Optical bistability with dispersion

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The purpose of this work is to study the behavior of atomic fluctuations and the spectrum of transmitted light by a nonlinear medium with absorption and dispersion. The authors make use of the system-size expansion and apply it to the quantum-mechanical Langevin's equations for the atomic operators and obtain the characteristic curves for bistable behavior, macroscopic atomic expectation values, and the spectrum of the atomic fluctuations. The calculated spectrum of transmitted light exhibits line narrowing near bistable thresholds, discontinuous formation of bands along the high-transmission bistable branch, and hysteresis effects, etc., for detuning $|\delta|$ less than a critical value δ_c. For $|\delta| > \delta_c$, the discontinuous character of all the physical variables of the system vanishes; the system no longer exhibits bistable behavior. The system, however, still shows cooperative and single-atom behavior separated by a critical value of the intensity of the incident coherent pump at which a critical slowing down of the medium occurs, exhibiting a phase transition.

I. INTRODUCTION

Recently there has been an increasing interest in the study of the optically bistable systems.[1-11] The idealized situation considered thus far in the literature comprises a homogeneously broadened absorptive medium placed inside and in tune with a Fabry-Perot cavity under the influence of a coherent external pump which is on resonance with a preferred mode of the cavity and thus is also in tune with the non-linear medium. The bistable behavior of such a system arises owing to the competition between the cooperative and single-atom decay mechanisms in the presence of an external coherent pump.[4,5] Assuming the same mechanism, we study the optical bistable behavior of a homogeneously broadened two-level (energy difference ω_0) dispersive medium which is detuned by an amount with the preferred mode of the cavity and the incident coherent pump (photon energy ω_L). In an earlier communication,[10] it has been shown that owing to the presence of dispersion, an increase in the transmitted intensity along the cooperative branch takes place. The bistable behavior of a dispersive homogeneously broadened medium depends on the detuning and the parameter C (the ratio of the cooperative decay rate to twice the single-atom transverse-decay rate), and it is found that there is a critical value of the detuning beyond which the system ceases to be bistable. In this paper we present a detailed quantum-mechanical analysis of a bistable system with both absorption and dispersion. We use the same method as in Ref. 6. We study in detail the transition of the system from bistable regime to nonbistable regime as $|\delta|$ is increased. For $|\delta|$ of the medium less than the critical value δ_c, the hysteresis cycles for the transmitted intensity, the incoherent part of transmitted intensity and the spectrum of the transmitted light, similar to those obtained for a purely absorptive bistable medium, are observed. As the detuning approaches the critical value, the sizes of the hysteresis cycles reduce. At $|\delta| = \delta_c$, the hysteresis cycles completely vanish. For $|\delta| > \delta_c$, the transmitted intensity is a single-valued function of the intensity of incident coherent pump. The discontinuous character for the appearance of dynamical Stark shift is also absent. The three-peak structure in the spectrum of the transmitted field appears gradually beyond a critical value of the intensity of the incident coherent pump at which one of the normal modes of the system becomes soft, but with finite lifetime, and the spectrum of fluctuations shows a narrow central peak. Similarly for $|\delta| \rangle \delta_c$, the incoherent part of the transmitted intensity is a single-valued function of incident intensity and shows a maximum at the same critical value of the incident intensity at which the three-peak structure starts appearing.

It is found that for a large sample of two-level atoms, introduction of the detuning of the medium which results in the increase of transmitted intensity along the cooperative branch, so much as to quench the bistable output of the medium, does not greatly affect the cooperative behavior of the medium. The system still has dynamically two distinct behaviors, viz., cooperative and single-atom behavior separated by a critical value of the incident intensity. Results of the study of atomic fluctuations and spectrum of transmitted light, etc., reported here are purely quantum mechanical. In Sec. II, the master equation taking account of dispersion of the medium is presented. The mean-field results which follow from the master equations are described. In Sec. III, we obtain the Langevin equations and the equations for the correlation functions involving atomic operators. The steady-state solution of the correlation matrix is determined. In Sec. IV, the spectrum of transmitted light is obtained. The results are discussed for various values of δ by choosing an example with $C=10$. In Sec. V, we discuss the variation in the incoherent part of transmitted light with detuning. A formula is given for the determination of the incident intensity at which the nonbistable system undergoes transition from cooperative to one-atom behavior.

II. MEAN-FIELD RESULTS AND THE MASTER EQUATION

In our model the incident coherent pump interacts with a detuned large sample of two-level atoms ($N \gg 1$). The atoms get excited to an upper level and decay spontaneously as well as via stimulated emission. Since the atoms are enclosed in a Fabry–Perot cavity, we consider the situation where one of the modes of the electromagnetic field inside the cavity plays a dominant role. We assume for simplicity perfect resonance between the incident coherent pump and the dominant mode of the cavity. A detuning $(\omega_0-\omega_L)$ between the two-level atoms and the incident coherent pump, however, is introduced. In this model one has both absorption and dispersion. The damping rate κ of the field in the cavity is assumed large compared to the single-atom relaxation rates. This enables us to eliminate the internal field adiabatically, and one finds, in the Born-Markov approximation and in a frame rotating at angular frequency of the incident coherent pump, that the master equation of the reduced atomic-density operator W is given by

$$\frac{dW}{dt}=-i[\Delta S^z,\,W]+i\Omega_I[S^+ + S^-,W]+\Lambda_S W+\Lambda_A W\,, \tag{2.1}$$

where Δ represents the detuning $\omega_0-\omega_L$, Ω_I represents the coupling with the coherent driving laser field, $S^{\pm}$, and S^z are the collective atomic operators. The terms Λ_S and Λ_A describe, respectively, the cooperative and the incoherent emission and are given by

$$\Lambda_S W=(2g^2/\kappa)(S^- W S^+ - \tfrac{1}{2}WS^+S^- - \tfrac{1}{2}S^+S^-W), \tag{2.2}$$

$$\Lambda_A W=\sum_{i=1}^{N}\gamma_\perp([S_i^-,\;WS_i^+]+\text{h.c.})\,, \tag{2.3}$$

where $2\gamma_\perp$ gives the single-atom decay rate.

