections for arbitrary incident fields characterized by coherence matrices. The coherence matrix for scattered fields is given. The equivalent electric and magnetic surface currents on a flat surface which would lead to the same expressions for first-order fields as those obtained for the rough surface, are calculated using the extinction theorem. The extinction cross section is calculated using an appropriately formulated optical theorem. This leads in a straightforward manner to a change in the value of the reflectivity as compared to the value on a flat surface. The experiment of Teng and Stern is discussed briefly in the light of the expressions for extinction cross sections. Finally, first-order and second-order fields are used to discuss Smith-Purcell radiation, i.e., the radiation emitted by an electron moving parallel to a grating surface. This case corresponds to the conversion of evanescent waves into homogeneous waves due to surface roughness. The relation of some of our results to those of Crowell, Elson, Ritchie, Juranek, Lalor, Marvin *et al.*, Maradudin, and Mills is also discussed.

I. INTRODUCTION

Recently the properties of surface polaritons have been studied extensively both experimentally and theoretically.[1,2] A number of techniques such as those involving frustrated total reflection, use of rough surfaces such as gratings,[3-8] using the fields produced by charged particles and excited atoms,[9] have been considered as possible ways to excite surface polaritons. The present theoretical study[10] about the interaction of electromagnetic waves with rough dielectric surfaces was motivated by the lack of a self-consistent vectorial theory. However, in the meantime a number of self-consistent vectorial theories have appeared. These are based on (i) the boundary matching method[11-14] of Rayleigh and Fano, (ii) the coordinate transformation to a curvilinear system of coordinates,[7b] (iii) the integral equations.[15,16] The results of all these studies are in agreement with our own results.[17-19]

The organization of this paper is as follows: In Sec. II we formulate the perturbation theory in terms of the Ewald-Oseen extinction theorem.[20-24] Explicit expressions for first- and second-order reflected and transmitted fields are obtained for the case of roughness on a perfectly flat surface. The surface is characterized by a structure function. The incident field may be as general a field as possible. In Sec. III we calculate the equivalent electric and magnetic surface currents which when placed on a perfectly flat surface, would lead to the same fields as the first-order fields on a rough surface. We find that both electric and magnetic surface currents are needed. In Sec. IV we specialize to the case of an incident plane wave, characterized by a single propagation vector and calculate the scattering cross sections for arbitrary state of polarization of the incident wave. We also give the relation between the coherence matrices of the first-order fields and the incident field.[25,26] This relation is useful in the calculation of the state of polarization of the scattered light and the Goos-Hänchen shifts.[27] In Sec. V we study the absorption of electromagnetic waves. An analog of the optical theorem is derived and is used to calculate the extinction cross section. The extinction cross section is shown to be the sum of the scattering cross section and the cross section corresponding to the energy absorption due to surface-polariton creation. The experiment of Teng and Stern[3] is analyzed. We also present the form of the extinction cross section when the incident beam is arbitrarily polarized. In Sec. VI we develop the theory of Smith-Purcell radiation.[28-31] The power radiated by an electron moving parallel to the metallic grating is calculated. The polarization characteristics and wave length of the emitted radiation are also calculated.

II. PERTURBATION THEORY VIA EWALD-OSEEN EXTINCTION THEOREM

In this section we formulate the perturbation theory, i.e., the calculation of the reflected and transmitted fields of different orders in the surface-roughness parameter, in terms of the Ewald-Oseen extinction theorem. The extinction theorem was originally derived for an isotropic, homogeneous, and spatially nondispersive dielectric medium.[20] Recently the general form of the extinction theorem has been obtained for a material medium with arbitrary constitutive relation.[21,22] This theorem along with Maxwell's equations has been used to solve a large number of problems in electrodynamics.[20-24] These include the study of reflection, refraction, and in general scattering problems as well as the study of the dispersion relations for radiative and nonradiative surface modes in various geometries. We also point out that sometime ago Waterman[32] developed another formulation of the electromagnetic scattering. Waterman's formulation is now known as the extended-boundary-condition method and has been used extensively by Ström and co-workers[33] to study the electromagnetic scattering in a variety of situations. We have, however, shown elsewhere[34] the equivalence of the extended-boundary-condition method and the Ewald-Oseen extinction theorem and hence in what follows, we restrict ourselves to developing a perturbation theory with the Ewald-Oseen extinction theorem.

For the sake of simplicity, we will assume that our dielectric medium is linear, homogeneous, isotropic, spatially nondispersive. Let the medium occupy the volume V bounded by the surface S and let $\vec{E}^{(i)}$ be the field incident on such a medium. We will assume that the field $\vec{E}^{(i)}$ is such that for any point inside the medium $\vec{\nabla}\cdot\vec{E}^{(i)}=0$. The Ewald-Oseen extinction theorem states that the electric field inside the medium, denoted by $\vec{E}_T(\vec{r})$, must be such that

$$\vec{E}^{(i)}(\vec{r})+\frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_S dS'\left(\vec{E}_T(\vec{r}')\frac{\partial G_0(\vec{r}-\vec{r}')}{\partial n}-G_0(\vec{r}-\vec{r}')\frac{\partial\vec{E}_T(\vec{r}')}{\partial n}\right)=0,\quad \vec{r}\in V, \tag{2.1}$$

where G_0 is the free-space Green's function defined by

$$G_0(\vec{r}-\vec{r}')=e^{ik_0|\vec{r}-\vec{r}'|}/|\vec{r}-\vec{r}'|,\quad k_0=\omega/c, \tag{2.2}$$

and $\vec{n}$ is the unit outward normal to the surface S. The magnetic field satisfies the relation similar to (2.1), i.e.,

$$\vec{H}^{(i)}(\vec{r})+\frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_S dS'\left(\vec{H}_T(\vec{r}')\frac{\partial G_0(\vec{r}-\vec{r}')}{\partial n}-G_0(\vec{r}-\vec{r}')\frac{\partial\vec{H}_T(r')}{\partial n}-4\pi ik_0[\vec{n}\times\vec{P}(\vec{r}')]G_0(\vec{r}-\vec{r}')\right)=0, \tag{2.3}$$

where $\vec{P}$ denotes the polarization of the medium. Once (2.1) is solved for the transmitted field, the reflected field can be obtained from

$$\vec{E}_R(\vec{r})=\frac{1}{4\pi k_0^2}\vec{\nabla}\times\vec{\nabla}\times\int_S dS'\left(\vec{E}_T(\vec{r}')\frac{\partial G_0(\vec{r}-\vec{r}')}{\partial n}-G_0(\vec{r}-\vec{r}')\frac{\partial\vec{E}_T(\vec{r}')}{\partial n}\right),\quad \vec{r}\notin V. \tag{2.4}$$

In Eqs. (2.1)–(2.4), all the fields carrying the superscript i denote the incident electromagnetic fields, i.e., the electromagnetic fields if the dielectric medium were absent. The fields carrying the subscript T and R represent, respectively, the transmitted and the reflected field. It should be noted that at any point outside the medium the total field is the sum of the incident field and the reflected field.

We will base our perturbation calculation on (2.1) and (2.4). We will write the equation of the surface as, $\rho(x,y,z)=\rho_0(x,y,z)+h\rho_1(x,y,z)=0$, where $\rho_0(x,y,z)=0$ is the equation for the corresponding surface without roughness and h is the perturbation parameter. We assume that the zeroth order problem can be solved exactly. For roughness on a flat surface (spherical surface) one has $\rho_0\equiv z$ $(\rho_0\equiv r-R)$, $\rho_1\equiv f(x,y)$ $[\rho_1\equiv f(\theta,\varphi)]$. The unit outward normal to the surface is defined (assuming that the medium occupies the domain $\rho\geq 0$).

$$\vec{n}=-\frac{\vec{\nabla}\rho}{|\vec{\nabla}\rho|}=-\frac{(\vec{\nabla}\rho_0+h\vec{\nabla}\rho_1)}{(|\vec{\nabla}\rho_0|^2+h^2|\vec{\nabla}\rho_1|^2+2h\vec{\nabla}\rho_0\cdot\vec{\nabla}\rho_1)^{1/2}}\,. \tag{2.5}$$

Our next step consists in expanding the fields in the following manner:

$$\vec{E}_T(\vec{r})=\sum_{n=0}^{\infty}\vec{E}_T^{(n)}(\vec{r})h^n,\qquad \vec{E}_R(\vec{r})=\sum_{n=0}^{\infty}h^n\vec{E}_R^{(n)}(\vec{r}). \tag{2.6}$$

The unit normal $\vec{n}$, the Green's function G_0, and various normal derivatives which appear in (2.1) and (2.4) are likewise expanded in powers of h. A further complication arises from the fact that (2.1) and (2.4) involve only the surface fields and hence $\vec{E}_T^{(n)}(\vec{r})|_{\rho=0}$ and other terms are to be expanded in powers of h as well. For example, in case of roughness on a flat surface one has

$$\vec{E}_T^{(n)}(\vec{r})|_{z+hf(x,y)=0} = \vec{E}_T^{(n)}(x,y,-hf)$$
$$= \vec{E}_T^{(n)}(x,y,0) - hf(x,y)\frac{d}{dz}\vec{E}_T^{(n)}(x,y,0) + \frac{h^2f^2}{2!}\frac{d^2}{dz^2}\vec{E}_T^{(n)}(x,y,0)\cdots \quad (2.7)$$

One then equates terms of each power of h in (2.1) and (2.4). This leads to a set of equations. $\vec{E}_T^{(n)}$, for example, is determined in terms of $\vec{E}_T^{(0)}$, $\vec{E}_T^{(1)}, \ldots, \vec{E}_T^{(n-1)}$.

