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The algebra and geometry of $SU(3)$ matrices

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Abstract. We give an elementary treatment of the defining representation and Lie algebra of the three-dimensional unitary unimodular group $SU(3)$. The geometrical properties of the Lie algebra, which is an eight dimensional real linear vector space, are developed in an $SU(3)$ covariant manner. The f and d symbols of $SU(3)$ lead to two ways of 'multiplying' two vectors to produce a third, and several useful geometric and algebraic identities are derived. The axis-angle parametrization of $SU(3)$ is developed as a generalization of that for $SU(2)$, and the specifically new features are brought out. Application to the dynamics of three-level systems is outlined.

Keywords. $SU(3)$ matrices; octet algebra; octet geometry; $SU(3)$ axis-angle parameters.

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1. Introduction

The unitary unimodular group $SU(2)$ in two complex dimensions is the simplest nontrivial example of a nonabelian compact Lie group. Its many uses in physics–spin of the electron, proton, neutron,..., isotopic spin of nucleons, description of two-level atoms and two-level quantum systems in general are very well known. At the same time its adjoint representation coincides with the three-dimensional real proper rotation group $SO(3)$, with its associated concepts of three dimensional vectors in $\mathcal{R}^3$ and their algebra. The Pauli matrices $\sigma_j, j = 1,2,3$ mediate in a natural way between the defining two-dimensional and the adjoint three-dimensional representations. Being the generators of the defining representation, the expression of a finite $SU(2)$ element as the exponential of a generator in closed form is also well known. Thus one has the familiar collection of results:

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\,\sigma_l,$$
$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}; \tag{1.1a}$$
$$\underline{a}\cdot\underline{\sigma}\,\underline{b}\cdot\underline{\sigma} = \underline{a}\cdot\underline{b} + i\,\underline{a}_{\wedge}\underline{b}\cdot\underline{\sigma}; \tag{1.1b}$$
$$a(\hat{\alpha},\theta) = \exp(i\theta\hat{\alpha}\cdot\underline{\sigma}) = \cos\theta + i\hat{\alpha}\cdot\underline{\sigma}\,\sin\theta \in SU(2),$$
$$|\hat{\alpha}| = 1, \quad 0 \le \theta \le 2\pi. \tag{1.1c}$$

Here $\underline{a}, \underline{b}$ are (real) three-dimensional vectors, ϵ_{jkl} is the Levi–Civita symbol (structure constants of $SU(2)$), and $(\hat{\alpha}, \theta)$ are axis-angle coordinates for the general element $a(\hat{\alpha}, \theta) \in SU(2)$. The two-to-one homomorphism $SU(2) \to SO(3)$ determines an element $R(a) \in SO(3)$ for each $a \in SU(2)$:

$$R(a)_{jk} = \tfrac{1}{2}\mathrm{Tr}(\sigma_j a \sigma_k a^\dagger),$$
$$R(a')R(a) = R(a'a), \quad a', a \in SU(2). \tag{1.2}$$

The next group after $SU(2)$ in the classical unitary family is the eight-dimensional group $SU(3)$ of unitary unimodular matrices in three complex dimensions. Many representation theoretic complications expected for compact Lie groups in general, but not yet seen with $SU(2)$, do show up with $SU(3)$, making it quite nontrivial in comparison. Its use in elementary particle physics [1, 2] often exploits the canonical subgroup chain $SU(3) \supset U(2) \supset SU(2) \supset U(1)$, while its use in the nuclear physics context involves the chain $SU(3) \supset SO(3) \supset SO(2)$. The description of three-level systems [3–5] in general quantum mechanics (atoms, for instance) also involves $SU(3)$.

The purpose of this paper is to present a generalization of relations of the forms (1.1, 1.2) from $SU(2)$ to $SU(3)$, bringing out the algebraic and geometric features of the eight-dimensional octet or adjoint representation of $SU(3)$. In doing so we describe the minimum and unavoidable new features in both algebraic and geometric aspects that one must accept in the $SU(2) \to SU(3)$ transition. Among real eight component vectors in $\mathcal{R}^8$, apart from the Euclidean inner product, two kinds of bilinear products of vectors leading again to vectors – one antisymmetric and the other symmetric – play important roles, and are essential in developing a formula generalizing equation (1.1c). One of our results will indeed be an axis-angle description of $SU(3)$ elements, namely a closed-form expansion of the exponential of a general matrix in the Lie algebra $\underline{SU(3)}$ of $SU(3)$ yielding a general finite $SU(3)$ matrix. This will be seen to be considerably more complicated than the $SU(2)$ result (1.1c). In general our aim is to develop useful identities which help in getting closed form expressions, and to build up geometric pictures in some situations.

The contents of this paper are arranged as follows. Section 2 recalls the definition of the group $SU(3)$ and the generators – the λ-matrices – in the defining representation. From their commutation and anticommutation relations the structure constants f_{rst} and symmetric invariant tensor d_{rst} can be read off. Their independent nonzero components are listed. Section 3 discusses the eight-dimensional adjoint or octet representation of $SU(3)$. Based on the available invariant tensors f_{rst}, d_{rst}, two kinds of vector products among octet vectors – elements of $\mathcal{R}^8$ – are defined: an antisymmetric wedge product and a symmetric star product. Both are $SU(3)$ covariant. Apart from the geometric expression of the trilinear Jacobi identity using wedge products, several other identities involving these products and the Euclidean scalar product on $\mathcal{R}^8$ are developed. In § 4 we take up a detailed analysis of the algebraic properties of a single generator matrix in the defining representation of $SU(3)$. The geometric tools of §3 are used to get convenient forms for products, inverses, powers, determinants and the minimal equation for a general three dimensional generator matrix. A convenient way of characterizing the eigenvalue spectrum of a (suitably normalized) generator matrix, and the notion of its 'rest frame' or specific diagonal form, are developed. At all stages the $SU(3)$ covariance of the procedures is kept in view. Section 5 introduces the concept of axis and angle parameters for $SU(3)$. This is a way of describing general one-parameter subgroups in the group. The essential differences compared to $SU(2)$ are emphasized. We also obtain a closed form expression for a general element of $SU(3)$ expressed as the exponential of a generator matrix; for this the method of going to the 'rest frame' is exploited. Section 6 briefly describes the features of Hamiltonian dynamics for three level quantum systems, based on a generalization of the Bloch spin equation familiar from two level systems. Section 7 contains some concluding remarks.

