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The real symplectic groups in quantum mechanics and optics

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Abstract. We present a utilitarian review of the family of matrix groups $\mathrm{Sp}(2n,\mathscr{R})$, in a form suited to various applications both in optics and quantum mechanics. We contrast these groups and their geometry with the much more familiar Euclidean and unitary geometries. Both the properties of finite group elements and of the Lie algebra are studied, and special attention is paid to the so-called unitary metaplectic representation of $\mathrm{Sp}(2n,\mathscr{R})$. Global decomposition theorems, interesting subgroups and their generators are described. Turning to n-mode quantum systems, we define and study their variance matrices in general states, the implications of the Heisenberg uncertainty principles, and develop a $\mathrm{U}(n)$-invariant squeezing criterion. The particular properties of Wigner distributions and Gaussian pure state wavefunctions under $\mathrm{Sp}(2n,\mathscr{R})$ action are delineated.

Keywords. Symplectic groups; symplectic geometry; Huyghens kernel; uncertainty principle; multimode squeezing; Gaussian states.

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1. Introduction

The symplectic groups form one of the three major families of classical semisimple Lie groups, the other two being the real orthogonal family and the complex unitary family [1]. Apart from the groups describing nonrelativistic and relativistic space-time geometries, namely the Galilei, Lorentz and Poincare groups, most of the Lie groups encountered in physical problems, for example as symmetry groups, belong to either the real orthogonal or the unitary families [2]. These are multidimensional as well as complex generalisations of the rotation group of Euclidean geometry characterising physical three-dimensional space. As a result, the intuitive geometrical ideas that go with real orthogonal or complex unitary geometries are quite familiar to most physicists.

It has been realised recently, however, that in several problems both in quantum mechanics and in optics, the real symplectic groups play an important role [3]. In the latter context, this is so in both classical and quantum theories. More generally, these groups come in very naturally through the canonical formalism of classical dynamics, and its counterpart in quantum mechanics.

Symplectic geometry, on the other hand, differs profoundly from the real or complex Euclidean variety [4]. Here the intuitively familiar concepts of length, angle, perpendicularity and the Pythagoras Theorem are all absent. In their place we have typically new concepts characteristic of canonical mechanics.

The purpose of this informal and utilitarian survey is to introduce methods based on the real symplectic groups to those who are otherwise familiar with the structures of quantum mechanics and/or the theory of partial coherence in optics. The intention is to "inform rather than astonish", and to describe the main features of the symplectic point of view, Our account will not contain complete proofs of all statements presented, but a motivated reader should easily be able to supply additional detail and proceed to make practical use of these methods.

The material of this article is organised as follows. Section 2 introduces the group $Sp(2n, \mathscr{R})$ as the group of linear transformations preserving the classical Poisson Brackets as well as the quantum commutation relations among n pairs of canonical variables. In the quantum case, the Stone-von Neumann theorem allows us to infer that these transformations are unitarily implementable. Section 3 develops some ideas related to real symplectic linear vector spaces, specially the concepts of symplectic complement and symplectic rank of a subspace, in order to contrast symplectic geometry with real Euclidean and complex unitary geometries. Some useful properties of the matrices occurring in the defining representation of $Sp(2n, \mathscr{R})$, and their complex form, are then explained in § 4. Here we also list several useful subgroups of $Sp(2n, \mathscr{R})$, and describe four global decomposition theorems – the polar, Euler, pre-Iwasawa and Iwasawa decompositions. Section 5 studies the Lie algebra $\underline{Sp(2n, \mathscr{R})}$, first in the defining representation and then in a general, possibly unitary, representation. Convenient ways of breaking up the generators into subsets, and generators of various subgroups, are described. In § 6 we set up and study the special unitary metaplectic representation of $Sp(2n, \mathscr{R})$ and relate it to the generalised Huyghens kernel in any number of dimensions. The characteristic differences between the compact and the noncompact generators of $Sp(2n, \mathscr{R})$ are seen in their dependences on mode annihilation and creation operators. The former (latter) conserve (do not conserve) the total number operator. Section 7 studies the relationship between the metaplectic unitary representation of $Sp(2n, \mathscr{R})$, and two often used descriptions of quantum mechanical operators, namely the Wigner function representation and the diagonal coherent state representation. While the former is covariant, i.e., transforms simply, under the full group $Sp(2n, \mathscr{R})$, the latter is covariant only under the maximal compact subgroup $K(n) = U(n)$ of $Sp(2n, \mathscr{R})$. Section 8 takes up the questions of defining the noise or variance matrix for any state of an n-mode quantum system, both in real and complex forms, and their behaviours under $Sp(2n, \mathscr{R})$. In § 9 we carry this analysis further to show that the Heisenberg uncertainty principles for any number of modes can be given in explicitly $Sp(2n, \mathscr{R})$ covariant forms; a key role here is played by Williamson's Theorem relating to normal forms of quadratic Hamiltonians. This study leads in § 10 to the setting up of an $U(n)$-invariant squeezing criterion for n-mode systems. This is the maximal physically reasonable invariance one could ask for in these systems, and it can be stated very elegantly in terms of the general variance matrix set up in § 8. Section 11 describes and motivates some interesting classes of variance matrices with distinctive group theoretic properties, and § 12 is devoted to a study of general centred and normalized pure Gaussian wavefunctions for n-mode systems. The transitive action of $Sp(2n, \mathscr{R})$ on these wavefunctions, via the metaplectic representation, and the emergence of a matrix form of the Mobius transformation, are described. Section 13 contains some concluding remarks.