The structure of the cooperative term $\Lambda_S W$ follows from the master equation of Bonifacio, Schwendimann, and Haake[12] describing the single-mode superradiance. The master equation (2.1) also follows from the master equation of Ref. 4d by adiabatically eliminating the cavity-field variables and by ignoring the detuning terms corresponding to the detuning between the cavity-field frequency and the coherent driving-field frequency. The master equation (2.1) differs from that of the purely absorptive medium in the appearance of the detuning terms. We will see that the detuning terms play a very important role in determining the bistable behavior.

The equations of motion for the atomic expectation values which follow from Eq. (2.1) are

$$\frac{d}{d\tau}\langle S^+\rangle=(i\delta-1)\langle S^+\rangle-\sqrt{2}\,iy\,\langle S^z\rangle+\frac{4c}{N}\langle S^+S^z\rangle\,, \tag{2.4a}$$

$$\frac{d}{d\tau}\langle S^-\rangle=(-i\delta-1)\langle S^-\rangle+\sqrt{2}\,iy\langle S^z\rangle+\frac{4c}{N}\langle S^zS^-\rangle\,, \tag{2.4b}$$

$$\frac{d}{d\tau}\langle S^z\rangle=\frac{-iy}{\sqrt{2}}\,(\langle S^+\rangle-\langle S^-\rangle)\,\frac{-4c}{N}\,\langle S^+S^-\rangle -2\left(\langle S^z\rangle+\frac{N}{2}\right), \tag{2.4c}$$

where

$$\tau=\gamma_\perp t\,,\quad \delta=\frac{\Delta}{\gamma_\perp}\,,\quad y=\frac{\sqrt{2}\,\Omega_I}{\gamma_\perp}\,,\quad \text{and } C=\frac{g^2N}{2\kappa\gamma_\perp}\,. \tag{2.4d}$$

The above equations can be solved in the mean-field approximation, i.e., by using the factorizations $\langle S^+S^z\rangle \simeq \langle S^+\rangle\langle S^z\rangle$, $\langle S^+S^-\rangle \approx \langle S^+\rangle\langle S^-\rangle$, which could be justified by using the system size expansion of Sec. III. We will indicate very briefly how the steady-state solutions of Eq. (2.4) can be obtained. If we define

$$\langle S^+\rangle=(iN/\sqrt{2})\Phi,\quad \langle S^z\rangle=-\tfrac{1}{2}N\Psi\,, \tag{2.5}$$

then Eqs. (2.4) reduce to

$$(y-2c\Phi)=(1-i\delta)(\Phi/\Psi)\,, \tag{2.6}$$

$$(\Psi-1)+\tfrac{1}{2}\Phi(y-2c\Phi^*)+\tfrac{1}{2}\Phi^*(y-2c\Phi)=0. \tag{2.7}$$

On subtituting Eq. (2.6) in (2.7), we get

$$|\Phi|^2/\Psi + (\Psi - 1) = 0 \ , \tag{2.8}$$

and hence we can write

$$\Psi = 1/(1+b^2), \quad \Phi = b/(1+b^2)e^{-i\nu} , \tag{2.9}$$

where the relationship between b and y can be obtained from Eq. (2.6):

$$y = be^{-i\nu}[1 - i\delta + 2c(1+b^2)^{-1}] \ . \tag{2.10}$$

We now introduce a complex field Z by

$$Z = be^{-i\nu}(1 - i\delta) \ , \tag{2.11}$$

so that when $C = 0$, $Z = y$. On combining Eqs.(2.10) and (2.11), we obtain

$$y = Z[1 + 2c(1+i\delta)/(1+\delta^2+|Z|^2)] \ , \tag{2.12}$$

which leads to the following relations for amplitude and phase θ of the complex field $Z = |Z| \, e^{-i\theta}$:

$$y^2 = |Z|^2\left(1 + \frac{4c^2(1+\delta^2)}{(1+\delta^2+|Z|^2)^2} + \frac{4c}{1+\delta^2+|Z|^2}\right) \ , \tag{2.13}$$

$$\tan\theta = \delta[(1+\delta^2+|Z|^2)/2c + 1]^{-1} \ . \tag{2.14}$$

In terms of the field Z, the atomic expectation values have macroscopic values given by

$$\langle S^+ \rangle = \frac{iN}{\sqrt{2}} \frac{Z(1+i\delta)}{1+\delta^2+|Z|^2} \ , \quad \langle S^z \rangle = \frac{-N}{2} \frac{1+\delta^2}{1+\delta^2+|Z|^2} \ . \tag{2.15}$$

The field Z is connected with the transmitted field. The internal field is $\langle a \rangle \sim (-ig/\kappa)\langle S^- \rangle$, and hence the transmitted field should be (apart from constants)

$$Z = \sqrt{2}\,\Omega_I/\gamma_\perp + (2g/\gamma_\perp)\langle a \rangle = (y - 2c\Phi),$$

which is equivalent to Eq. (2.12). The relation (2.13) determines the values of the transmitted intensity in terms of the incident intensity. In general it is a multivalued function of y. We will refer to Eq. (2.13) as the characteristic equation for bistability. The characteristic equation (2.12) has been derived by a number of authors.[4d,8,9a,10] The most general form of the characteristic equation, including the effects of inhomogeneous broadening and cavity detuning, is given in Ref. 4d. The existence of the multiple solutions depends on two parameters C and δ.