2. The defining representation of $SU(3)$ and the λ-matrices

The defining representation of the group $SU(3)$ is given by [6, 7]

$$SU(3) = \{A = 3 \times 3 \text{ complex matrix } | A^\dagger A = 1, \ \det A = 1\}. \tag{2.1}$$

This is an eight-parameter compact Lie group. The $U(2)$ and $SO(3)$ subgroups are identified (up to conjugation) by [8]

$$U(2) = \left\{ A(u) = \begin{pmatrix} u & 0 \\ 0 & (\det u)^{-1} \end{pmatrix} \middle| u \in U(2) \right\} \subset SU(3); \tag{2.2a}$$

$$SO(3) = \{A = 3 \times 3 \text{ real matrix } | A^T A = 1, \ \det A = 1\} \subset SU(3). \tag{2.2b}$$

The generalization of the Pauli matrices σ_j, in a form adapted to the $U(2)$ subgroup, leads to the eight hermitian traceless generators λ_r, $r = 1, 2, \ldots, 8$ defined as follows [6, 7]:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.3}$$

These are trace-orthonormal in the sense

$$\mathrm{Tr}(\lambda_r \lambda_s) = 2\delta_{rs}, r, s = 1, 2, \ldots, 8. \tag{2.4}$$

The commutators and anticommutators among the λ's, lead to the completely antisymmetric structure constants f_{rst} of $SU(3)$ and to the completely symmetric d-symbols (for which there are no $SU(2)$ analogues):

$$[\lambda_r, \lambda_s] = 2i f_{rst} \lambda_t,$$
$$\{\lambda_r, \lambda_s\} = \tfrac{4}{3}\delta_{rs} + 2d_{rst} \lambda_t; \tag{2.5a}$$

$$f_{123}=1;\quad f_{456}=f_{678}=\frac{\sqrt{3}}{2};$$
$$f_{147}=f_{246}=f_{257}=f_{345}=f_{516}=f_{637}=1/2; \tag{2.5b}$$
$$d_{118}=d_{228}=d_{338}=-d_{888}=1/\sqrt{3};$$
$$d_{146}=d_{157}=-d_{247}=d_{256}=d_{344}=d_{355}=-d_{366}=-d_{377}=1/2;$$
$$d_{448}=d_{558}=d_{668}=d_{778}=-1/2\sqrt{3}. \tag{2.5c}$$

(Here only the independent nonvanishing components of f_{rst} and d_{rst} are given). The product of two λ's involves three kinds of terms:

$$\lambda_r\lambda_s=\tfrac{2}{3}\delta_{rs}+(d_{rst}+i\,f_{rst})\lambda_t. \tag{2.6}$$

The $U(2)$ subgroup generators are λ_1, λ_2, λ_3 (for $SU(2)$) and λ_8, while those of $SO(3)$ are λ_2, λ_5 and λ_7.

3. The adjoint representation of $SU(3)$ and the geometry of octet vectors

The adjoint representation of $SU(3)$[7] arises upon conjugation of the λ's by general $A\in SU(3)$ and expressing the result in terms of the λ's. It is a faithful representation, not of $SU(3)$, but of the quotient $SU(3)/\mathcal{Z}_3$, where $\mathcal{Z}_3$ is the centre of $SU(3)$:

$$\mathcal{Z}_3=\{A=e^{i\omega}\cdot 1|\omega=0,2\pi/3,4\pi/3\}\subset SU(3). \tag{3.1}$$

Thus we have a three-to-one homomorphism $SU(3)\rightarrow SU(3)/\mathcal{Z}_3$. Each $A\in SU(3)$ is mapped onto an eight dimensional real orthogonal matrix $D(A)=(D(A)_{rs})\in SO(8)$, whose matrix elements are easy to calculate:

$$\begin{aligned}A\in SU(3)\rightarrow A\lambda_rA^{-1}&=D(A)_{sr}\lambda_s,\\ D(A)_{sr}&=\tfrac{1}{2}\mathrm{Tr}(\lambda_sA\lambda_rA^{\dagger}),\\ D(A')D(A)&=D(A'A).\end{aligned} \tag{3.2}$$

Thus these matrices $D(A)$ form a very small part of the full twenty-eight dimensional group $SO(8)$. In comparison, the adjoint representation of $SU(2)$ is the same as $SO(3)$.

Let us denote general real eight component vectors in $\mathcal{R}^8$ – octet vectors – by $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma},\ldots$. Among them we have the usual Euclidean inner product

$$\underline{\alpha}\cdot\underline{\beta}=\alpha_r\beta_r. \tag{3.3}$$

We now define two 'vector products', one an antisymmetric wedge and the other a symmetric star, both of which lead to octet vectors again:

$$\begin{aligned}\underline{\alpha},\underline{\beta}\in\mathcal{R}^8:(\underline{\alpha}_\wedge\underline{\beta})_r&=f_{rst}\,\alpha_s\beta_t,\\ \underline{\alpha}_\wedge\underline{\beta}&=-\underline{\beta}_\wedge\underline{\alpha};\\ (\underline{\alpha}*\underline{\beta})_r&=\sqrt{3}\,d_{rst}\,\alpha_s\beta_t,\\ \underline{\alpha}*\underline{\beta}&=\underline{\beta}*\underline{\alpha}.\end{aligned} \tag{3.4}$$