To illustrate how the method works, we consider explicitly the case of roughness on a flat surface.[35] Because of the assumed character of the material medium, each of the fields $\vec{E}_R$ and $\vec{E}_T$ satisfies the Helmholtz equation and hence these can be represented as a plane-wave expansion (angular spectrum):

$$\vec{E}_T(\vec{r},\omega) = \int d^2\kappa\, \vec{\mathcal{E}}_T(\vec{\kappa},\omega)e^{i\vec{\kappa}\cdot\vec{r}+iWz}, \quad \vec{K}\cdot\vec{\mathcal{E}}_T = 0,$$
$$\vec{E}_R(\vec{r},\omega) = \int d^2\kappa\, \vec{\mathcal{E}}_R(\vec{\kappa},\omega)e^{i\vec{\kappa}\cdot\vec{r}-iW_0z}, \quad \vec{K}_0'\cdot\vec{\mathcal{E}}_R = 0, \quad (2.8)$$
$$\vec{K} = (\vec{\kappa}, W), \quad \vec{K}_0 = (\vec{\kappa}, W_0), \quad \vec{K}_0' = (\vec{\kappa}, -W_0),$$
$$W^2 = k_0^2\epsilon - \kappa^2, \quad W_0^2 = k_0^2 - \kappa^2.$$

The square roots in (2.8) are chosen such that $\operatorname{Im} W > 0, \operatorname{Im} W_0 > 0$. The Green's function G_0 can be represented as (Weyl representation[36])

$$G_0 = \frac{i}{2\pi}\int \frac{d^2\kappa}{W_0}\exp[i\vec{\kappa}\cdot(\vec{r}-\vec{r}\,') + iW_0|z-z'|]. \quad (2.9)$$

In the above equations $\vec{\kappa}$ is a two-dimensional vector parallel to the surface $z=0$. On using (2.5)–(2.9) in (2.1) we obtain for transmitted fields, after some algebra, the following:

$$\vec{\mathcal{E}}^{(i)}(\vec{\kappa}) + \frac{1}{2W_0k_0^2}(W+W_0)\vec{K}_0\times\vec{K}_0\times\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}) = 0, \quad (2.10)$$
$$\vec{E}^{(i)}(\vec{r},\omega) = \int d^2\kappa\, \vec{\mathcal{E}}^{(i)}(\vec{\kappa},\omega)e^{i\vec{\kappa}\cdot\vec{r}+iW_0z},$$
$$\vec{K}_0\times\vec{K}_0\times\left((W+W_0)\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}) - ik_0^2(\epsilon-1)\int d^2\kappa_0\, F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0)\right) = 0, \quad (2.11)$$
$$\vec{K}_0\times\vec{K}_0\times\Big((W+W_0)\vec{\mathcal{E}}_T^{(2)}(\vec{\kappa}) - ik_0^2(\epsilon-1)\int d^2\kappa_0\, F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}_0)$$
$$-\int d^2\kappa'\, d^2\kappa''\, F(\vec{\kappa}'')F(\vec{\kappa}-\vec{\kappa}'-\vec{\kappa}'')\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}')[\vec{\kappa}''\cdot(\vec{\kappa}+\vec{\kappa}')(W_0-W')$$
$$-\tfrac{1}{2}\vec{\kappa}''\cdot(\vec{\kappa}-\vec{\kappa}'-\vec{\kappa}'')(W_0+W') + \tfrac{1}{2}(W_0+W')(W_0-W')^2]\Big) = 0, \quad W'^2 = k_0^2\epsilon - \kappa'^2, \quad (2.12)$$

and for the reflected fields

$$\vec{\mathcal{E}}_R^{(0)}(\vec{\kappa}) = (1/2W_0k_0^2)(W-W_0)\vec{K}_0'\times\vec{K}_0'\times\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}), \quad (2.13)$$
$$\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}) = \frac{1}{2W_0k_0^2}\vec{K}_0'\times\vec{K}_0'\times\left((W-W_0)\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}) - ik_0^2(\epsilon-1)\int d^2\kappa_0\, F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0)\right), \quad (2.14)$$
$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}) = \frac{1}{2W_0k_0^2}\vec{K}_0'\times\vec{K}_0'\times\Big((W-W_0)\vec{\mathcal{E}}_T^{(2)}(\vec{\kappa}) - ik_0^2(\epsilon-1)\int d^2\kappa_0\, F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}_0)$$
$$-\int d^2\kappa'\, d^2\kappa''\, F(\vec{\kappa}'')F(\vec{\kappa}-\vec{\kappa}'-\vec{\kappa}'')\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}')$$
$$\times[-\vec{\kappa}''\cdot(\vec{\kappa}+\vec{\kappa}')(W_0+W') + \tfrac{1}{2}\vec{\kappa}''\cdot(\vec{\kappa}-\vec{\kappa}'-\vec{\kappa}'')(W_0-W') - \tfrac{1}{2}(W_0-W')(W_0+W')^2]\Big). \quad (2.15)$$

The higher-order fields are given by similar but more complicated expressions. In these equations $F(\vec{\kappa})$ is the Fourier transform of the surface roughness function,

$$f(x,y) = \int d^2\kappa\, F(\vec{\kappa})e^{i\vec{\kappa}\cdot\vec{r}}. \quad (2.16)$$

We first simplify (2.10)–(2.12). The solution of these equations is easily obtained by the transverse nature of the fields. If we write any of the above equations in the form

$$\vec{K}_0\times\vec{K}_0\times\vec{A}(\vec{\kappa}) + \vec{B} = 0, \quad \vec{K}\cdot\vec{A}(\vec{\kappa}) = 0, \quad (2.17)$$

then it is clear that

$$\vec{K}_0(\vec{K}_0\cdot\vec{A}) - k_0^2\vec{A} + \vec{B} = 0 \Rightarrow (\vec{K}\cdot\vec{K}_0)(\vec{K}_0\cdot\vec{A}) + (\vec{K}\cdot\vec{B}) = 0,$$

and hence

$$\vec{A} = \vec{B}/k_0^2 - \vec{K}_0(\vec{K}\cdot\vec{B})/(\vec{K}\cdot\vec{K}_0). \tag{2.18}$$

Thus the explicit solution can be constructed in terms of the known inhomogeneity $\vec{B}$. One has from (2.10), (2.11), and (2.18) the following explicit results:

$$\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}) = \frac{2W_0}{W+W_0}\,\vec{\mathcal{E}}^{(i)}(\vec{\kappa}) - \frac{2W_0[\vec{K}\cdot\vec{\mathcal{E}}^{(i)}(\vec{\kappa})]\vec{K}_0}{k_0^2(W_0\epsilon + W)}\,, \tag{2.19}$$

$$\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}) = \int i(\epsilon - 1)F(\vec{\kappa}-\vec{\kappa}_0)\times\{k_0^2(W+W_0)^{-1}\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0) - [\vec{K}\cdot\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0)]\vec{K}_0(W_0\epsilon+W)^{-1}\}\,d^2\kappa_0. \tag{2.20}$$

It is now a simple matter to substitute (2.19) and (2.20) in (2.13) and (2.14) and to obtain the following results for the zeroth- and first-order scattered (reflected) fields:

$$\vec{\mathcal{E}}_R^{(0)}(\vec{\kappa}) = -\left(\frac{W-W_0}{W+W_0}\,\vec{\mathcal{E}}^{(i)}(\vec{\kappa}) + \frac{2(W-W_0)}{k_0^2(W_0\epsilon+W)}\times(W_0\vec{\kappa} - WW_0\hat{z})\vec{\mathcal{E}}_z^{(i)}(\vec{\kappa})\right)\,, \tag{2.21}$$

$$\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}) = \int d^2\kappa_0\, i(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}_0)(W_0\epsilon+W)^{-1}\times\{(\kappa^2+WW_0)\vec{\mathcal{E}}_{T\parallel}^{(0)}(\vec{\kappa}_0) + (W\hat{z}-\vec{\kappa})[\vec{\kappa}\cdot\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0)] + \epsilon(W_0\vec{\kappa}+\hat{z}\kappa^2)\vec{\mathcal{E}}_{Tz}^{(0)}(\vec{\kappa}_0)\}\,. \tag{2.22}$$

The second-order fields are similarly calculated using Eqs. (2.12) and (2.15). We do not present the explicit expressions for second-order fields. In a latter section we shall need the expressions for the second-order fields only in particular directions and hence we leave their detailed discussion to Sec. V.

III. ROUGHNESS-INDUCED FIRST-ORDER SCATTERED AND TRANSMITTED FIELDS AND THE EQUIVALENT SURFACE CURRENTS

We now show that the first-order fields (owing to surface roughness) can be regarded as those due to equivalent electric and magnetic surface currents existing on the corresponding smooth surface. This equivalence can be easily established in the frame work of the extinction theorem.