It is found that: (1) for $C < 4$, $|Z|$ is a single-valued function of y for all values of δ; (ii) for $C > 4$ and $|\delta| > \delta_c$, $|Z|$ is a single-valued function of y, where δ_c gives the critical value of the detuning determined by

$$1 + \delta_c^2 = (4/27)[(C-1)^3/C] \ ; \tag{2.16}$$

and (iii) for $C > 4$ and $|\delta| < \delta_c$, $|Z|$ is single valued for $y < y_2$ and $y > y_1$ and multivalued for $y_2 \leq y \leq y_1$, where y_1 and y_2 are the solutions of the equation

$$\frac{dy}{d|Z|} = 0 \ . \tag{2.17}$$

We thus see that the bistable behavior exists only for $C > 4$, $y_2 \leq y \leq y_1$, and if the detuning is less than a critical value determined by Eq. (2.16). In Fig. 1, we show that behavior of the transmitted intensity as a function of the incident intensity for $C = 10$ and for values of the detuning both below and above the critical value.

A. $C > 4$, $|\delta| < \delta_c$

We have a low-transmission branch called the cooperative branch and a high-transmission branch called the one-atom branch.[4a] Along the low-transmission branch the atomic expectation values are not an extensive function of the number of atoms, since $y \gg |Z|$ and $|Z|$ is critically dependent on the density via C. The intensity, however, is proportional to $(1+\delta^2)$ along this branch. This is the increase in intensity of the transmitted light due to the presence of dispersion. Along the high-transmission branch and for large values of y^2, the atomic expectation values are directly proportional to the number of atoms in the cavity since $y \sim |Z|$, single-atom behavior dominates. The transmitted intensity shows a small reduction exhibiting the fact that system does not undergo complete saturation as in the absence of dispersion. As y is increased beyond y_1 along the coop-

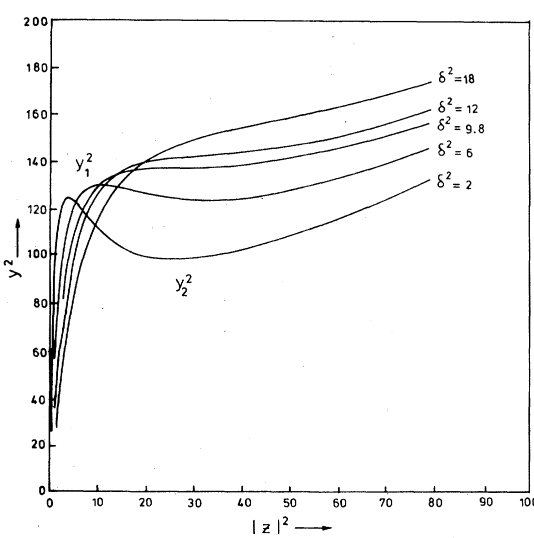

FIG. 1. The incident intensities vs the transmitted intensities are shown for various detunings of nonlinear medium for $C = 10$.

erative branch, or decreased past y_2 along the "one-atom" branch, the transmitted intensity undergoes a sharp discontinuity. The discontinuity depends on δ. At $|\delta| = \delta_c$, the lower and upper bistability thresholds meet and the hysteresis in the output cycle disappears.

B. $C>4, |\delta|>\delta_c$

The transmitted intensity is a monotonic function of the intensity of the incident coherent pump. No region of this curve has a negative slope, so none of the steady states of the system is unstable.

In this case one would like to understand if the system still has two distinct behaviors, viz., cooperative and one-atom behavior, what happens to the spectrum of fluctuations, at what incident intensity the system shifts from cooperative to one-atom behavior, etc. Some of these questions may be discussed within the framework of the relaxation matrix.[11]

In Sec. III we examine many of the problems posed in this section quantum mechanically. In particular we will calculate the spectrum of fluctuations.

III. LANGEVIN'S EQUATION AND CORRELATION MATRIX

The system-size-expansion technique as used by Agarwal *et al.*[6] is applied here to analyze the fluctuations around the steady state. We will follow very closely Ref. 6 and present the results in brief only. The Lagevin equations for our problem in terms of the scaled atomic operators $x^{(\pm)} = (1/N)S^{(\pm)}$, $x^{(3)} = (1/N)S^{(Z)}$ can be obtained by using the master equation (2.1) and are

$$\dot{x}^{(+)} = -(1-i\delta)x^{(+)} + 4cx^{(+)}x^{(3)} + \sqrt{2}\,iyx^{(3)} + (1/\sqrt{N})F^{(+)}, \quad (3.1a)$$

$$\dot{x}^{(-)} = -(1+i\delta)x^{(-)} + 4cx^{(3)}x^{(-)} - \sqrt{2}\,iyx^{(3)} + (1/\sqrt{N})F^{-}, \quad (3.1b)$$

$$\dot{x}^{(3)} = -2(x^{(3)}+\tfrac{1}{2}) - 4c\,x^{(+)}x^{(-)} + \frac{iy}{\sqrt{2}}(x^{(+)}-x^{(-)}) + (1/\sqrt{N})F^{(3)}, \quad (3.1c)$$

where $F^{(\pm)}$, $F^{(3)}$ are δ-correlated stochastic operator forces. These are found to have the correlation properties

$$\langle F^{(\pm)}\rangle = \langle F^{(3)}\rangle = 0,$$

$$\langle F^{(+)}(t)F^{(-)}(t')\rangle = 2\langle D_{+-}(t)\rangle\delta(t-t'),$$

$$\langle F^{(-)}(t)F^{(-)}(t')\rangle = 2\langle D_{--}(t)\rangle\delta(t-t'),$$

$$\langle F^{(+)}(t)F^{(3)}(t')\rangle = 2\langle D_{+3}(t)\rangle\delta(t-t'); \quad (3.2)$$

$$\langle D_{+-}(t)\rangle = \langle D_{++}(t)\rangle = \langle D_{+3}(t)\rangle = 0,$$

$$\langle D_{33}(t)\rangle = \langle\tfrac{1}{2} + x^{(3)}(t)\rangle + 2c\langle x^{(+)}(t)x^{(-)}(t)\rangle. \quad (3.3)$$