The basic $SU(3)$ covariant properties of these products follow from the fact that f_{rst} and d_{rst} are invariant tensors:

$$A \in SU(3): D(A)\underline{\alpha} \wedge D(A)\underline{\beta} = D(A)(\underline{\alpha} \wedge \underline{\beta}),$$
$$D(A)\underline{\alpha} * D(A)\underline{\beta} = D(A)(\underline{\alpha} * \underline{\beta}). \tag{3.5}$$

To express the components of $\underline{\alpha} \wedge \underline{\beta}$ and $\underline{\alpha} * \underline{\beta}$ in convenient forms in terms of those of $\underline{\alpha}$ and $\underline{\beta}$, it is useful to assemble the components α_4, α_5, α_6, α_7 of $\underline{\alpha}$ into a two-component complex column vector $\psi(\underline{\alpha})$:

$$\underline{\alpha} \in \mathcal{R}^8: \psi(\underline{\alpha}) = \begin{pmatrix} \alpha_4 - i\alpha_5 \\ \alpha_6 - i\alpha_7 \end{pmatrix}. \tag{3.6}$$

(This is related to the fact that the λ's are adapted to the $U(2)$ subgroup (2.29) of $SU(3)$). Then we have these expressions for the components of $\underline{\alpha}_\wedge \underline{\beta}$:

$$(\underline{\alpha}_\wedge \underline{\beta})_j = \epsilon_{jkl}\, \alpha_k \beta_l + \tfrac{1}{2} \mathrm{Im}\, \psi(\underline{\beta})^\dagger \sigma_j\, \psi(\underline{\alpha});$$
$$(\underline{\alpha}_\wedge \underline{\beta})_8 = \frac{\sqrt{3}}{2}\, \mathrm{Im}\, \psi(\underline{\beta})^\dagger \psi(\underline{\alpha});$$
$$\psi(\underline{\alpha}_\wedge \underline{\beta}) = \frac{i}{2} \begin{pmatrix} \beta_3 + \sqrt{3}\beta_8 & \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 & -\beta_3 + \sqrt{3}\beta_8 \end{pmatrix} \psi(\underline{\alpha}) - (\underline{\alpha} \leftrightarrow \underline{\beta}). \tag{3.7}$$

Similarly for the components of $\underline{\alpha} * \underline{\beta}$ we have:

$$(\underline{\alpha} * \underline{\beta})_j = \alpha_8 \beta_j + \beta_8 \alpha_j + \frac{\sqrt{3}}{2} \mathrm{Re} \psi(\underline{\beta})^\dagger \sigma_j\, \psi(\underline{\alpha});$$
$$(\underline{\alpha} * \underline{\beta})_8 = \alpha_j \beta_j - \alpha_8 \beta_8 - \frac{1}{2} \mathrm{Re}\, \psi(\underline{\beta})^\dagger \psi(\underline{\alpha});$$
$$\psi(\underline{\alpha} * \underline{\beta}) = \frac{1}{2} \begin{pmatrix} \sqrt{3}\beta_3 - \beta_8 & \sqrt{3}(\beta_1 - i\beta_2) \\ \sqrt{3}(\beta_1 + i\beta_2) & -\sqrt{3}\beta_3 - \beta_8 \end{pmatrix} \psi(\underline{\alpha}) + (\underline{\alpha} \leftrightarrow \underline{\beta}). \tag{3.8}$$

Now we consider some cubic relations, identities involving triple vector products, with wedges and stars in various combinations. The first of these is just a statement of the Jacobi identity for the structure constants f_{rst} and involves two wedge products:

$$\underline{\alpha}_\wedge(\underline{\beta}_\wedge \underline{\gamma}) + \underline{\beta}_\wedge(\underline{\gamma}_\wedge \underline{\alpha}) + \underline{\gamma}_\wedge(\underline{\alpha}_\wedge \underline{\beta}) = 0. \tag{3.9}$$

Other relations arise by calculating the triple product $\underline{\alpha} \cdot \underline{\lambda}\, \underline{\beta} \cdot \underline{\lambda}\, \underline{\gamma} \cdot \underline{\lambda}$ in two ways and comparing the results. We have the equality

$$\begin{aligned}
\underline{\alpha} \cdot \underline{\lambda}\underline{\beta} \cdot \underline{\lambda}\underline{\gamma} \cdot \underline{\lambda} = {} & \frac{2}{3\sqrt{3}} \underline{\alpha} * \underline{\beta} \cdot \underline{\gamma} + \frac{2i}{3} \underline{\alpha}_\wedge \underline{\beta} \cdot \underline{\gamma} + \Bigg[\frac{2}{3} \underline{\alpha} \cdot \underline{\beta}\, \underline{\gamma} + \frac{1}{3} (\underline{\alpha} * \underline{\beta}) * \underline{\gamma} \\
& - (\underline{\alpha}_\wedge \underline{\beta})_\wedge \underline{\gamma} + \frac{i}{\sqrt{3}} ((\underline{\alpha}_\wedge \underline{\beta}) * \underline{\gamma} + (\underline{\alpha} * \underline{\beta})_\wedge \underline{\gamma}) \Bigg] \cdot \underline{\lambda} \\
= {} & \frac{2}{3\sqrt{3}} \underline{\alpha} \cdot \underline{\beta} * \underline{\gamma} + \frac{2i}{3} \underline{\alpha} \cdot \underline{\beta}_\wedge \underline{\gamma} + \Bigg[\frac{2}{3} \underline{\beta} \cdot \underline{\gamma}\underline{\alpha} + \frac{1}{3} \underline{\alpha} * (\underline{\beta} * \underline{\gamma}) \\
& - \underline{\alpha}_\wedge(\underline{\beta}_\wedge \underline{\gamma}) + \frac{i}{\sqrt{3}} (\underline{\alpha}_\wedge(\underline{\beta} * \underline{\gamma}) + \underline{\alpha} * (\underline{\beta}_\wedge \underline{\gamma})) \Bigg] \cdot \underline{\lambda}.
\end{aligned} \tag{3.10}$$