Pattanayak and Wolf find that if there are electric and magnetic surface currents $\vec{J}_e, \vec{J}_m$ at the surface of an isotropic medium, then the Ewald-Oseen extinction theorem reads

$$\frac{1}{4\pi k_0^2}\,\vec{\nabla}\times\vec{\nabla}\times\int_S dS'\left(\vec{E}_T(\vec{r}')\frac{\partial G_0(\vec{r}-\vec{r}')}{\partial n} - G_0(\vec{r}-\vec{r}')\frac{\partial \vec{E}_T(\vec{r}')}{\partial n} + \frac{4\pi}{c}[ik_0\vec{J}_e(\vec{r}') - \vec{J}_m(\vec{r}')\times\vec{\nabla}']G_0(\vec{r}-\vec{r}')\right) = 0,\quad \vec{r}\in V, \tag{3.1}$$

where it is assumed that there are no other sources of the fields. To obtain the forms of $\vec{J}_e, \vec{J}_m$, we consider the case when the surface S is a flat surface. Since the currents are nonzero only at the surface, the field $\vec{E}$ still satisfies Helmholtz equation. Hence on using the angular spectrum for $\vec{E}$ and the Weyl representation (2.9) for G_0, we find that (3.1) reduces to

$$\vec{K}_0\times\vec{K}_0\times\{(W+W_0)\vec{\mathcal{E}}_T(\vec{\kappa}) + (4\pi/c)[k_0\vec{J}_e(\vec{\kappa}) + \vec{J}_m(\vec{\kappa})\times\vec{K}_0]\} = 0, \tag{3.2}$$

where $\vec{J}(\vec{\kappa})$ is the two-dimensional Fourier transform of $\vec{J}(\vec{r})$

$$\vec{J}(\vec{r}) = \int d^2\kappa\,\vec{J}(\vec{\kappa})e^{i\vec{\kappa}\cdot\vec{r}}\,. \tag{3.3}$$

It is clear from the definition of surface currents that their normal components (z component in the present case) are zero. The scattered field generated by surface currents is given by

$$\vec{E}_R(\vec{r}) = \frac{1}{4\pi k_0^2}\,\vec{\nabla}\times\vec{\nabla}\times\int_S dS'\left(\vec{E}_T(\vec{r}')\frac{\partial G_0(\vec{r}-\vec{r}')}{\partial n} - G_0(\vec{r}-\vec{r}')\frac{\partial E_T(\vec{r}')}{\partial n} + \frac{4\pi}{c}[ik_0\vec{J}_e(\vec{r}') - \vec{J}_m(\vec{r}')\times\vec{\nabla}']G_0(\vec{r}-\vec{r}')\right),\quad \vec{r}\in V, \tag{3.4}$$

which on simplification leads, for the present case, to

$$\vec{\mathcal{E}}_R(\vec{\kappa}) = (1/2W_0k_0^2)\vec{K}_0'\times\vec{K}_0'\times\{(W-W_0)\vec{\mathcal{E}}_T(\vec{\kappa}) + (4\pi/c)[k_0\vec{J}_e(\vec{\kappa}) + \vec{J}_m(\vec{\kappa})\times\vec{K}_0']\}\,. \tag{3.5}$$

A comparison of (3.2) and (3.5) with the first-order scattered and transmitted fields (2.11) and (2.14) leads to the relations

$$-(4\pi/c)k_0\vec{K}_0\times\vec{K}_0\times\vec{J}_e(\vec{\kappa})+(4\pi/c)k_0^2\vec{J}_m(\vec{\kappa})\times\vec{K}_0 = \vec{K}_0\times\vec{K}_0\times\mathfrak{A}(\vec{\kappa}), \quad (3.6)$$

$$-(4\pi/c)k_0\vec{K}_0'\times\vec{K}_0'\times\vec{J}_e(\vec{\kappa})+(4\pi/c)k_0^2\vec{J}_m(\vec{\kappa})\times\vec{K}_0' = \vec{K}_0'\times\vec{K}_0'\times\mathfrak{A}(\vec{\kappa}), \quad (3.7)$$

where $\mathfrak{A}$ is defined by

$$\vec{\mathfrak{A}}(\vec{\kappa}) = ik_0^2(\epsilon-1)\int d^2\kappa_0\,F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}_0). \quad (3.8)$$

On substracting (3.7) from (3.6) and noting that $J_{mz}=J_{ez}=0$, we obtain

$$-(4\pi/c)(2W_0)\{k_0^2[\vec{J}_m(\vec{\kappa})\times\hat{z}]-k_0[\vec{\kappa}\cdot\vec{J}_e(\vec{\kappa})]\hat{z}\} = -[\vec{\kappa}\cdot\vec{\mathfrak{A}}(\vec{\kappa})]2W_0\hat{z}-2W_0\vec{\kappa}\mathfrak{A}_z(\vec{\kappa}), \quad (3.9)$$

which immediately leads to

$$\vec{J}_m(\vec{\kappa}) = (c/4\pi k_0^2)(\hat{z}\times\vec{\kappa})\mathfrak{A}_z(\vec{\kappa}) = \frac{ic}{4\pi}(\epsilon-1)(\hat{z}\times\vec{\kappa})\int d^2k_0\,F(\vec{\kappa}-\vec{\kappa}_0)\mathcal{E}_{Tz}^{(0)}(\vec{\kappa}_0). \quad (3.10)$$

On substituting (3.10) in (3.6) we obtain, for the electric surface current, the expression

$$\vec{J}_e(\vec{\kappa}) = -\frac{c}{4\pi k_0}\vec{\mathfrak{A}}_{\parallel}(\vec{\kappa}) = -\frac{ick_0}{4\pi}(\epsilon-1)\int d^2\kappa_0\,F(\vec{\kappa}-\vec{\kappa}_0)\vec{\mathcal{E}}_{T\parallel}^{(0)}(\vec{\kappa}_0). \quad (3.11)$$

We have thus shown that the first-order transmitted and scattered fields can be regarded as due to the distribution (3.10) and (3.11) of "magnetic" and electric surface currents, i.e.,

$$\begin{aligned}\hat{z}\times[\vec{E}_T^{(1)}(\vec{r})-\vec{E}_R^{(1)}(\vec{r})]_{z=0} &= -(4\pi/c)\vec{J}_m(\vec{r}),\\ \hat{z}\times[\vec{H}_T^{(1)}(\vec{r})-\vec{H}_R^{(1)}(\vec{r})]_{z=0} &= (4\pi/c)\vec{J}_e(\vec{r}),\end{aligned} \quad (3.12)$$

with $\vec{J}_m, \vec{J}_e$ given by (3.3), (3.10), and (3.11).

The nature of the equivalent surface currents depends on the properties of the surface roughness as well as the incident field. The scattering has been studied from the view point of surface currents by Stern,[4] Kretschmann and Kröger[8] and Juranek.[13] It would be noticed that since our theory involves both electric and magnetic surface currents, which have only tangential components, the characteristic difficulties of the earlier theories, about the placement of surface currents, do not arise. It should be borne in mind that the above surface currents are only the mathematical entities. In reality, the magnetic surface currents do not exist.

IV. SCATTERING CROSS SECTION AND POLARIZATION CHARACTERISTICS OF THE SCATTERED LIGHT (COHERENCE MATRICES OF THE SCATTERED LIGHT)

In this section we examine many of the properties of the first-order fields. We first note that relations (2.20) and (2.22) have been derived without any assumption about the polarization as well as the propogation characteristics of the incident beam. Equations (2.20) and (2.22) give the angular spectrum of the first-order fields in terms of the angular spectrum of the zeroth-order fields. This generality enables us to study a wide variety of situations.

For a single plane wave incident on the medium (2.20), (2.22) reduced to

$$\vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}) = i(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})[k_0^2(W+W_0)^{-1}\vec{\mathcal{E}}_T^{(0)} - (W_0\epsilon+W)^{-1}(\vec{K}\cdot\vec{\mathcal{E}}_T^{(0)})\vec{K}_0], \quad (4.1)$$

$$\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}) = i(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})(W_0\epsilon+W)^{-1} \times[(\kappa^2+WW_0)\vec{\mathcal{E}}_{T\parallel}^{(0)}+(W\hat{z}-\vec{\kappa})(\vec{\kappa}\cdot\vec{\mathcal{E}}_T^{(0)}) +\epsilon(W_0\vec{\kappa}+\hat{z}\kappa^2)\mathcal{E}_{Tz}^{(0)}], \quad (4.2)$$

where the angular spectrum of the incident and the transmitted waves is given by

$$\begin{aligned}\vec{\mathcal{E}}^{(i)}(\vec{\kappa}) &= \vec{\mathcal{E}}^{(i)}\delta(\vec{\kappa}-\vec{\kappa}^{(0)}),\\ \vec{\mathcal{E}}_T^{(0)}(\vec{\kappa}) &= \vec{\mathcal{E}}_T^{(0)}\delta(\vec{\kappa}-\vec{\kappa}^{(0)}).\end{aligned} \quad (4.3)$$

Here $\vec{\kappa}^{(0)}$ is the component of the incident wave vector parallel to the surface $z=0$. The amplitude $\vec{\mathcal{E}}^{(i)}, \vec{\mathcal{E}}_T^{(0)}$ are related by (2.19) with $\vec{\kappa}$ replaced by $\vec{\kappa}^{(0)}$. It is clear that the allowed $\vec{\kappa}$ vectors of the first-order fields are determined by the structure function $F(\vec{\kappa}-\vec{\kappa}^{(0)})$ of the surface. In particular for a periodic surface, the allowed wave vectors of first-order fields are given by

$$\vec{\kappa} = \vec{\kappa}^{(0)}+\vec{g}, \quad (4.4a)$$

where we have defined $\vec{g}$

$$F(\vec{\kappa}) = \sum_{\vec{g}}\rho_{\vec{g}}\delta(\vec{\kappa}-\vec{g}). \quad (4.4b)$$

Equation (4.4a) is just the grating equation. Thus in general, the first-order field will be both homogeneous ($\kappa<k_0$) and evanescent ($\kappa>k_0$) even if the incident field is homogeneous. The precise nature depends on the directions and magnitudes of $\vec{\kappa}^{(0)}, \vec{g}$. An important aspect of the first-order fields is that they contain the factor $(W_0\epsilon+W)^{-1}$. The vanishing of $(W_0\epsilon+W)$ gives the dispersion relation for surface polaritons in the geometry—the dielectric medium occupying the domain $0\le z\le\infty$ and the region $-\infty\le z\le 0$ is vacuum. The surface mode dispersion relation in this geometry, can be written

$$W_0\epsilon + W = 0 \Rightarrow \kappa^2 + WW_0 = 0 \Rightarrow \kappa^2 c^2/\omega^2 = \epsilon/(\epsilon+1). \tag{4.5}$$

The resonant character of the first-order fields thus implies that surface polaritons can be excited by homogeneous incident waves. This excitation leads to the absorption of the energy from the incident electromagnetic waves and is responsible for a number of interesting effects at rough surfaces such as Wood anomalies,[37] reflectance drop,[3,6] etc. We study the absorption in detail in Sec. V.