Expanding all operators of Eqs. (3.1) and (3.2) in the spirit of system size expansion and comparing the zeroth-order terms in $1/\sqrt{N}$, one gets a set of equations independent of random forces. These have exactly the structure as those obtained from the master equation in the mean-field approximation in Sec. II. The first-order terms in $(1/\sqrt{N})$ $[x^{(\pm)} = x_0^{(\pm)} + (1/\sqrt{N})y^{(\pm)}]$ give the equations of fluctuation operators and are

$$\frac{d}{d\tau}\begin{bmatrix} y_0^{(+)} \\ y_0^{(-)} \\ y_0^{(3)} \end{bmatrix} = M(\tau)\begin{bmatrix} y_0^{(+)} \\ y_0^{(-)} \\ y_0^{(3)} \end{bmatrix} + \begin{bmatrix} F_0^{(+)} \\ F_0^{(-)} \\ F_0^{(3)} \end{bmatrix}, \quad (3.4)$$

$$M(\tau) = \begin{bmatrix} -1+i\delta+4cx_0^{(3)}(\tau) & 0 & 4cx_0^{(+)}(\tau)-\sqrt{2}\,iy \\ 0 & -1-i\delta+4cx_0^{(3)}(\tau) & 4cx_0^{(-)}(\tau)+\sqrt{2}\,iy \\ -4cx_0^{(-)}(\tau)-\frac{iy}{\sqrt{2}} & -4cx_0^{(+)}(\tau)+\frac{iy}{\sqrt{2}} & -2 \end{bmatrix},$$

where the new δ-correlated forces have their properties similar to those in Eq. (3.2), but with x replaced by x_0. From Eq. (3.4) one can easily construct, as was done in Ref. 6, the equations for the correlation functions involving atomic operators. In particular the steady-state correlations are found to obey the equation

$$\frac{d}{d\tau}\begin{pmatrix} \langle y_0^{(+)}(\tau)y_0^{(-)}\rangle \\ \langle y_0^{(-)}(\tau)y_0^{(-)}\rangle \\ \langle y_0^{(3)}(\tau)y_0^{(-)}\rangle \end{pmatrix} = M\begin{pmatrix} \langle y_0^{(+)}(\tau)y_0^{(-)}\rangle \\ \langle y_0^{(-)}(\tau)y_0^{(-)}\rangle \\ \langle y_0^{(3)}(\tau)y_0^{(-)}\rangle \end{pmatrix}, \quad (3.5)$$

where

$$\langle A(\tau)B\rangle = \lim_{t\to\infty}\langle A(t+\tau)B(t)\rangle \text{ and } M = \lim_{t\to\infty} M(t). \quad (3.6)$$

On using Eq. (3.4) and the expectation values (2.15), we find the following for the relaxation matrix:

$$M = \begin{bmatrix} -A+i\delta & 0 & -\sqrt{2}\,iZ \\ 0 & -A-i\delta & \sqrt{2}\,iZ^* \\ -\sqrt{2}\,iZ^*\Lambda & +\sqrt{2}\,iZ\Lambda^* & -2 \end{bmatrix}, \quad (3.7)$$

where

$$A=[1+2C(1+\delta^2)/(1+\delta^2+|Z|^2)];\ \ \Lambda=1-y/2Z^*. \tag{3.8}$$

The solution of Eq. (3.5) can be easily obtained. However, we need the equal-time correlation functions to solve Eq. (3.5). For this one constructs the equation of the equal-time correlation matrix[6] which follows from the Langevin equation (3.4):

$$\dot{\sigma}^{(N)}(t)=M(t)\sigma^{(N)}(t)+\sigma^{(N)}(t)\tilde{M}(t)+2\mathfrak{D}(t), \tag{3.9a}$$

where $\sigma^{(N)}(t)$ is defined by

$$\sigma^{(N)}(t)=\begin{bmatrix}\langle y_0^+(t)y_0^+(t)\rangle & \langle y_0^{(+)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(+)}(t)y_0^{(3)}(t)\rangle\\ \langle y_0^+(t)y_0^{(-)}(t)\rangle & \langle y_0^{(-)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(-)}(t)\rangle\\ \langle y_0^{(+)}(t)y_0^{(3)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(-)}(t)\rangle & \langle y_0^{(3)}(t)y_0^{(3)}(t)\rangle\end{bmatrix}. \tag{3.9b}$$

In what follows we only need the steady-state values and hence we only quote the diffusion matrix in this limit,

$$2\mathfrak{D}=\lim_{t\to\infty}2\mathfrak{D}(t)=\begin{bmatrix}\dfrac{ZZ(1+i\delta)}{1+\delta^2+|Z|^2} & 0 & 0\\ 0 & \dfrac{Z^*Z^*(1-i\delta)}{1+\delta^2+|Z|^2} & 0\\ 0 & 0 & \dfrac{2|Z|^2}{1+\delta^2+|Z|^2}\end{bmatrix}. \tag{3.10}$$

Equation (3.9) can be easily solved for the steady-state values.

For the purpose of the calculation of the spectrum of transmitted light, we need only three correlation functions, viz., $\langle y_0^{(+)}y_0^{(-)}\rangle$, $\langle y_0^{(-)}y_0^{(-)}\rangle$, $\langle y_0^{(3)}y_0^{(-)}\rangle$, and these are found to be

$$\langle y_0^{(+)}y_0^{(-)}\rangle=\frac{|Z|^4}{D(1+\delta^2+|Z|^2)}\left\{\mathrm{Re}\left[\left(\frac{\Lambda^*(1-i\delta)+A+i\delta}{A+i\delta}\right)\left(A-i\delta+2+\frac{2|Z|^2\Lambda}{A-i\delta}\right)\right]-2|Z|^2(\mathrm{Im}\Lambda)\,\mathrm{Im}\left(\frac{\Lambda^*(1-i\delta)+A+i\delta}{A+i\delta}\right)\right\}, \tag{3.11}$$