The $\underline{\lambda}$-independent terms lead to the obvious results

$$\underline{\alpha} \cdot \underline{\beta}_\wedge \underline{\gamma} = \underline{\alpha}_\wedge \underline{\beta} \cdot \underline{\gamma}, \tag{3.11a}$$

$$\underline{\alpha} \cdot \underline{\beta} * \underline{\gamma} = \underline{\alpha} * \underline{\beta} \cdot \underline{\gamma}, \tag{3.11b}$$

similar to properties of the triple scalar product of vectors in $\mathcal{R}^3$. The $\underline{\lambda}$-dependent terms, upon separation of real and imaginary parts and some rearrangement, lead to further relations at the vector level:

$$\underline{\alpha} * (\underline{\beta} * \underline{\gamma}) - \underline{\gamma} * (\underline{\beta} * \underline{\alpha}) = 2(\underline{\alpha} \cdot \underline{\beta}\, \underline{\gamma} - \underline{\gamma} \cdot \underline{\beta} \underline{\alpha}) + 3 \underline{\beta}_\wedge (\underline{\alpha}_\wedge \underline{\gamma}), \tag{3.12a}$$

$$\underline{\alpha}_\wedge (\underline{\beta} * \underline{\gamma}) + \underline{\gamma}_\wedge (\underline{\beta} * \underline{\alpha}) = \underline{\alpha} * (\underline{\gamma}_\wedge \underline{\beta}) + \underline{\gamma} * (\underline{\alpha}_\wedge \underline{\beta}). \tag{3.12b}$$

These are respectively antisymmetric and symmetric in the pair $\underline{\alpha}$, $\underline{\gamma}$.

The relations (3.9,11,12) are the basic $SU(3)$ covariant cubic vector relations involving three independent octet vectors.

4. Algebraic relations for $SU(3)$ generator matrices

A general traceless hermitian three dimensional matrix is of the form $\underline{\alpha} \cdot \underline{\lambda}$, $\underline{\alpha} \in \mathcal{R}^8$. For a pair of such matrices we have from eq. (2.6) the product (and square) rules

$$\begin{aligned} \underline{\alpha} \cdot \underline{\lambda} \underline{\beta} \cdot \underline{\lambda} &= \frac{2}{3}\, \underline{\alpha} \cdot \underline{\beta} + \frac{1}{\sqrt{3}} \underline{\alpha} * \underline{\beta} \cdot \underline{\lambda} + i \underline{\alpha}_\wedge \underline{\beta} \cdot \underline{\lambda}, \\ (\underline{\alpha} \cdot \underline{\lambda})^2 &= \frac{2}{3}\, \underline{\alpha}^2 + \frac{1}{\sqrt{3}} \underline{\alpha} * \underline{\alpha} \cdot \underline{\lambda}, \end{aligned} \tag{4.1}$$

which generalize eq. (1.1b).

Now we develop the properties of a single matrix $\underline{\alpha} \cdot \underline{\lambda}$ in some detail. The determinant is easily worked out in terms of the star product:

$$\det \underline{\alpha} \cdot \underline{\lambda} = \frac{2}{3\sqrt{3}}\, \underline{\alpha} \cdot \underline{\alpha} * \underline{\alpha}. \tag{4.2}$$

If α_0 is a ninth 'scalar', from the $SU(3)$ covariance property

$$\begin{aligned} A(\alpha_0 + \underline{\alpha} \cdot \underline{\lambda}) A^{-1} &= \alpha_0 + \underline{\alpha}' \cdot \underline{\lambda}, \\ \underline{\alpha}' &= D(A) \underline{\alpha}, \end{aligned} \tag{4.3}$$

and invariance of the determinant we see that we must necessarily have

$$\det(\alpha_0 + \underline{\alpha} \cdot \underline{\lambda}) = \alpha_0^3 + c\alpha_0 \underline{\alpha}^2 + \frac{2}{3\sqrt{3}} \underline{\alpha} \cdot \underline{\alpha} * \underline{\alpha}, \tag{4.4}$$

with no term quadratic in α_0, and with some constant c. Let us now diagonalize $\underline{\alpha} \cdot \underline{\lambda}$ using a suitable $SU(3)$ transformation. We shall refer to this as 'putting $\underline{\alpha} \cdot \underline{\lambda}$ into its rest frame', and will refine this notion in the sequel. Then

$$\begin{aligned} \alpha_0 + \underline{\alpha} \cdot \underline{\lambda} &= \mathrm{diag}(\alpha_0 + \alpha_3 + \alpha_8/\sqrt{3}, \alpha_0 - \alpha_3 + \alpha_8/\sqrt{3}, \alpha_0 - 2\alpha_8/\sqrt{3}), \\ \det(\alpha_0 + \underline{\alpha} \cdot \underline{\lambda}) &= \alpha_0^3 - \alpha_0(\alpha_3^2 + \alpha_8^2) + \frac{2}{\sqrt{3}} \alpha_8 (\alpha_3^2 - \alpha_8^2/3). \end{aligned} \tag{4.5}$$

This fixes $c = -1$, so we have the general relation

$$\det(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda}) = \alpha_0^3 - \alpha_0\underline{\alpha}^2 + \frac{2}{3\sqrt{3}}\underline{\alpha}\cdot\underline{\alpha}*\underline{\alpha} \tag{4.6}$$

valid in any 'frame'.