Here we consider the scattering of electromagnetic waves. Corresponding to the direction of propagation $\vec{K}_0'$ of the scattered wave, we introduce two orthogonal unit vectors $\vec{s}_{R1}(\vec{\kappa})$, $\vec{s}_{R2}(\vec{\kappa})$ defined by

$$\begin{aligned} \vec{s}_{R2}(\vec{\kappa}) &= (\hat{z}\times\vec{\kappa})/\kappa, \\ \vec{s}_{R1}(\vec{\kappa}) &= [\vec{s}_{R2}(\vec{\kappa})\times\vec{K}_0']/k_0. \end{aligned} \tag{4.6}$$

It is evident that the vectors $\vec{s}_{R1}$, $\vec{s}_{R2}$, $\vec{K}_0'/k_0$ form a right-handed coordinate system. The components of the reflected wave along $\vec{s}_{R2}$ and $\vec{s}_{R1}$ are given by

$$\begin{aligned} \mathcal{E}_{R2}(\vec{\kappa}) &\equiv \vec{s}_{R2}(\vec{\kappa})\cdot\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}) \\ &= \frac{ik_0^2(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})}{W+W_0}\,\vec{s}_{R2}(\vec{\kappa})\cdot\vec{\mathcal{E}}_T^{(0)}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \mathcal{E}_{R1}(\vec{\kappa}) &\equiv \vec{s}_{R1}(\vec{\kappa})\cdot\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}) \\ &= -\frac{ik_0(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})}{\kappa(W_0\epsilon+W)}\,[\kappa^2\epsilon\,\mathcal{E}_{Tz}^{(0)} + W(\vec{\kappa}\cdot\vec{\mathcal{E}}_T^{(0)})]. \end{aligned} \tag{4.8}$$

Equations (4.7) and (4.8) give, respectively, the s and p components of the scattered beam.

We now introduce the s and p components of the zeroth-order transmitted beam by

$$\begin{aligned} \vec{\mathcal{E}}_T^{(0)} &= \vec{s}_{T2}(\vec{\kappa}^{(0)})\mathcal{E}_{T2}^{(0)} + \vec{s}_{T1}(\vec{\kappa}^{(0)})\mathcal{E}_{T1}^{(0)}, \\ \vec{s}_{T2}(\vec{\kappa}^{(0)}) &= (\hat{z}\times\vec{\kappa}^{(0)})/\kappa^{(0)}, \\ \vec{s}_{T1}(\vec{\kappa}^{(0)}) &= [\vec{s}_{T2}(\vec{\kappa}^{(0)})\times\vec{K}^{(0)}]/k_0(\epsilon)^{1/2}, \\ \vec{K}^{(0)} &= (\vec{\kappa}^{(0)}, W^{(0)}), \end{aligned} \tag{4.9}$$

and those of the incident beam by

$$\begin{aligned} \vec{\mathcal{E}}^{(i)} &= \vec{s}_2^{(i)}(\vec{\kappa}^{(0)})\mathcal{E}_2^{(i)} + \vec{s}_1^{(i)}(\vec{\kappa}^{(0)})\mathcal{E}_1^{(i)}, \\ \vec{s}_2^{(i)}(\vec{\kappa}^{(0)}) &= (\hat{z}\times\vec{\kappa}^{(0)})/\kappa^{(0)}, \\ \vec{s}_1^{(i)}(\vec{\kappa}^{(0)}) &= [\vec{s}_2^{(i)}(\vec{\kappa}^{(0)})\times\vec{K}_0^{(0)}]/k_0, \\ \vec{K}_0^{(0)} &= (\vec{\kappa}^{(0)}, W_0^{(0)}), \end{aligned} \tag{4.10}$$

where $\mathcal{E}_j^{(i)}$, $\mathcal{E}_{Tj}^{(0)}$, are related by [cf. Eqs. (2.19) and (2.21)]

$$\mathcal{E}_{T2}^{(0)} = \frac{2W_0^{(0)}}{W^{(0)}+W_0^{(0)}}\mathcal{E}_2^{(i)}, \quad \mathcal{E}_{T1}^{(0)} = \frac{2W_0^{(0)}(\epsilon)^{1/2}}{W_0^{(0)}\epsilon+W}\mathcal{E}_1^{(i)}, \tag{4.11}$$

and where

$$(W^{(0)})^2 = k_0^2\epsilon - (\kappa^{(0)})^2, \quad (W_0^{(0)})^2 = k_0^2 - (\kappa^{(0)})^2.$$

On using (4.9) in (4.7) and (4.8) we obtain for the s and p components of the scattered beam

$$\begin{aligned} \mathcal{E}_{R2}(\vec{\kappa}) &= \frac{ik_0^2(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})}{W+W_0} \\ &\quad\times\left(\cos\tilde{\varphi}\,\mathcal{E}_{T2}^{(0)} + \frac{W^{(0)}}{k_0(\epsilon)^{1/2}}\sin\tilde{\varphi}\,\mathcal{E}_{T1}^{(0)}\right), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathcal{E}_{R1}(\vec{\kappa}) &= -\frac{i(\epsilon-1)F(\vec{\kappa}-\vec{\kappa}^{(0)})}{W_0\epsilon+W} \\ &\quad\times\left(\frac{(WW^{(0)}\cos\tilde{\varphi}-\epsilon\kappa\kappa^{(0)})\mathcal{E}_{T1}^{(0)}}{(\epsilon)^{1/2}} + k_0W\sin\tilde{\varphi}\,\mathcal{E}_{T2}^{(0)}\right), \end{aligned} \tag{4.13}$$

where $\tilde{\varphi}$ is the angle between $\vec{\kappa}$ and $\vec{\kappa}^{(0)}$.

The scattering cross sections can now be obtained from the Poynting vector considerations in the far zone. The asymptotic expansion of the angular spectrum is well known[38] and one finds that the radiation field in the far zone can be written

$$\begin{aligned} \vec{E}_R^{(1)}(\vec{r},\omega) &= -2\pi ik_0|\cos\theta|(e^{ik_0r}/r)\vec{\mathcal{E}}_R^{(1)} \\ &\quad\times(k_0\sin\theta\cos\varphi, k_0\sin\theta, \sin\varphi), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \vec{H}_R^{(1)}(\vec{r},\omega) &= -2\pi ik_0|\cos\theta|\vec{n}\times\vec{\mathcal{E}}_R^{(1)} \\ &\quad\times(k_0\sin\theta\cos\varphi, k_0\sin\theta\sin\varphi)(e^{ik_0r}/r), \\ &\quad (\cos\theta<0), \end{aligned} \tag{4.15}$$

where $\vec{n}$ is the direction of observation $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The time averaged power radiated, per unit solid angle, per unit incident flux, along z axis is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{(c/8\pi)\,\mathrm{Re}\,\vec{n}\cdot(\vec{E}\times\vec{H}^*)r^2}{(c/8\pi)\cos\theta_0|\mathcal{E}^{(i)}|^2} \\ &= 4\pi^2k_0^2\cos^2\theta|\vec{\mathcal{E}}_R^{(1)}(k_0\sin\theta\cos\varphi, k_0\sin\theta\sin\varphi)|^2/\cos\theta_0|\mathcal{E}^{(i)}|^2, \end{aligned} \tag{4.16}$$

$$\frac{dP}{d\Omega} = \frac{4\pi^2k_0^2\cos^2\theta}{\cos\theta_0|\mathcal{E}^{(i)}|^2}(|\mathcal{E}_{R1}|^2 + |\mathcal{E}_{R2}|^2). \tag{4.17}$$

On substituting (4.11)–(4.13) in (4.17) we obtain the following results for different types of scattering:

$$\frac{dP_{s\to s}}{d\Omega} = 16\pi^2k_0^4|F|^2\cos^2\tilde{\varphi}\cos^2\theta\cos\theta_0\left|\frac{(\epsilon-1)}{\{[\cos\theta+(\epsilon-\sin^2\theta)^{1/2}]\;[\cos\theta_0+(\epsilon-\sin^2\theta_0)^{1/2}]\}}\right|^2, \tag{4.18}$$

$$\frac{dP_{s\to p}}{d\Omega}=16\pi^2k_0^4|F|^2\sin^2\tilde{\varphi}\cos^2\theta\cos\theta_0\left|\frac{(\epsilon-1)(\epsilon-\sin^2\theta)^{1/2}}{\{[\epsilon\cos\theta+(\epsilon-\sin^2\theta)^{1/2}][\cos\theta_0+(\epsilon-\sin^2\theta_0)^{1/2}]\}}\right|^2, \tag{4.19}$$

$$\frac{dP_{p\to s}}{d\Omega}=16\pi^2k_0^4|F|^2\sin^2\tilde{\varphi}\cos^2\theta\cos\theta_0\left|\frac{(\epsilon-1)(\epsilon-\sin^2\theta_0)^{1/2}}{\{[\cos\theta+(\epsilon-\sin^2\theta)^{1/2}]\;[\epsilon\cos\theta_0+(\epsilon-\sin^2\theta_0)^{1/2}]\}}\right|^2 \tag{4.20}$$

$$\frac{dP_{p\to p}}{d\Omega}=16\pi^2k_0^4|F|^2\cos^2\theta\cos\theta_0\left|\frac{[\cos\tilde{\varphi}(\epsilon-\sin^2\theta)^{1/2}(\epsilon-\sin^2\theta_0)^{1/2}-\epsilon\sin\theta\sin\theta_0]\quad(\epsilon-1)}{[\epsilon\cos\theta+(\epsilon-\sin^2\theta)^{1/2}][\epsilon\cos\theta_0+(\epsilon-\sin^2\theta_0)^{1/2}]}\right|^2, \tag{4.21}$$

where the angles are defined by

$$\begin{aligned}&W_0=k_0\cos\theta,\quad W_0^{(0)}=k_0\cos\theta_0,\quad W_0,W_0^{(0)}>0,\\&\kappa=k_0\sin\theta,\quad \kappa^{(0)}=k_0\sin\theta_0.\end{aligned} \tag{4.22}$$