$$\langle y_0^{(3)}y_0^{(-)}\rangle=-\frac{iZ^*|Z|^4}{\sqrt{2}\,D(1+\delta^2+|Z|^2)}\left[2iA\Lambda\,\mathrm{Im}\left(\frac{\Lambda(1+i\delta)+A-i\delta}{A-i\delta}\right)+2i\,\mathrm{Im}\left(\Lambda\frac{\Lambda(1+i\delta)+A-i\delta}{(A-i\delta)}\right)\right.$$
$$\left.-\frac{A}{|Z|^2}\left((A-i\delta+2)+\frac{2|Z|^2\Lambda}{(A-i\delta)}\right)\left(\frac{\Lambda^*(1-i\delta)+A+i\delta}{A+i\delta}\right)\right], \tag{3.12}$$

$$\langle y_0^{(-)}y_0^{(-)}\rangle=[\sqrt{2}\,iZ^*/(A+i\delta)]\langle y_0^{(3)}y_0^{(-)}\rangle+Z^{*2}(1-i\delta)/2(A+i\delta)(1+\delta^2+|Z|^2), \tag{3.13}$$

where

$$D=A\left|(A-i\delta+2)+\frac{2|Z|^2\Lambda}{A-i\delta}\right|^2+2|Z|^2\,\mathrm{Re}\left(A(A-i\delta+2)\Lambda^*+(A+i\delta+2)\Lambda^*+\frac{2|Z|^2A\Lambda\Lambda^*}{A-i\delta}+\frac{2|Z|^2\Lambda^*\Lambda^*}{A+i\delta}\right)$$
$$-4|Z|^4(\mathrm{Im}\Lambda^*)^2. \tag{3.14}$$

These expectation values will be used to calculate, in Sec. IV, the spectrum of the transmitted light and, in Sec. V, the incoherent intensity.

IV. SPECTRUM OF TRANSMITTED LIGHT

The spectrum $S_{\rm in}$ of the incoherent part of the transmitted light is given by

$$S_{\rm in}(\omega)=N\,\mathrm{Re}\int_0^\infty d\tau\,e^{-i\omega\tau}\langle y_0^{(+)}(\tau)y_0^{(-)}\rangle$$
$$\doteq N\,\mathrm{Re}\hat{X}_1(p)|_{p=i\omega}. \tag{4.1}$$

The Laplace transform $\hat{X}_1$ is easily found from Eq. (3.5):

$$\hat{X}_1(p) = \mathcal{P}^{-1}(p)\bigg[[(p+A+i\delta)(p+2)+2|Z|^2\Lambda^*]X_1(0) + \sqrt{2}\, iZ\left(\frac{2|Z|^2\Lambda^*}{A+i\delta} - (p+A+i\delta)\right)X_3(0) + \frac{\Lambda^*|Z|^4(1-i\,\delta)}{(1+\delta^2+|Z|^2)(A+i\delta)}\bigg], \tag{4.2}$$

where

$$\mathcal{P}(p) = (p+A-i\delta)(p+A+i\delta)(p+2) + 2(p+A-i\delta)|Z|^2\Lambda^* + 2|Z|^2\Lambda(p+A+i\delta), \tag{4.3}$$

and where $X_1(0)$, $X_3(0)$, and $X_2(0)$ are given, respectively, by Eqs. (3.11), (3.12), and (3.13). Note that the zeroes of $\mathcal{P}(p)$ give the eigenvalues of the relaxation matrix M. These eigenvalues are critical in determining the widths of the various peaks in the scattered spectrum. A discussion of the eigenvalues of M in the bistable region can be found in an earlier conference presentation.[11] It can, for example, be shown that

$$\det M \propto \frac{dy}{d|Z|}, \tag{4.4}$$

and hence it is obvious that at least one mode will become soft and will have infinite lifetime at both the upper and lower bistability thresholds. For the region $|\delta| > \delta_c$, no mode will have zero eigenfrequency; however, we will see later that one of the modes starts becoming very close to soft, however, with finite lifetime.

The analytical expression for the spectrum is too complicated. We will now present numerical results. We have chosen C as 10. For $C = 10$, we find from Eq. (2.16) that the critical value of the

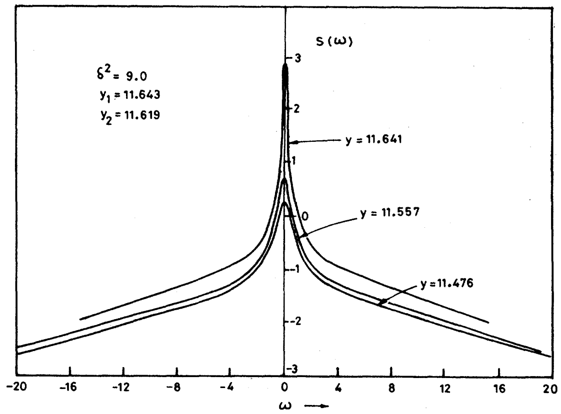

FIG. 3. Same as in Fig. 2 but for $\delta^2 = 9$.

detuning is $\delta_c^2 = 9.8$. The results of the numerical computations are given in Figs. 2–9 for various values of the parameters, all the spectra are plotted on a logarithmic scale.

A. $|\delta| < \delta_c$

Figures 2 and 3 and 4 and 5 show the plots of $S(\omega)$ for the cooperative and one-atom branches of our system with $\delta^2 = 6$ and 9. Different curves in a particular figure correspond to various values of the intensity of incident coherent pump approaching the upper bistability threshold on the cooperative branch and the lower bistability thresholds for the one-atom branch. For the cooperative branch (Figs. 2 and 3) we observe the general features—the spectrum is a single central peak, the peak height increases, and the width decreases as the incident intensity approaches the upper bistability threshold. This type of behavior of line narrowing is known to occur at the threshold of a phase transition and is connected with the softening of one of the modes [see Eq.