Turning to matrix inverses, and assuming α_0 is not an eigenvalue of $-\underline{\alpha}\cdot\underline{\lambda}$, again $SU(3)$ covariance dictates the general structure

$$(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda})^{-1} = (c_1\alpha_0^2 + c_2\underline{\alpha}^2 + c_3\alpha_0\underline{\alpha}\cdot\underline{\lambda} + c_4\underline{\alpha}*\underline{\alpha}\cdot\underline{\lambda})/\det(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda}), \tag{4.7}$$

for some constants $c_1, \ldots, c_4$. Using eq. (4.6) and transposing terms we have

$$(c_1\alpha_0^2 + c_2\,\underline{\alpha}^2 + c_3\alpha_0\,\underline{\alpha}\cdot\underline{\lambda} + c_4\,\underline{\alpha}*\underline{\alpha}\cdot\underline{\lambda})(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda}) = \det(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda}). \tag{4.8}$$

Comparing powers of α_0 gives $c_1 = 1$, $c_2 = -1/3$, $c_3 = -1$, $c_4 = 1/\sqrt{3}$ and also the relation

$$\underline{\alpha}*(\underline{\alpha}*\underline{\alpha}) = \underline{\alpha}^2\underline{\alpha} - i\sqrt{3}(\underline{\alpha}*\underline{\alpha})_\wedge\underline{\alpha} \tag{4.9}$$

which we shall understand in another way in a moment. So for matrix inverses we have the $SU(3)$ covariant result

$$(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda})^{-1} = \left(\alpha_0^2 - \frac{1}{3}\underline{\alpha}^2 - \alpha_0\underline{\alpha}\cdot\underline{\lambda} + \frac{1}{\sqrt{3}}\underline{\alpha}*\underline{\alpha}\cdot\underline{\lambda}\right)\Big/\det(\alpha_0 + \underline{\alpha}\cdot\underline{\lambda}). \tag{4.10}$$

From the determinant relation (4.6) we see that the minimal (cubic) equation for $\underline{\alpha}\cdot\underline{\lambda}$ is

$$(\underline{\alpha}\cdot\underline{\lambda})^3 = \underline{\alpha}^2\underline{\alpha}\cdot\underline{\lambda} + \frac{2}{3\sqrt{3}}\underline{\alpha}\cdot\underline{\alpha}*\underline{\alpha}. \tag{4.11}$$

If we substitute (4.1) here for $(\underline{\alpha}\cdot\underline{\lambda})^2$ and compare coefficients we get the two relations

$$\underline{\alpha}_\wedge(\underline{\alpha}*\underline{\alpha}) = 0, \tag{4.12a}$$

$$\underline{\alpha}*(\underline{\alpha}*\underline{\alpha}) = \underline{\alpha}^2\underline{\alpha}, \tag{4.12b}$$

which explain the earlier result (4.9). Incidentally the first result above is obtainable from eq. (3.12b) by setting $\underline{\alpha} = \underline{\beta} = \underline{\gamma}$. We also obtain the following useful property of octet vectors, which generalizes the result in three dimensions that $\underline{a}_\wedge\underline{b}$ vanishes only if $\underline{b}$ is parallel to $\underline{a}$:

$$\underline{\alpha}, \underline{\beta} \in \mathcal{R}^8, \underline{\alpha}_\wedge\underline{\beta} = 0 \Leftrightarrow \underline{\beta} = c_1\underline{\alpha} + c_2\underline{\alpha}*\underline{\alpha}, c_{1,2} \text{ constants}. \tag{4.13}$$

That there are no more terms here follows from the relationship of octet vectors to traceless hermitian matrices in *three* dimensions.

Let us now trace the consequences of the minimal equation (4.11) in more detail. For brevity, denote $\underline{\alpha}*\underline{\alpha}$ by $\underline{\alpha}'$ for the moment. Then eqs (4.1), (4.11) read

$$(\underline{\alpha}\cdot\underline{\lambda})^2 = \frac{2}{3}\underline{\alpha}^2 + \frac{1}{\sqrt{3}}\underline{\alpha}'\cdot\underline{\lambda},$$

$$(\underline{\alpha}' \cdot \underline{\lambda})^2 = \frac{2}{3}\underline{\alpha}'^2 + \frac{1}{\sqrt{3}}\underline{\alpha}' * \underline{\alpha}' \cdot \underline{\lambda},$$
$$(\underline{\alpha} \cdot \underline{\lambda})^3 = \underline{\alpha}^2\underline{\alpha} \cdot \underline{\lambda} + \frac{2}{3\sqrt{3}}\underline{\alpha} \cdot \underline{\alpha}'. \quad (4.14)$$

Now we substitute the first relation here into the second and keep simplifying till we have only linear terms in $\underline{\alpha} \cdot \underline{\lambda}$ and $\underline{\alpha}' \cdot \underline{\lambda}$:

$$\begin{aligned}\frac{2}{3}\underline{\alpha}'^2 + \frac{1}{\sqrt{3}}\underline{\alpha}' * \underline{\alpha}' \cdot \underline{\lambda} &= 3\left((\underline{\alpha} \cdot \underline{\lambda})^2 - \frac{2}{3}\underline{\alpha}^2\right)^2 \\ &= 3(\underline{\alpha} \cdot \underline{\lambda})^4 - 4\underline{\alpha}^2(\underline{\alpha} \cdot \underline{\lambda})^2 + \frac{4}{3}(\underline{\alpha}^2)^2 \\ &= \frac{2}{3}(\underline{\alpha}^2)^2 + \frac{2}{\sqrt{3}}\underline{\alpha} \cdot \underline{\alpha}'\underline{\alpha} \cdot \underline{\lambda} - \frac{\underline{\alpha}^2}{\sqrt{3}}\underline{\alpha}' \cdot \underline{\lambda}. \end{aligned} \quad (4.15)$$

This implies as consequences for any $\underline{\alpha}$, upon substituting $\underline{\alpha}' = \underline{\alpha} * \underline{\alpha}$:

$$(\underline{\alpha} * \underline{\alpha})^2 = (\underline{\alpha}^2)^2, \quad (4.16a)$$
$$(\underline{\alpha} * \underline{\alpha}) * (\underline{\alpha} * \underline{\alpha}) = 2\underline{\alpha} \cdot \underline{\alpha} * \underline{\alpha}\,\underline{\alpha} - \underline{\alpha}^2\underline{\alpha} * \underline{\alpha}. \quad (4.16b)$$