The above results are valid for complex ϵ. In the special case when $\epsilon\leqslant-1$ and real, above results reduce to those obtained recently by Marvin *et al.*[12] using boundary matching method if we further assume that $F(\vec{\kappa})$ is nonvanishing only for $\vec{\kappa}$ vectors along the y axis. For normal incidence ($\theta_0=0$), (4.20) and (4.21) reduce to the results of Elson and Ritchie[7b] obtained by coordinate transformation method and differ somewhat from their earlier results.[7a]

It is quite clear from the above analysis that an incident field of a given polarization produces, in general, scattered fields of both the polarizations. The depolarization ratio, defined as the ratio of the intensities along s_1 and s_2, i.e.,

$$\Delta=|\mathcal{E}_{R2}|^2/|\mathcal{E}_{R1}|^2, \tag{4.23}$$

can be easily calculated using (4.12) and (4.13). For incident beams of arbitrary polarization described in terms of coherence matrices (or equivalently in terms of Stokes parameters), we have the relations between the coherence matrices of the incident and zeroth order transmitted beams[26]

$$J_T^{(0)}=4\begin{pmatrix}|W_0^{(0)}(\epsilon)^{1/2}/(W_0^{(0)}\epsilon+W^{(0)})|^2J_{11}^{(i)} & |W_0^{(0)}|^2(\epsilon^*)^{1/2}J_{12}^{(i)}/(W_0^{(0)}\epsilon+W^{(0)})^*(W^{(0)}+W_0^{(0)})\\ |W_0^{(0)}|^2(\epsilon)^{1/2}J_{21}^{(i)}/(W_0^{(0)}\epsilon+W^{(0)})\,(W_0^{(0)}+W^{(0)})^* & |W_0^{(0)}/(W^{(0)}+W_0^{(0)})|^2J_{22}^{(i)}\end{pmatrix}, \tag{4.24}$$

where $J_{\alpha\beta}^{(i)}$ are the components of the coherence matrix of the incident beam

$$J_{\alpha\beta}^{(i)}=\langle\mathcal{E}_\alpha^{(i)*}\,\mathcal{E}_\beta^{(i)}\rangle. \tag{4.25}$$

The coherence matrix of the scattered beam, in any direction, can be constructed from (4.12), (4.13), and (4.24). The result is

$$J_R^{(1)}=4|W_0^{(0)}|^2|\epsilon-1|^2|F(\vec{\kappa}-\vec{\kappa}^{(0)})|^2\mathfrak{M}J^{(i)}\mathfrak{M}^\dagger, \tag{4.26}$$

where the matrix $\mathfrak{M}$ is given by

$$\mathfrak{M}=\begin{pmatrix}(\epsilon\kappa\kappa^{(0)}-WW^{(0)}\cos\tilde{\varphi})/(W_0\epsilon+W)(W_0^{(0)}\epsilon+W^{(0)}) & -k_0W\sin\tilde{\varphi}/(W_0\epsilon+W)(W^{(0)}+W_0^{(0)})\\ k_0W^{(0)}\sin\tilde{\varphi}/(W+W_0)(W_0^{(0)}\epsilon+W^{(0)}) & k_0^2\cos\tilde{\varphi}/(W+W_0)(W^{(0)}+W_0^{(0)})\end{pmatrix}. \tag{4.27}$$

Once the coherence matrix is known all the polarization characteristics can be calculated. The degree of polarization, for example, defined by

$$P=[1-4\det J/(\mathrm{Tr}\,J)^2]^{1/2}. \tag{4.28}$$

can be easily found. It is clear from (4.26) that if the incident beam is fully polarized, i.e.,

$$J^{(i)}=\begin{pmatrix}I_1 & (I_1I_2)^{1/2}e^{-i\delta}\\(I_1I_2)^{1/2}e^{i\delta} & I_2\end{pmatrix}, \tag{4.29}$$

then the scattered beam is also completely polarized, i.e.,

$$\det J^{(i)}=0\Rightarrow\det J_R^{(1)}=0, \tag{4.30}$$

however, the plane of polarization of the scattered light is different. The diagonal elements of $J_R^{(1)}$ lead to the cross sections for arbitrary polarization of the incident beam. Finally note that these coherence matrices will be useful in the study of Goos-Hänchen shifts of the evanescent waves produced at rough surfaces.

V. ABSORPTION OF ELECTROMAGNETIC WAVES DUE TO SURFACE-POLARITON EXCITATION

We have seen in Sec. IV that evanescent waves are produced at rough surfaces due to incident homogeneous waves. If the frequency of the incident wave is in the region where the real part of the dielectric function can be negative, then the surface polaritons (nonradiative surface modes in our case) can be excited in the medium and this leads to the absorption of energy even if there is no true absorption in the dielectric medium (i.e., imaginary part of $\epsilon = 0$). To calculate the absorption of energy we will make use of the optical theorem which directly gives the extinction cross section. The extinction cross section shall be seen to be the sum of absorption cross section and scattering cross section. From Maxwell's equations one finds that the absorption of energy by the dielectric is given by

$$W^{\text{abs}} = \frac{\omega}{2\pi} \int d^3r\, |\vec{E}|^2 \epsilon''(\vec{r}, \omega)$$
$$= -\text{Re} \int dS\, \vec{n} \cdot \vec{S}, \quad \vec{S} = \frac{c}{2\pi} \text{Re}(\vec{E} \times \vec{H}^*), \tag{5.1}$$

where the electric field has been written in the form $\vec{E}(\vec{r}, t) = \vec{E} e^{-i\omega t} + \text{c.c.}$. The integral in (5.1) is over the surface at infinity. We shall write the field in the form

$$\vec{E} = \begin{cases} \vec{E}^{(i)}(\vec{r}) + \vec{E}_R^{(0)}(\vec{r}) + \vec{E}^{(s)}(\vec{r}), & z < 0, \\ \vec{E}_T^{(0)}(\vec{r}) + \Delta \vec{E}_T(\vec{r}), & z > 0, \end{cases} \tag{5.2}$$

and specifically assume that the dielectric function is such that in range of frequencies of interest, its real part is negative and hence all the waves in the medium are decaying waves and do not lead to any energy flow at infinity ($z > 0$). We define the total scattered flux as

$$W^{\text{sc}} = \frac{c}{2\pi} \text{Re} \int dS\, \vec{n} \cdot (\vec{E}^{(s)} \times \vec{H}^{(s)*}). \tag{5.3}$$

On substituting (5.2) in (5.1) and doing the surface integral involving cross terms like $\vec{E}^{(i)} \times \vec{H}^{(s)*}$, $\vec{E}_R^{(0)} \times \vec{H}^{(s)*}$ by the method of stationary phase[39] we obtain the result

$$W^{\text{abs}} + W^{\text{sc}} = -4\pi c |\cos\theta_0| \text{Re}[\vec{\mathcal{E}}_R^{(0)*} \cdot \vec{\mathcal{E}}^{(\text{sc})}(\vec{\kappa}^{(0)})]. \tag{5.4}$$

Defining the extinction cross section by normalizing (5.4) with respect to the incident flux $(c/2\pi)|\mathcal{E}^{(i)}|^2 \cos\theta_0$, we get

$$Q^{\text{ext}} = -8\pi^2 \text{Re}[\vec{\mathcal{E}}_R^{(0)*} \cdot \vec{\mathcal{E}}^{(\text{sc})}(\vec{\kappa}^{(0)})] / |\mathcal{E}^{(i)}|^2, \tag{5.5}$$

$$Q^{\text{ext}} = \int_{z<0} \frac{dP}{d\Omega} d\Omega + \frac{W^{\text{abs}}}{(c/2\pi)|\mathcal{E}^{(i)}|^2 \cos\theta_0}. \tag{5.6}$$

If $\epsilon''(\omega) \approx 0$ and $\epsilon'(\omega) \leq -1$, then Q^{ext} can also be written

$$Q^{\text{ext}} = -8\pi^2 \text{Re}[\vec{\mathcal{E}}_R^{(0)*} \cdot \vec{\mathcal{E}}^{(\text{sc})}(\vec{\kappa}^{(0)})] / |\mathcal{E}_R^{(0)}|^2. \tag{5.7}$$

Thus the extinction cross section is determined from the component of the scattered field in the direction of the zeroth-order reflected field. This was expected, as due to roughness, part of the energy, which would have appeared in the reflected beam, does not appear there but appears in contributions to the diffuse scattering and surface polariton creation. Equation (5.7) gives the so-called reflectance drop at rough surfaces.