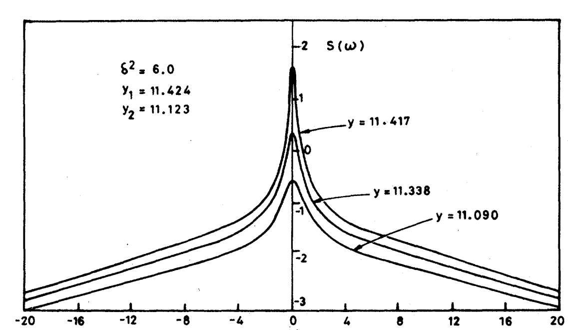

FIG. 2. Spectrum of transmitted light along the cooperative branch for $\delta^2 = 6$. The values of y are increasingly closer to the switching threshold y_1. The spectra are characterized by a central peak with decreasing half width.

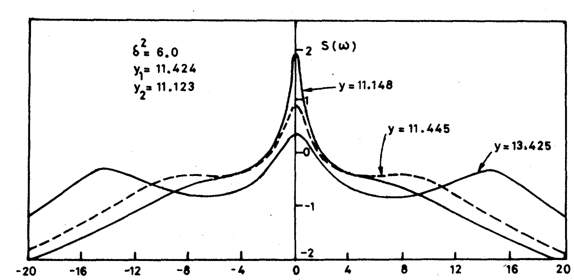

FIG. 4. Spectrum of transmitted light on the single-atom branch for $\delta^2 = 6$. The values of y are gradually decreasing towards the lower threshold y_2. The side bands merge into the central peak, which grows increasingly higher and narrows as the lower threshold is approached.

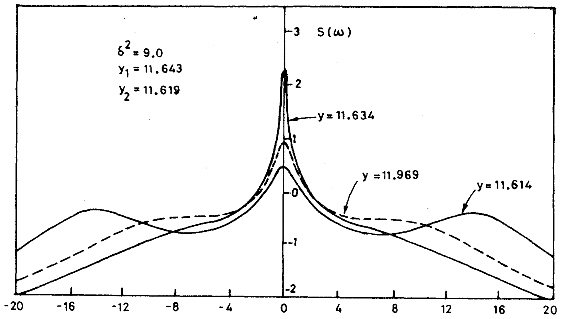

FIG. 5. Same as in Fig. 4 but for $\delta^2 = 9$.

(4.4)], the critical slowing down of dynamics, etc. On increasing the incident intensity y greater than y_1 the system jumps suddenly to the one-atom branch. The transmitted intensity is shown in Fig. 1 and the spectrum of incoherent part of the transmitted intensity is given in Figs. 4 and 5. On the one-atom branch we immediately note that the spectrum has a dominant central peak and two side bands placed roughly at the Rabi frequency, one on each side of the central peak. In Figs. 4 and 5 the following general features may be noted, as the intensity of incident coherent pump is reduced to approach the lower bistability threshold: For $y \gg y_1$, the side bands appear well separated from central peak; the height of central peak and the side bands have the same ratio as that of the well-known single-atom spectrum.[13] For $y < y_1$ and $y > y_2$, the side bands move towards the central peak. The peaks of the side bands get increasingly slowly mixed with the shoulders of the central peak, and the height of the central peak increases. In the neighborhood of y_2, the side bands are almost absent; as the central peak height increases, its width decreases. Once again one observes the line narrowing effect depicting a critical threshold of a phase transition. For $y < y_2$ the system jumps suddenly to the cooperative branch, and the spectrum of fluctuation is a central single peak with its height very much reduced. This behavior demonstrates the presence of the first-order phase transition at $y = y_2$ when the dispersive bistable system is on the one-atom branch. Now if the intensity of the incident light is increased, the system remains on the cooperative branch and the spectrum is also the single central peak. These results show that dispersive system has similar dynamical behavior in its bistable domain ($|\delta| < \delta_c$) as does a purely absorptive bistable system. One feature of difference with the absorptive model is observed on the one-atom branch near the lower bistability threshold. The interesting feature of the side bands becoming more dominant than the central peak in the pure absorptive model is quite absent in the bistable model with dispersion.

B. $|\delta|^2 = \delta_c^2 = 9.8$

Figures 6 and 7 show the spectrum of fluctuations in this case. From Eq. (2.5) one knows that $y_1 = y_2$ for $|\delta| = \delta_c$. We note in Fig. 4(a) that as $y < y_1 = y_2$ value, the central peak shows line narrowing and hence an approach to the threshold of a phase transition. Fig. 7 shows that for $y > y_1 = y_2$ the spectrum is three peaked. As $y \to y_1 = y_2$ from larger side, the side bands merge into the shoulders of the central peak. And the height of the central peak increases with its width gradually decreasing. One again observes a line narrowing and hence a phase transition. But now we note that the system does not jump suddenly from a single-peak spectrum to the three-peaked spectrum or vice versa. There is a shift from one state of system to another at $y = y_1$. For increasing values of y the single-peaked spectrum terminates at y_1, and the three-peaked spectrum is initiated, whereas the reverse happend for decreasing y from very large values ($y > y_1$) to smaller values.