This last relation is to be contrasted with the result of taking the star product of (4.12b) with $\underline{\alpha}$, which is

$$\underline{\alpha} * (\underline{\alpha} * (\underline{\alpha} * \underline{\alpha})) = \underline{\alpha}^2\underline{\alpha} * \underline{\alpha}. \quad (4.17)$$

We are now in a position to deal in more detail with the eigenvalue spectrum of a single generator $\underline{\alpha} \cdot \underline{\lambda}$, and in the process refine the idea of the 'rest frame' form of $\underline{\alpha} \cdot \underline{\lambda}$. Let us hereafter assume $\underline{\alpha}$ is a unit vector, $\hat{\alpha}^2 = 1$. Then the eigenvalues of $\hat{\alpha} \cdot \underline{\lambda}$ are μ_1, μ_2, μ_3 obeying

$$\mu_1 + \mu_2 + \mu_3 = 0,$$
$$\mu_1^2 + \mu_2^2 + \mu_3^2 = \mathrm{Tr}(\hat{\alpha} \cdot \underline{\lambda})^2 = 2. \quad (4.18)$$

We can easily see that they can be ordered according to $\mu_1 \geq \mu_2 \geq \mu_3$ and the ranges can be fixed as follows:

$$\mu_{1,3} = -\frac{1}{2}\mu_2 \pm \sqrt{1 - \frac{3}{4}\mu_2^2},$$
$$-\frac{2}{\sqrt{3}} \leq \mu_3 \leq -\frac{1}{\sqrt{3}} \leq \mu_2 \leq \frac{1}{\sqrt{3}} \leq \mu_1 \leq \frac{2}{\sqrt{3}}. \quad (4.19)$$

A convenient parametrization of all three eigenvalues is by an angle φ in the range $[\pi/6, \pi/2]$ as follows:

$$\mu_1 = \frac{2}{\sqrt{3}}\sin\varphi, \quad \mu_2 = \frac{2}{\sqrt{3}}\sin(\varphi + 2\pi/3), \quad \mu_3 = \frac{2}{\sqrt{3}}\sin(\varphi + 4\pi/3). \quad (4.20)$$

The three angles occurring here are in strictly non-overlapping regions. We now define the 'rest frame' of $\hat{\alpha} \cdot \underline{\lambda}$ to be that unique diagonal form in which we have

$$\hat{\alpha}\cdot\underline{\lambda} = \text{diag}\,\frac{2}{\sqrt{3}}(\sin\varphi, \sin(\varphi+2\pi/3), \sin(\varphi+4\pi/3)),$$
$$\hat{\alpha}_3 = \cos(\varphi+4\pi/3), \hat{\alpha}_8 = -\sin(\varphi+4\pi/3),$$
$$\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_4 = \hat{\alpha}_5 = \hat{\alpha}_6 = \hat{\alpha}_7 = 0, \quad \pi/6 \le \varphi \le \pi/2. \tag{4.21}$$

In this 'frame' for $\hat{\alpha}$, we find that $\hat{\alpha}*\hat{\alpha}$ has similar nonzero components and is by eq. (4.16a) also a unit vector, but is not in its own 'rest frame':

$$\hat{\alpha}*\hat{\alpha}\cdot\underline{\lambda} = \text{diag}\,\frac{2}{\sqrt{3}}(\sin(2\varphi-\pi/2), \sin(2\varphi+5\pi/6), \sin(2\varphi+\pi/6)),$$
$$(\hat{\alpha}*\hat{\alpha})_3 = 2\hat{\alpha}_3\hat{\alpha}_8 = -\sin(2\varphi+2\pi/3),$$
$$(\hat{\alpha}*\hat{\alpha})_8 = \hat{\alpha}_3^2 - \hat{\alpha}_8^2 = \cos(2\varphi+2\pi/3),$$
$$(\hat{\alpha}*\hat{\alpha})_r = 0 \quad \text{for } r = 1,2,4,5,6,7. \tag{4.22}$$

The angle φ can be inferred from the value of the invariant $\xi = \hat{\alpha}\cdot\hat{\alpha}*\hat{\alpha} = -\sin 3\varphi$: since $-1 \le \xi \le 1$ and $\pi/2 \le 3\varphi \le 3\pi/2, \xi$ fixes the value of φ uniquely.

5. Axis-angle parameters and finite elements of $SU(3)$

In this section we derive the $SU(3)$ analogue of eq. (1.1c) for $SU(2)$. It is easily seen, for instance by going to the diagonal form, that every $A \in SU(3)$ can be obtained by exponentiating a suitable traceless antihermitian matrix. We now compute in closed form the matrix

$$A(\hat{\alpha},\theta) = \exp(i\theta\hat{\alpha}\cdot\underline{\lambda}) \tag{5.1}$$

set up in analogy to eq. (1.1c) for $SU(2)$. The unit octet vector $\hat{\alpha}$ is the axis, and θ is the angle, for the element $A(\hat{\alpha},\theta)$. The range for θ is discussed below. The sole $SU(3)$ scalar we can form from $\hat{\alpha}$ is the angle φ given by

$$\xi = \hat{\alpha}\cdot\hat{\alpha}*\hat{\alpha} = -\sin 3\varphi,$$
$$\pi/6 \le \varphi \le \pi/2. \tag{5.2}$$