It is easily seen that the first-order scattered fields do not contribute to the right-hand side of (5.7) for it follows from (4.1) and (4.2) that the first-order reflected field in the direction of the zeroth-order reflected field is

$$\vec{\mathcal{E}}_R^{(1)}(\vec{\kappa}^{(0)}) = iF(\vec{0}) 2 W_0^{(0)} \vec{\mathcal{E}}_R^{(0)}, \tag{5.8}$$

which when substituted in (5.7) leads to zero as $F(0), W_0^{(0)}$ are real. In view of this, the approximate expression for the extinction cross section becomes

$$Q^{\text{ext}} \simeq -8\pi^2 \text{Re}[\vec{\mathcal{E}}_R^{(0)*} \cdot \vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}^{(0)}) / |\mathcal{E}_R^{(0)}|^2. \tag{5.9}$$

Thus the second-order reflected fields only in the direction of $\vec{\kappa}^{(0)}$ are needed. The second-order fields in the direction $\vec{\kappa}^{(0)}$ are easily shown to be from (2.12) and (2.15) (letting $\vec{\kappa}^{(0)} \to \vec{\kappa}$)

$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}) = -2WW_0 \int d^2\kappa_0 |F(\vec{\kappa}_0)|^2 \left(1 - \frac{4\vec{\kappa} \cdot \vec{\kappa}_0}{k_0^2(\epsilon - 1)}\right) \vec{\mathcal{E}}_R^{(0)}$$
$$+ i(\epsilon - 1) \int d^2\vec{\kappa}_0 |F(\vec{k} - \vec{k}_0)|^2 (W_0\epsilon + W)^{-1} \{(\kappa^2 + WW_0) \vec{\mathcal{E}}_{T\parallel}^{(1)}(\vec{\kappa}_0) + [\vec{\kappa} \cdot \vec{\mathcal{E}}_T^{(1)}(\vec{\kappa}_0)](W\hat{z} - \vec{\kappa}) + \epsilon(W_0\vec{\kappa} + \hat{z}\kappa^2) \mathcal{E}_{Tz}^{(1)}(\vec{\kappa}_0)\}, \tag{5.10}$$

where the expressions for the first-order transmitted fields are to be obtained from (4.1). Equation (5.10) after further simplification can be written

$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}) = -2WW_0 \int d^2\kappa_0 |F(\vec{\kappa}_0)|^2 \left(1 - \frac{4\vec{\kappa} \cdot \vec{\kappa}_0}{k_0^2(\epsilon - 1)}\right) \vec{\mathcal{E}}_R^{(0)} + \int d^2\kappa_0 |F(\vec{\kappa} - \vec{\kappa}_0)|^2 (W^{(0)} - W_0^{(0)}) 2W_0 \vec{\mathcal{E}}_R^{(0)}$$
$$+ (\epsilon - 1)^2 \int d^2\kappa_0 |F(\vec{\kappa} - \vec{\kappa}_0)|^2 \quad (W_0\epsilon + W)^{-1}(W^{(0)} - W)(W_0\vec{\kappa} + \hat{z}\kappa^2) \mathcal{E}_{Tz}^{(0)}$$
$$+ (\epsilon - 1)^2 \int d^2\kappa_0 |F(\vec{\kappa} - \vec{\kappa}_0)|^2 \quad (W_0\epsilon + W)^{-1}(W_0^{(0)}\epsilon + W^{(0)})^{-1}(\vec{K}^{(0)} \cdot \vec{\mathcal{E}}_T^{(0)})$$
$$\times [(\kappa^2 + WW_0)\vec{\kappa}_0 + (W\hat{z} - \vec{\kappa}) \quad (\vec{\kappa} \cdot \vec{\kappa}_0) - W^{(0)}(W_0\vec{\kappa} + \hat{z}\kappa^2)]. \tag{5.11}$$

As mentioned in Sec. IV, the surface polariton contribution arises only from the resonant denominator $(W_0^{(0)}\epsilon + W^{(0)})$ and hence it comes only from the last integral in (5.11) and that too for values of κ_0 such that $\kappa_0 \geq k_0$. Denoting this contribution by $Q_{\rm sp}^{\rm ext}$, we have

$$Q_{\rm sp}^{\rm ext} = -\frac{8\pi^2}{|\mathcal{E}_R^{(0)}|^2}\,\mathrm{Re}\bigg((\epsilon-1)^2\int_{\kappa_0\geq k_0} d^2\kappa_0\,|F(\vec{\kappa}-\vec{\kappa}_0)|^2(W_0\epsilon+W)^{-1}(W_0^{(0)}\epsilon+W^{(0)})^{-1}(\vec{\mathrm{K}}^{(0)}\cdot\vec{\mathcal{E}}_T^{(0)})\vec{\mathcal{E}}_R^{(0)*}$$
$$\times[(\kappa^2+WW_0)\vec{\kappa}_0+(W\hat{z}-\vec{\kappa})(\vec{\kappa}\cdot\vec{\kappa}_0)-W^{(0)}(W_0\vec{\kappa}+\hat{z}\kappa^2)]\bigg). \tag{5.12}$$

To simplify (5.12) further, we discuss the two cases of the incident polarization separately.

A. Incident s polarization

For an incident wave which is s polarized, one has

$$\vec{\mathcal{E}}_R^{(0)}(\vec{\kappa}) = \frac{\hat{z}\times\vec{\kappa}}{\kappa}\,\mathcal{E}_{R2}^{(0)}(\vec{\kappa}), \quad \mathcal{E}_{T2}^{(0)}(\vec{\kappa}) = -\frac{2W_0}{W-W_0}\,\mathcal{E}_{R2}^{(0)}(\vec{\kappa}), \tag{5.13}$$

and hence (5.12) becomes

$$Q_{\rm sp,2}^{\rm ext} = -8\pi^2\,\mathrm{Re}\bigg\{(\epsilon-1)^2\int_{\kappa_0\geq k_0} d^2\kappa_0\,|F(\vec{\kappa}-\vec{\kappa}_0)|^2(W_0\epsilon+W)^{-1}(W_0^{(0)}\epsilon+W^{(0)})^{-1}\left(\frac{-2W_0}{W-W_0}\right)$$
$$\times\left[\vec{\mathrm{K}}^{(0)}\cdot\left(\frac{\hat{z}\times\vec{\kappa}}{\kappa}\right)\right]\left(\frac{\hat{z}\times\vec{\kappa}}{\kappa}\right)\cdot[(\kappa^2+WW_0)\vec{\kappa}_0+(W\hat{z}-\vec{\kappa})(\vec{\kappa}\cdot\vec{\kappa}_0)-W^{(0)}(W_0\vec{\kappa}+\hat{z}\kappa^2)]\bigg\}$$
$$=16\pi^2W_0\,\mathrm{Re}\bigg((\epsilon-1)\int_{\kappa_0\geq k_0} d^2\kappa_0\,|F(\vec{\kappa}-\vec{\kappa}_0)|^2\,\frac{(\vec{\kappa}\times\vec{\kappa}_0)^2}{\kappa^2}(W_0^{(0)}\epsilon+W^{(0)})^{-1}\bigg). \tag{5.14}$$

It can be easily shown that (under the assumptions: $\mathrm{Re}\epsilon \leq -1$, ϵ real)

$$\mathrm{Re}(W_0^{(0)}\epsilon+W^{(0)})^{-1} = [\pi|\epsilon|^{3/2}/\epsilon(|\epsilon|-1)(|\epsilon|+1)]\,\delta(\kappa_0-\kappa_s), \tag{5.15}$$

where κ_s is the wave vector of the surface polariton defined by

$$\kappa_s^2 = (\omega^2/c^2)[\epsilon/(\epsilon+1)] = (\omega^2/c^2)[|\epsilon|/(|\epsilon|-1)]. \tag{5.16}$$

On substituting (5.15) in (5.14), and on simplification we obtain for the reflectance drop, due to surface polariton excitation, the expression

$$Q_{\rm sp,2}^{\rm ext} = 16\pi^4\cos\theta\,\kappa_s^5|\epsilon|^{-1/2}\frac{1}{\pi k_0}\int_0^{2\pi} d\varphi\,\sin^2(\varphi-\varphi_0)|F(\kappa_s\cos\varphi-k_0\sin\theta\cos\varphi_0,\kappa_s\sin\varphi-k_0\sin\theta\sin\varphi_0)|^2, \tag{5.17}$$

where φ_0 defines the incident plane wave $\kappa_x = k_0\sin\theta\cos\varphi_0$.

B. Incident p polarization

For this polarization we have the relations

$$\vec{\mathcal{E}}_R^{(0)} = \{[(\hat{z}\times\vec{\kappa})/\kappa]\times\vec{\mathrm{K}}_0'/k_0\}\,\mathcal{E}_{R1}^{(0)}, \quad \mathcal{E}_{T1}^{(0)} = [2W_0(\epsilon)^{1/2}/(W_0\epsilon-W)]\,\mathcal{E}_{R1}^{(0)}. \tag{5.18}$$

On substituting (5.18) and (4.9) in (5.12) we get

$$Q_{\rm sp,1}^{\rm ext} = -16\pi^2W_0\,\mathrm{Re}\bigg[(\epsilon-1)^2(\epsilon)^{1/2}\int_{\kappa_0\geq k_0} d^2\kappa_0\,|F(\vec{\kappa}-\vec{\kappa}_0)|^2(W_0^2\epsilon^2-W^2)^{-1}(W_0^{(0)}\epsilon+W^{(0)})^{-1}$$
$$\times\left(\vec{\mathrm{K}}^{(0)}\cdot\frac{(\hat{z}\times\vec{\kappa})\times\vec{\mathrm{K}}}{k_0\kappa(\epsilon)^{1/2}}\right)\left(\frac{(\hat{z}\times\vec{\kappa})\times\vec{\mathrm{K}}_0'}{k_0\kappa}\right)$$
$$\cdot[(\kappa^2+WW_0)\vec{\kappa}_0+(W\hat{z}-\vec{\kappa})(\vec{\kappa}\cdot\vec{\kappa}_0)-W^{(0)}(W_0\vec{\kappa}+\hat{z}\kappa^2)]\bigg], \tag{5.19}$$

which on simplification becomes

$$Q_{\rm sp,1}^{\rm ext} = 16\pi^2W_0\,\mathrm{Re}\bigg[(\epsilon-1)^2\int_{\kappa_0\geq k_0} d^2\kappa_0\,|F(\vec{\kappa}-\vec{\kappa}_0)|^2(W_0^2\epsilon^2-W^2)^{-1}(W_0^{(0)}\epsilon+W^{(0)})^{-1}\left(W\frac{(\vec{\kappa}\cdot\vec{\kappa}_0)}{\kappa}-\kappa W^{(0)}\right)^2\bigg]. \tag{5.20}$$

On using (5.15), (5.20) reduces to

$$Q^{\text{ext}}_{\text{sp},1}=\frac{16\pi^4\kappa_s^5\cos\theta}{|\epsilon|^{1/2}(\sin^2\theta+|\epsilon|\cos^2\theta)}\frac{1}{\pi k_0}\int_0^{2\pi}d\varphi\,[\cos(\varphi-\varphi_0)(|\epsilon|+\sin^2\theta)^{1/2}-|\epsilon|^{1/2}\sin\theta]^2$$
$$\times|F(\kappa_s\cos\varphi-k_0\sin\theta\cos\varphi_0,\kappa_s\sin\varphi-k_0\sin\theta\sin\varphi_0)|^2. \tag{5.21}$$