C. $|\delta| > \delta_c$

Figures 8 and 9 demonstrate the spectrum of fluctuation for $\delta^2 = 12$. Here again, as in the case $|\delta| = \delta_c$, we observe that there exists a critical value of the incident intensity y_c below which the spectrum is a single peaked, exhibiting cooperative behavior in the system, and above which the spectrum is a three-peaked dynamical Stark-split spectrum exhibiting one-atom behavior. y_c^2 is the

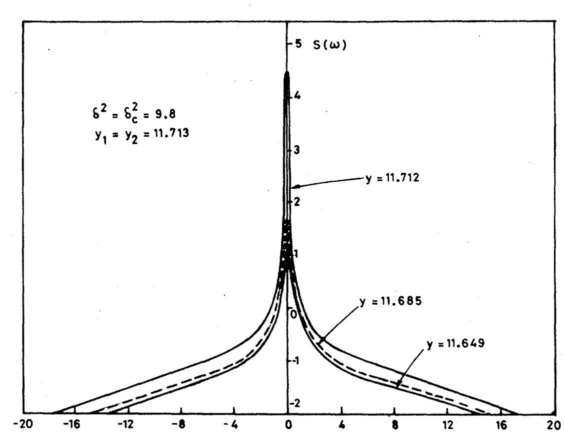

FIG. 6. Spectrum of transmitted light along the cooperative branch for the critical value of detuning $\delta_c^2 = 9.8$. The values of y are increasingly closer to the threshold $y_1 = y_2$. The spectra are characterized by a central peak with decreasing half width.

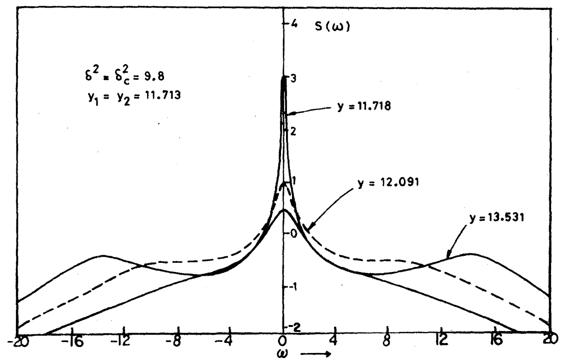

FIG. 7. Spectrum of transmitted light along the single-atom branch for $\delta_c^2 = 9.8$.

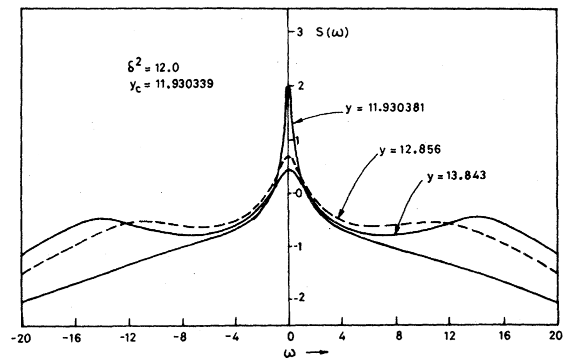

FIG. 9. Spectrum of transmitted light along the single-atom region for $\delta^2 = 12 > \delta_c^2 = 9.8$. The values of y are gradually decreasing towards the critical incident field y_c (Eq. 5.1). The side bands merge into the central peak which grows higher and narrows to a finite minimum as the threshold is approached.

intensity at which the phenomenon of line narrowing occurs, and one of the normal modes of the system becomes *soft with finite lifetime*. We show in Fig. 10 the behavior of the eigenvalues of the relaxation matrix for $\delta^2 = 12$, $C = 10$. It is seen that one eigenvalue (which is real) has a minimum as a function of y. This is the mode in the problem which becomes soft with a finite lifetime. As is well known and can be seen from Eq. (4.2) and the fact that the zeroes of the polynomial $\mathcal{P}(p)$ just give the eigenvalues of the relaxation matrix, the width of the central peak is given by $-\lambda_{\text{soft}}$ and the width of the side peaks are given by $-\text{Re}\lambda$ for values of y on the single-atom branch. We find from our numerical calculations that this is indeed the case. Note that the spectra are plotted on logarithmic scale.

V. INCOHERENT TRANSMITTED INTENSITY AND CONCLUSIONS

We know from the pure-absorptive bistable model[6] that the incoherent part of the transmitted intensity is not an extensive function of N, the number of atoms. It therefore shows wide variation along the cooperative branch. On the other hand, this correlation function is an extensive function of N on the one-atom branch and shows mild variation along it. For our model the incoherent part $N\langle y_0^{(+)} y_0^{(-)} \rangle$ is given by Eq. (3.11). In Fig. 11, we present its behavior on a logarithmic scale for a wide variety of parameters. In the bistable region of a dispersive medium, it is observed that as y is increased along cooperative branch, the incoherent part of the transmitted intensity varies sharply approaching an infinity at $y = y_1$. For $y > y_1$, the intensity suddenly jumps to finite value. As y is varied now, the value of the function remains almost constant. However, if y is decreased towards y_2, the value first increases very mildly

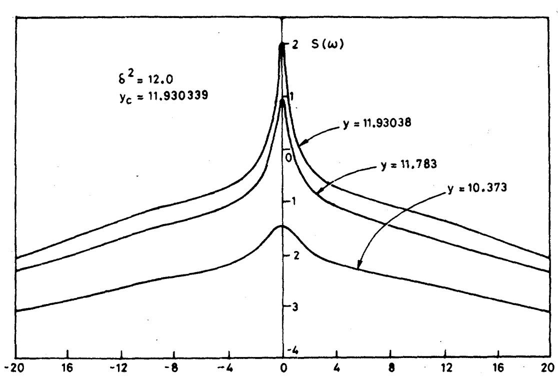

FIG. 8. Spectrum of transmitted light along the cooperative region for $\delta^2 = 12 > \delta_c^2 = 9.8$. The values of y are increasingly closer to the critical incident field y_c [Eq. (5.1)]. The spectra are characterized by a central peak with half width decreasing to a finite minimum.

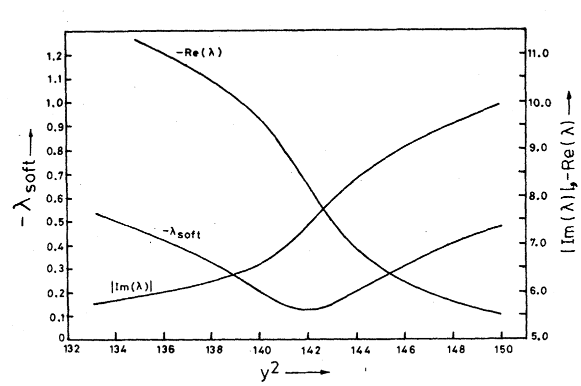

FIG. 10. Eigenvalues of the relaxation matrix M for $\delta^2 = 12 > \delta_c^2 = 9.8$. As the incident intensity approaches a critical value, one of the modes becomes longer lived. The dynamical Stark shift appears gradually beyond a different critical incident field determined by the condition $|\text{Im}\lambda| = \text{Re}\lambda$.