Upon expanding the exponential in eq. (5.1) and using (4.1, 4.12b), we see that the only terms that arise are multiples of the unit matrix, of $\hat{\alpha}\cdot\underline{\lambda}$ and $\hat{\alpha}*\hat{\alpha}\cdot\underline{\lambda}$. We therefore write

$$A(\hat{\alpha},\theta) = \frac{2}{\sqrt{3}}c(\theta,\varphi) + a(\theta,\varphi)\hat{\alpha}\cdot\underline{\lambda} + b(\theta,\varphi)\hat{\alpha}*\hat{\alpha}\cdot\underline{\lambda}, \tag{5.3}$$

and proceed to determine the three $SU(3)$ scalar coefficients. For this we go to the rest frame (4.21) of $\hat{\alpha}\cdot\underline{\lambda}$ – then $\hat{\alpha}*\hat{\alpha}\cdot\underline{\lambda}$ is also diagonal (cf. eq. (4.22)) and so is $A(\hat{\alpha},\theta)$. Equation (5.3) reduces to

$$\Delta(\varphi)\begin{pmatrix} a(\theta,\varphi) \\ b(\theta,\varphi) \\ c(\theta,\varphi) \end{pmatrix} = \frac{\sqrt{3}}{2}\begin{pmatrix} \exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin\varphi\right) \\ \exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+2\pi/3)\right) \\ \exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+4\pi/3)\right) \end{pmatrix},$$

$$\Delta(\varphi)=\begin{pmatrix}\sin\varphi & \sin(2\varphi-\pi/2) & 1\\ \sin(\varphi+2\pi/3) & \sin(2\varphi+5\pi/6) & 1\\ \sin(\varphi+4\pi/3) & \sin(2\varphi+\pi/6) & 1\end{pmatrix}. \tag{5.4}$$

The determinant of $\Delta(\varphi)$ is

$$\det\Delta(\varphi)=\frac{3\sqrt{3}}{2}\cos 3\varphi, \tag{5.5}$$

so $\Delta(\varphi)$ is nonsingular for $\pi/6<\varphi<\pi/2$. In this range we have

$$\Delta(\varphi)^{-1}=\frac{2}{3\cos 3\varphi}\begin{pmatrix}-\sin 2\varphi & \sin(2\varphi+\pi/3) & \sin(2\varphi-\pi/3)\\ -\cos\varphi & \sin(\varphi+\pi/6) & -\sin(\varphi-\pi/6)\\ \frac{1}{2}\cos 3\varphi & \frac{1}{2}\cos 3\varphi & \frac{1}{2}\cos 3\varphi\end{pmatrix}, \tag{5.6}$$

and the solution for the coefficients $a(\theta,\varphi)$, $b(\theta,\varphi)$, $c(\theta,\varphi)$ is

$$a(\theta,\varphi)=\left\{-\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin\varphi\right)\cdot\sin 2\varphi+\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+2\pi/3)\right)\right.$$
$$\left.\times\sin(2\varphi+\pi/3)+\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+4\pi/3)\right)\sin(2\varphi-\pi/3)\right\}\Big/$$
$$\sqrt{3}\cos 3\varphi,$$

$$b(\theta,\varphi)=\left\{-\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin\varphi\right)\cdot\cos\varphi+\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+2\pi/3)\right)\right.$$
$$\left.\times\sin(\varphi+\pi/6)-\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+4\pi/3)\right)\sin(\varphi-\pi/6)\right\}\Big/$$
$$\sqrt{3}\cos 3\varphi,$$

$$c(\theta,\varphi)=\left\{\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin\varphi\right)+\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+2\pi/3)\right)\right.$$
$$\left.+\exp\left(\frac{2\theta}{\sqrt{3}}\cdot i\sin(\varphi+4\pi/3)\right)\right\}\Big/2\sqrt{3}. \tag{5.7}$$

With this the computation of $A(\hat{\alpha},\theta)$ in closed form in the generic case is complete.

The two limiting cases $\varphi=\pi/6$ and $\varphi=\pi/2$ correspond respectively to $\hat{\alpha}*\hat{\alpha}=-\hat{\alpha}$ and $\hat{\alpha}*\hat{\alpha}=\hat{\alpha}$. In these cases we find after some algebra:

$$\varphi=\pi/6: A(\hat{\alpha},\theta)=\frac{2}{\sqrt{3}}c(\theta,\pi/6)+a(\theta,\pi/6)\hat{\alpha}\cdot\underline{\lambda},$$
$$a(\theta,\pi/6)=\{e^{i\theta/\sqrt{3}}-e^{-2i\theta/\sqrt{3}}\}/\sqrt{3},$$
$$c(\theta,\pi/6)=\{2e^{i\theta/\sqrt{3}}+e^{-2i\theta/\sqrt{3}}\}/2\sqrt{3};$$
$$\varphi=\pi/2: A(\hat{\alpha},\theta)=\frac{2}{\sqrt{3}}c(\theta,\pi/2)+a(\theta,\pi/2)\hat{\alpha}\cdot\underline{\lambda},$$

$$a(\theta,\pi/2) = -\{e^{-i\theta/\sqrt{3}} - e^{2i\theta/\sqrt{3}}\}/\sqrt{3},$$
$$c(\theta,\pi/2) = \{2e^{-i\theta/\sqrt{3}} + e^{2i\theta/\sqrt{3}}\}/2\sqrt{3}. \tag{5.8}$$