For normal incidence $\theta=0$, the expressions (5.17) and (5.21) reduce to

$$Q^{\text{ext}}_{\text{sp},2}=\frac{16\pi^4\kappa_s^5}{|\epsilon|^{1/2}\pi k_0}\int_0^{2\pi}d\varphi\,\sin^2\varphi\times|F(\kappa_s\cos\varphi,\kappa_s\sin\varphi)|^2, \tag{5.22}$$

$$Q^{\text{ext}}_{\text{sp},1}=\frac{16\pi^4\kappa_s^5}{|\epsilon|^{1/2}\pi k_0}\int_0^{2\pi}d\varphi\,\cos^2\varphi\times|F(\kappa_s\cos\varphi,\kappa_s\sin\varphi)|^2. \tag{5.23}$$

If one further assumes that the structure function $|F(\vec{\kappa})|^2$ depends only on $|\vec{\kappa}|$, then (5.22) and (5.23) become identical (note that in this case the energy absorption should be independent of the polarization of the normally incident light)

$$Q^{\text{ext}}_{\text{sp}}=16\pi^4\kappa_s^5|F(\kappa_s)|^2/k_0|\epsilon|^{1/2}, \tag{5.24}$$

which is the result quoted in our earlier communication. In the special case when

$$F(\vec{\kappa})=\frac{L}{\pi}\sum\rho_G\delta(\kappa_y-G)\delta(\kappa_x),$$

the results (5.17), (5.21) are in agreement with those obtained by Marvin *et al.* by boundary matching method.[12]

The energy-loss expressions can either be plotted as a function of the incident angle θ (for a given ω) or the frequency ω (for a given θ). The experiment of Teng and Stern concerns the dependence of the reflectance drop as a function of θ. In their experiment a diffraction grating was used. Let the rulings of the grating make an angle δ with x axis, then $|F|^2$ can be written

$$|F(\vec{\kappa})|^2=\alpha_0\delta(\vec{\kappa})\delta(\vec{\kappa})+\alpha\delta(\vec{\kappa}-\vec{g})\delta(\vec{\kappa}-\vec{g})+\alpha^*\delta(\vec{\kappa}+\vec{g})\delta(\vec{\kappa}+\vec{g})+\text{other terms}. \tag{5.25}$$

A straightforward analysis now shows that (5.17) and (5.21) reduce to

$$Q^{\text{ext}}_{\text{sp},2}=16\pi^4\cos\theta\,\kappa_s^5|\epsilon|^{-1/2}\frac{|\alpha|^2}{(\kappa_s^2-g^2\sin^2\delta)^{1/2}}\left(\frac{g^2\sin^2\delta}{\kappa_s^2}\right)\mathcal{B}, \tag{5.26}$$

$$\mathcal{B}=\frac{1}{4\pi^3k_0}[\delta(g\cos\delta+(\kappa_s^2-g^2\sin^2\delta)^{1/2}-k_0\sin\theta)+\delta(|g\cos\delta-(\kappa_s^2-g^2\sin^2\delta)^{1/2}|-k_0\sin\theta)],$$

$$Q^{\text{ext}}_{\text{sp},1}=\frac{16\pi^4\kappa_s^5|\epsilon|^{-1/2}\cos\theta}{(\sin^2\theta+|\epsilon|\cos^2\theta)}\frac{|\alpha|^2}{(\kappa_s^2-g^2\sin^2\delta)^{1/2}}\left[\left(1-\frac{g^2\sin^2\delta}{\kappa_s^2}\right)^{1/2}(|\epsilon|+\sin^2\theta)^{1/2}-|\epsilon|^{1/2}\sin\theta\right]^2\mathcal{B}, \tag{5.27}$$

where Q now gives the extinction cross section per unit area of the surface. In deriving these relations we have assumed that δ lies between 0 to $\frac{1}{2}\pi$. Similar expressions can be obtained for other range of values of δ. As a function of θ, the above expressions show in general two peaks. If δ is close to zero, then one will see appreciable peaks only for the incident p polarization.[3]

In the above analysis we have assumed that the incident light is either s polarized or p polarized. The results for arbitrary incident polarization can be obtained in terms of the coherence matrix of the incident beam. The method of derivation is similar to that used above. We quote the final result without proof

$$Q^{\text{ext}}_{\text{sp}}=\frac{1}{I^{(i)}}\Bigg(Q^{\text{ext}}_{\text{sp},2}J^{(i)}_{22}+J^{(i)}_{11}Q^{\text{ext}}_{\text{sp},1}+16\pi^2k_0^2W_0\,\text{Re}\Bigg[\int d^2\kappa_0|F(\vec{\kappa}-\vec{\kappa}_0)|^2(W_0^{(0)}\epsilon+W^{(0)})^{-1}\left(\frac{\vec{\kappa}\times\vec{\kappa}_0}{\kappa}\right)(|\epsilon|+1)\kappa^2(\kappa^4-W^2W_0^2)^{-1}$$
$$\times\left(\frac{(\vec{\kappa}\cdot\vec{\kappa}_0)W-\kappa^2W^{(0)}}{\kappa k_0}\right)[J^{(i)}_{12}(\kappa^2+WW_0)-J^{(i)}_{21}(\kappa^2-WW_0)]\Bigg]\Bigg). \tag{5.28}$$

The relation (5.28) cannot be simplified any further. Note also that, in general, the imaginary part of $(W_0^{(0)}\epsilon+W^{(0)})^{-1}$ (principal part) also contributes. The situation is simple for the circularly polarized light which is characterized by the coherence matrix $J^{(i)}_{11}=J^{(i)}_{22}=\frac{1}{2}I^{(i)}$, $J^{(i)}_{12}=J^{(i)*}_{21}=\pm\frac{1}{2}iI^{(i)}$ [the $+(-)$ sign for left- (right-) handed polarization]. The result for the surface polariton contribution to extinction cross section is

$$Q_{\rm sp}^{\rm ext}=\tfrac{1}{2}(Q_{\rm sp,2}^{\rm ext}+Q_{\rm sp,1}^{\rm ext})\pm\frac{16\pi^4\kappa_s^5}{|\epsilon|^{1/2}}\;\frac{\cos\theta\,\sin^2\theta}{(\sin^2\theta+|\epsilon|\cos^2\theta)}$$
$$\times\frac{1}{\pi k_0}\int_0^{2\pi}d\varphi\,\sin\varphi[\cos\varphi\,(|\epsilon|+\sin^2\theta)^{1/2}-|\epsilon|^{1/2}\sin\theta]\,|F(\kappa_s\cos\varphi-k_0\sin\theta,\kappa_s\sin\varphi)|^2. \qquad (5.29)$$

VI. THEORY OF SMITH-PURCELL RADIATION—CONVERSION OF EVANESCENT WAVES INTO HOMOGENEOUS WAVES

In Secs. IV and V we have discussed at length what happens when an homogeneous wave is incident on a dielectric medium with rough surface. We have seen that, in particular, the incident homogeneous waves give rise to evanescent waves, which in special cases correspond to the excitation of surface-polariton modes. We now discuss the reverse problem. It is well known that an electron beam moving close to a perfectly conducting flat surface does not radiate, the reason being that the fields for this particular configuration involve only the evanescent waves. Smith and Purcell[28] observed that an electron moving parallel to a metallic grating leads to radiation, which is now known as Smith-Purcell radiation. The reason for this radiation is that the grating structure transforms some of the evanescent waves into homogeneous waves which then radiate. Several classical and quantum mechanical treatments[29-31] of Smith-Purcell radiation have appeared previously. We include this here as the results on the emitted radiation follow very easily from our treatments of Secs. II and V. We will treat the metallic grating as a perfect conductor as this was the case Smith-Purcell had considered. In the dielectric case even the zeroth-order fields lead to radiation,[40] which is known as the Čerenkov radiation.

On taking the limit of infinite conductivity, we find that the first-order reflected fields (4.2) become (note that $\mathcal{E}_{T\parallel}^{(0)}\sim 1/(\epsilon)^{1/2}$, $\mathcal{E}_{Tz}^{(0)}\sim 1/\epsilon$)

$$\vec{\mathcal{E}}_R^{(1)}(\vec\kappa)=2iF(\vec\kappa-\vec\kappa^{(0)})\left[(W_0^{(0)}\vec{\mathcal{E}}_\parallel^{(i)}-\vec\kappa^{(0)}\mathcal{E}_z^{(i)})+\vec\kappa\mathcal{E}_z^{(i)}+\hat z\left(\frac{\kappa^2}{W_0}\mathcal{E}_z^{(i)}+\frac{W_0^{(0)}}{W_0}\vec\kappa\cdot\vec{\mathcal{E}}^{(i)}-\frac{\vec\kappa\cdot\vec\kappa_0}{W_0}\mathcal{E}_z^{(i)}\right)\right]. \qquad (6.1)$$

The second-order fields in the direction $\vec\kappa$ of the incident field become [limit of Eq. (5.11)]

$$\vec{\mathcal{E}}_R^{(2)}(\vec\kappa)=2W_0\int d^2\kappa_0|F(\vec\kappa-\vec\kappa_0)|^2W_0^{(0)}(\vec{\mathcal{E}}^{(i)}-2\mathcal{E}_z^{(i)}\hat z)$$
$$+2\int\frac{d^2\kappa_0}{W_0^{(0)}}|F(\vec\kappa-\vec\kappa_0)|^2\left((\vec\kappa_0-\vec\kappa)+\frac{\hat z}{W_0}(\vec\kappa\cdot\vec\kappa_0-\kappa^2)\right)[W_0\vec\kappa_0\cdot\vec{\mathcal{E}}^{(i)}-\mathcal{E}_z^{(i)}(\vec\kappa\cdot\vec\kappa_0)+k_0^2\mathcal{E}_z^{(i)}]. \qquad (6.2)$$