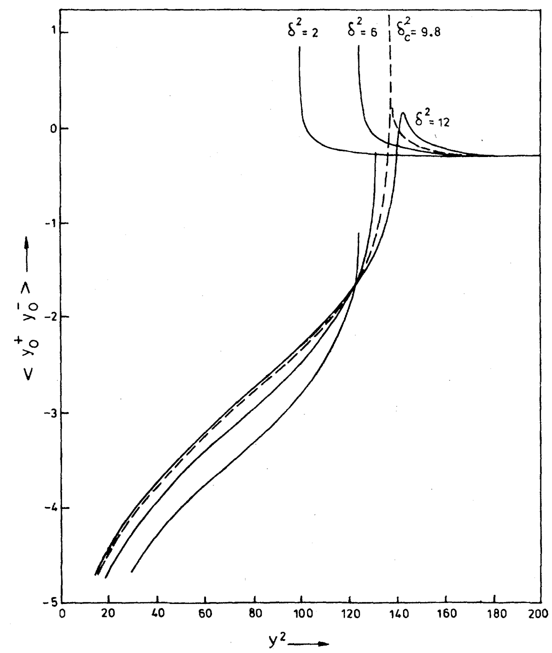

FIG. 11. The total transmitted incoherent intensity as a function of incident intensity for various values of detunings of the nonlinear medium with $C=10$.

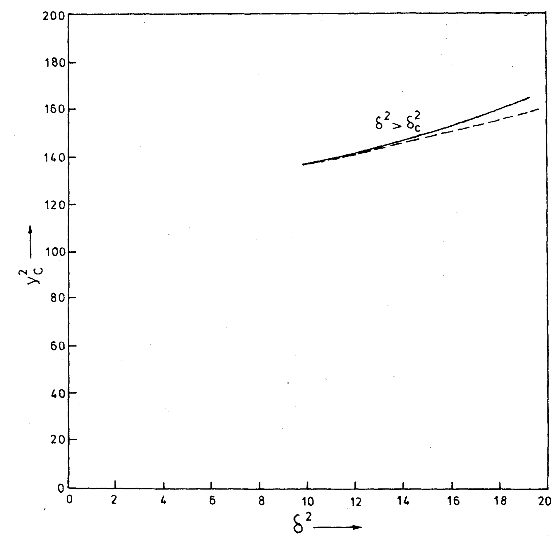

FIG. 12. The critical incident intensity for $\delta^2>\delta_c^2$ at which the system shifts from collective behavior to single-atom behavior. The dotted curve gives the values calculated on the basis of Eq. (5.1).

but in the neighborhood of y_2 changes sharply to become infinity at y_2. For $|\delta|\rightarrow\delta_c$ the size of the hysteresis cycle decreases, in that the upper bistability threshold infinity and the lower bistability threshold infinity approach each other. For $|\delta| = \delta_c$, the hysteresis cycle vanishes. The two infinities merge into a single infinity. We have the cooperative and one-atom behavior distinctly separated by this infinity. For $|\delta|>\delta_c$, the hysteresis cycles do not appear. The infinities also do not appear. Instead, a single maximum appears, which separates the cooperative behavior of the system from the one-atom behavior. The intensity of the incident coherent pump at which this maximum occurs is in the neighborhood of the critical intensity y_c, where one of the normal modes of the system becomes soft. A value of y_c therefore can be obtained by locating the maximum of this curve. The expression for $N\langle y_0^{(+)} y_0^{(-)}\rangle$ is too complicated and prohibits a direct calculation of the maxima. The y_c, is therefore calculated numerically. A plot of y_c with δ^2 for $\delta^2>9.8$ is shown in Fig. 12. A rough estimate of this y_c can, in view of Eq. (4.4), be had by the condition $d^2y/d|Z|^2=0$. An estimate is given by the expression

$$y_c^2=[(2C+1)/9(C-1)]\,[(2C+1)^2+9\delta^2]\;. \qquad (5.1)$$

In Fig. 12 the dashed line represents the value of y_c^2 calculated by Eq. (5.1). The value given by Eq. (5.1) differs from the exact numerical result by only few percent.

The above results indicate that the bistable behavior of a large sample of two-level atoms can be completely quenched by introducing large detunings of the medium. The transmitted intensity no longer remains a multivalued function of incident intensity. It is further shown that discontinuous character in physical variables of the system for $|\delta|>\delta_c$ disappears. The system, however, still shows two distinct dynamical behaviors, viz., (a) the collective behavior below a critical value of the intensity of incident radiation, and (b) the single-atom behavior above this critical intensity of coherent pump intensity. The transition from collective to single-atom behavior occurs for $|\delta|>\delta_c$, through a state at which one of the normal modes of the system becomes soft, accompanied by line narrowing. This also implies critical slowing down of dynamics. This behavior of our system is similar to that of a system undergoing a phase transition. Our dispersive model for $|\delta|\geq\delta_c$ shows second-order phase-transition behavior. Finally, we mention that linear stability analysis of the problem gives that, for the medium has an unstable branch in the negative-slope region of the bistability curve (Fig. 1). For $|\delta|\geq\delta_c$, we find no region with negative slope, and the system therefore never becomes unstable for $|\delta|\geq\delta_c$. The root of the relaxation matrix corresponding to soft mode never becomes positive for $|\delta| > \delta_c$ but acquires a maximum negative value at the critical value (y_c) of the intensity of the incident coherent pump.

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