Now we consider the question of the range of the angle variable θ. Here the situation is more intricate than in the $SU(2)$ case. With Pauli matrices, $\hat{\alpha}\cdot\underline{\sigma}$ always has eigenvalues ± 1 independent of $\hat{\alpha}$, so in eq. (1.1c) we have a fixed range $0 \leq \theta \leq 2\pi$. Equivalently, all $U(1)$ subgroups of $SU(2)$ are conjugate to one another. With $SU(3)$, however, there are infinitely many inequivalent $U(1)$ subgroups available. This means that while every $A \in SU(3)$ lies on some one-parameter subgroup and can be written as $A = A(\hat{\alpha},\theta)$ for some $\hat{\alpha}$ and θ, the range of θ depends on $\hat{\alpha}$, more specifically on the $SU(3)$ invariant angle φ associated with $\hat{\alpha}$. The determining factor is the nature of the eigenvalues μ_1, μ_2, μ_3 of $\hat{\alpha}\cdot\underline{\lambda}$. The case $\mu_2 = 0$, $\mu_1 = -\mu_3 = 1$ leads to the range $0 \leq \theta \leq 2\pi$, which occurs for $\varphi = \pi/3$. For $\mu_2 \neq 0$, we need to distinguish between the cases of rational and irrational ratios μ_1/μ_2. For rational μ_1/μ_2, when μ_3/μ_2 is also rational, the elements $A(\hat{\alpha},\theta)$ lie on a cyclic $U(1)$ subgroup in $SU(3)$ and θ can be taken to be in some finite interval from zero to a maximum determined by μ_2. For irrational μ_1/μ_2, when μ_3/μ_2 is also irrational, the character of the one-parameter subgroup is very different – it is the real line $\mathcal{R}$, not a cyclic $U(1)$, so $\theta \in (-\infty,\infty)$. This then is an essentially new feature with axis angle parameters for $SU(3)$ as compared to $SU(2)$.

6. Hamiltonian dynamics of three-level systems

Consider a three-level quantum system whose state is represented by a density matrix ρ. The properties

$$\rho^\dagger = \rho \geq 0, \quad \mathrm{Tr}\,\rho = 1 \tag{6.1}$$

allow us to expand ρ in terms of the λ's, bringing in a scalar c and a unit octet vector $\hat{n}$:

$$\rho = \tfrac{1}{3}(1 + c\hat{n}\cdot\underline{\lambda}),$$
$$c \leq \frac{\sqrt{3}}{2}\mathrm{cosec}(\varphi + \pi/3), \tag{6.2}$$

where the angle $\varphi \in [\pi/6, \pi/2]$ is determined by eq. (5.2): $\hat{n}\cdot\hat{n}*\hat{n} = -\sin 3\varphi$. The pure state case corresponds to $\varphi = \pi/2$ and $c = \sqrt{3}$: then $\hat{n}*\hat{n} = \hat{n}$ and the eigenvalues of ρ are (1,0,0).

Let H be a general (time-independent) Hamiltonian which we express in terms of an octet vector $\underline{h}$ and a scalar h_0:

$$H = \tfrac{1}{2}(h_0 + \underline{h}\cdot\underline{\lambda}). \tag{6.3}$$

The equation of motion for ρ (with c and $\hat{n}$ regarded as functions of time),

$$i\frac{\mathrm{d}\rho}{\mathrm{d}t} = [H,\rho], \tag{6.4}$$

is independent of h_0 and at first leads to

$$\dot{c}\hat{n} + c\dot{\hat{n}} = c\underline{h}_\wedge\hat{n}. \tag{6.5}$$

However since $\dot{\hat{n}}$ and $\underline{h}_\wedge\hat{n}$ are both orthogonal to $\hat{n}$, we get $\dot{c} = 0$ as expected, and the equation of motion for $\hat{n}$ becomes:

$$\dot{\hat{n}} = \underline{h}_\wedge\hat{n}. \tag{6.6}$$

This is the three level version of the Bloch equation for the spin vector familiar from two-level systems. One can now use some of the identities derived in earlier sections to verify that $\hat{n} \cdot \hat{n} * \hat{n}$ is a constant of motion:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{n} \cdot \hat{n} * \hat{n} &= \dot{\hat{n}} \cdot \hat{n} * \hat{n} + 2\hat{n} \cdot \hat{n} * \dot{\hat{n}} \\ &= \underline{h}_\wedge\hat{n} \cdot (\hat{n} * \hat{n}) + 2\hat{n} \cdot \hat{n} * (\underline{h}_\wedge\hat{n}) \\ &= \underline{h} \cdot \hat{n}_\wedge(\hat{n} * \hat{n}) + 2\hat{n} \cdot (\hat{n}_\wedge(\hat{n} * \underline{h}) + \underline{h}_\wedge(\hat{n} * \hat{n})) \\ &= 2\underline{h} \cdot (\hat{n} * \hat{n})_\wedge\hat{n} \\ &= 0. \end{aligned} \tag{6.7}$$

Here we used eqs (3.11a, 4.12a) and eq. (3.12b) with $\underline{\alpha} = \underline{\beta} = \hat{n}, \underline{\gamma} = \underline{h}$ to simplify terms at various stages. Thus, as expected, both the scalar c in ρ and the $SU(3)$ scalar angle φ characteristic of $\hat{n}$ are constant in time. This is consistent with the fact that $\hat{n}(t)$ evolves essentially according to the octet representation matrix $D(A(t, \hat{h}))$ where in the notation of eq. (5.1) the time t is the angle and the vector $\hat{h}$ the axis of 'rotation'.

7. Concluding remarks

We have presented a set of practical tools for carrying out calculations with finite matrices of $SU(3)$ as well as with its Lie algebra, exploiting both algebraic and geometric aspects of the situation. The space of octet vectors bears the same relation to $SU(3)$ as does ordinary Euclidean three-dimensional space to $SU(2)$. The constructions we have given for working with these vectors should prove useful in dealing with three level system dynamics. The existence and interpretation of the cubic invariant $\underline{\alpha} \cdot \underline{\alpha} * \underline{\alpha}$ and the general solution (4.13) to $\underline{\alpha}_\wedge\underline{\beta} = 0$ are noteworthy. We have also brought out the fact that in contrast to $SU(2)$, there are infinitely many distinct kinds of one-parameter subgroups in $SU(3)$, and this shows up in the axis-angle description in this case.

The main new feature in the $SU(3)$ situation, absent with $SU(2)$, is the occurrence of the d-symbols. However, for all groups $SU(n), n \geq 4$, nothing new apart from such a d-symbol is expected since one cannot in any case go beyond the commutators and anticommutators of the generators in the defining representation. It therefore is to be expected that the methods of this paper can be systematically extended to these higher dimensional groups as well.

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