These results can also be directly obtained from the Ewald-Oseen extinction theorem for a perfectly conducting body. In this case it reads

$$\vec{\mathrm{E}}^{(i)}(\vec{\mathrm{r}})+\frac{i}{k_0c}\vec\nabla\times\vec\nabla\times\int\vec{\mathrm{J}}(\vec{\mathrm{r}}')G_0(\vec{\mathrm{r}}-\vec{\mathrm{r}}')\,dS'=0,\qquad \vec{\mathrm{r}}\in V. \qquad (6.3)$$

The surface current $\vec{\mathrm{J}}$ has the property $\vec{\mathrm{n}}\cdot\vec{\mathrm{J}}=0$. Once the integral equation (6.3) is solved for the surface current $\vec{\mathrm{J}}$, the external (reflected field) can be obtained from

$$\vec{\mathrm{E}}^{(R)}(\vec{\mathrm{r}})=\frac{i}{k_0c}\vec\nabla\times\vec\nabla\times\int\vec{\mathrm{J}}(\vec{\mathrm{r}}')G_0(\vec{\mathrm{r}}-\vec{\mathrm{r}}')\,dS',\qquad \vec{\mathrm{r}}\in V. \qquad (6.4)$$

Using (6.3) and (6.4) we have done the perturbation calculation in the same way as in Sec. II, i.e., by expanding the surface current $\vec{\mathrm{J}}$, G_0, etc., in powers of h and equating the terms of equal power in h. We find that the results for the first- and second-order reflected field as calculated from (6.4) agrees with (6.1) and (6.2).

We now write for the current associated with the moving electron beam in the form

$$\vec{\mathrm{J}}(\vec{\mathrm{r}},t)=eVg(x-Vt)\delta(y)\delta(z+a)\hat x. \qquad (6.5)$$

It is now a simple exercise to show, that the incident electric field associated with the above current distribution can be written in the form of angular spectrum as

$$\vec{\mathrm{E}}(\vec{\mathrm{r}},\omega)=\int_{-\infty}^{+\infty}dt\,\vec{\mathrm{E}}(\vec{\mathrm{r}},t)e^{i\omega t}$$
$$=\int d^2\kappa\,\mathcal{E}^{(i)}(\vec\kappa,\omega)e^{i\vec\kappa\cdot\vec{\mathrm{r}}+iW_0z}, \qquad (6.6)$$

where

$$\vec{\mathcal{E}}^{(i)}(\vec\kappa,\omega)=g(-\omega/V)(e/W_0)\delta(u-\omega/V)\times e^{iW_0a}[\vec{\mathrm{K}}_0/V-(k_0/c)\hat x]. \qquad (6.7)$$

The representation (6.6) is valid for $z\geq -a$. The function $g(\omega)$ is the Fourier transform of $g(x)$. Since $V<c$, it is clear that $\vec{\mathcal{E}}^{(i)}$ contains only the

evanescent components $\kappa \geq k_0$. The zeroth-order reflected field is easily shown to be

$$\vec{\mathcal{E}}_R^{(0)}(\vec{\kappa}, \omega) = g(-\omega/V)(e/W_0)\delta(u - \omega/V) \times e^{iW_0 a}[-\vec{\kappa}/V + (k_0/c)\hat{x} + W_0\hat{z}/V], \quad (6.8)$$

which likewise also consists of only the evanescent waves. When (6.7) is substituted in (6.1) and (6.2), we obtain results in agreement with those of Lalor. In what follows we shall put $g = 1$. The case considered by Hessel[31] corresponds to $g(\omega) \sim \delta(\omega - \omega_0)$.

As is well known the energy loss per unit length can be expressed as

$$\frac{dW}{dl} = -\frac{e}{\pi}\int_0^\infty d\omega \, \mathrm{Re} E_x(Vt, 0, -a, \omega)e^{-i\omega t}. \quad (6.9)$$

It is clear that the second-order fields have the structure

$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}) = \int d^2\kappa' \, d^2\kappa'' \, F(\vec{\kappa} - \vec{\kappa}')F(\vec{\kappa} - \vec{\kappa}'') \times \vec{\Omega}(\vec{\kappa}, \vec{\kappa}') \cdot \vec{\Lambda}(\vec{\kappa}', \vec{\kappa}'') \cdot \vec{\mathcal{E}}^{(i)}(\vec{\kappa}''). \quad (6.10)$$

In the special case of a grating characterized by

$$F(\vec{\kappa}) = \tfrac{1}{2}\{\delta(\kappa_x) + \tfrac{1}{2}[\delta(\kappa_x - g) + \delta(\kappa_x + g)]\}\delta(\kappa_y), \quad (6.11)$$

(6.10) reduces to

$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}) = \tfrac{1}{4}[\vec{\Omega}(\vec{\kappa}, \vec{\kappa}) \cdot \vec{\Lambda}(\vec{\kappa}, \vec{\kappa}) + \tfrac{1}{4}\vec{\Omega}(\vec{\kappa}, \vec{\kappa} - \vec{g}) \cdot \vec{\Lambda}(\vec{\kappa} - \vec{g}, \vec{\kappa}) + \tfrac{1}{4}\vec{\Omega}(\vec{\kappa}, \vec{\kappa} + \vec{g}) \cdot \vec{\Lambda}(\vec{\kappa} + \vec{g}, \vec{\kappa})] \cdot \vec{\mathcal{E}}^{(i)}(\vec{\kappa}) + \cdots, \quad g_y \equiv 0, \quad (6.12)$$

where the dots stand for other terms involving different components in the angular spectrum of $\vec{\mathcal{E}}^{(i)}$. On substituting (6.12) in (6.9) and on ignoring the terms dependent on time, retaining only the non-vanishing contributions, we get up to terms of second order in h.

$$\frac{d^2W}{dl\,d\omega} \simeq -\frac{e}{16\pi}\,\mathrm{Re}\int d^2\kappa \, \Omega_{x\beta}(\vec{\kappa}, \vec{\kappa} - \vec{g}) \times \Lambda_{\beta\gamma}(\vec{\kappa} - \vec{g}, \vec{\kappa})\mathcal{E}_\gamma^{(i)}(\vec{\kappa}), \quad g_x > 0, \; g_y = 0, \quad (6.13)$$

where now $d^2W/dl\,d\omega$ gives the energy loss per unit length per unit frequency interval. It may be noted that the first-order fields do not contribute to (6.13). It may also be noticed from (6.10) that if $\mathcal{E}_\gamma^{(i)}(\vec{\kappa}') = \mathcal{E}_\gamma^{(i)}\delta^{(2)}(\vec{\kappa}' - \vec{\kappa}_0)$, then

$$\vec{\mathcal{E}}_R^{(2)}(\vec{\kappa}_0) = \int d^2\kappa' \, |F(\vec{\kappa}' - \vec{\kappa}_0)|^2 \times \vec{\Omega}(\vec{\kappa}_0, \vec{\kappa}') \cdot \vec{\Lambda}(\vec{\kappa}', \vec{\kappa}_0) \cdot \mathcal{E}^{(i)}. \quad (6.14)$$

The relevant functions which appear in (6.13) can be obtained from comparison of (6.14) and (6.2). On using (6.7), (6.2), and (6.14) and on simplification, we find that (6.13) reduces to[41]

$$\frac{d^2W}{dl\,d\omega} = \frac{e^2k_0^3}{8\pi c}\int_{-\pi/2}^{+\pi/2} d\varphi \left[\left(1 - \frac{1}{\beta^2}\right)\sin^2\theta_0\cos^2\varphi + \frac{g^2}{k_0^2}\right]\exp\left\{-2ak_0\left[\left(\frac{1}{\beta^2} - 1\right) + \sin^2\theta_0\sin^2\varphi\right]^{1/2}\right\}, \quad (6.15)$$

where

$$\sin\theta_0 = [1 - (1/\beta - g/k_0)^2]^{1/2}, \quad \beta = V/c. \quad (6.16)$$

(6.15) gives the loss of the energy of the electron. Since we have not included any other source of absorption, this energy should appear as radiation. Hence (6.15) is the power radiated by the electron per unit frequency interval and per unit length. θ_0 gives the direction of propagation of waves as is evident from the considerations of the asymptotic expansion of the first-order fields. We carry out asymptotic expansion in the fixed direction in any place $x = \text{const}$ as $\rho \equiv (y^2 + z^2)^{1/2} \to \infty$. Then we find that[28]

$$\vec{E}^{(1)}(\vec{r}, \omega) \sim (e\,e^{i\pi/4}/2^{3/2}c\pi^{1/2})[k_0^{3/2}(\sin\theta_0)^{1/2}/\rho^{1/2}]\exp[-k_0(c^2/V^2 - 1 + \sin^2\theta_0\sin^2\varphi)^{1/2}a] \times ((c/V\cos\theta_0\cos\varphi - \cos\varphi, (c/V)\sin\theta_0\sin\varphi\cos\varphi, [(c/V)\sin^2\theta_0\cos^2\varphi + \cos\theta_0 - c/V]/\sin\theta_0) \times \exp[ik_0(x\cos\theta_0 + \rho\sin\theta_0)], \quad \cos\varphi = z/\rho. \quad (6.17)$$

Thus it is clear that θ_0 defines the direction of propagation. Equation (6.16) then gives the wavelength of the emitted radiation for a given direction of the radiation. It is clear from the structure of (6.17) that the radiation emitted in the forward direction is strongly polarized in z direction—which is in agreement with the experimental observation of Smith-Purcell.[42] In the more general case of dielectric gratings, one should use the expressions (5.11). We plan to discuss this in a separate publication.

Note added in proof. Since this paper was submitted for publication, we have been able to generalize the present approach to include the effects of spatial dispersion and the time dependence of the rough surface (interaction of Rayleigh waves and light waves). These generalizations will be discussed elsewhere.

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