2. The real symplectic groups $\mathrm{Sp}(2n, \mathscr{R})$

We consider a classical or quantum canonical system with n degrees of freedom, that is, n pairs of mutually conjugate canonical variables. In the classical case these are numerical variables written as $q_r, p_r, r = 1, 2, \ldots, n$. In quantum mechanics we have an irreducible set of hermitian operators $\hat{q}_r, \hat{p}_r$ acting on a suitable Hilbert space $\mathscr{H}$. The basic kinematic structure is given by Poisson brackets (PB) in one case and by the Heisenberg commutation relations (CR) in the other. To express them both compactly and elegantly, we introduce the following notation. We assemble the q's and p's into $2n$-component vectors $\xi, \hat{\xi}$

$$\xi = (\xi_a) = (q_1 \cdots q_n p_1 \cdots p_n)^T,$$
$$\hat{\xi} = (\hat{\xi}_a) = (\hat{q}_1 \cdots \hat{q}_n \hat{p}_1 \cdots \hat{p}_n)^T, \quad a = 1, 2, \ldots, 2n. \tag{2.1}$$

Then the classical PB's and the quantum CR's are, respectively

$$\{\xi_a, \xi_b\} = \beta_{ab},$$
$$[\hat{\xi}_a, \hat{\xi}_b] = i\hbar\beta_{ab},$$
$$\beta = (\beta_{ab}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{2.2}$$

The usual relations stated separately in terms of q's and p's are all contained here; and the even-dimensional real antisymmetric matrix β will play an important role.

We assume always that all the ξ_a, $\hat{\xi}_a$ are of the Cartesian type: the natural range (spectrum) for each of them is the entire real line $\mathscr{R}$.

Changes of $\xi, \hat{\xi}$ to new quantities $\xi', \hat{\xi}'$ given as functions of the old ones such that the basic kinematic relations are preserved may be called canonical transformations in both situations

$$\xi' = \text{numerical functions of } \xi: \{\xi'_a, \xi'_b\} = \beta_{ab};$$
$$\hat{\xi}' = \text{operator functions of } \hat{\xi}: [\hat{\xi}'_a, \hat{\xi}'_b] = i\hbar\beta_{ab}. \tag{2.3}$$

Apart from C-number translations (shift of origin) the simplest such transformations are the linear homogeneous ones. Each such transformation may be specified by a real $2n$-dimensional matrix S, the actions being

$$S = (S_{ab}): \xi'_a = S_{ab}\xi_b.$$
$$\hat{\xi}'_a = S_{ab}\hat{\xi}_b. \tag{2.4}$$

In either case, the requirements (2.3) lead to a matrix condition on S [1]

$$S\beta S^T = \beta. \tag{2.5}$$

This is the defining condition for the real symplectic group in $2n$ dimensions

$$\mathrm{Sp}(2n, \mathscr{R}) = \{S = \text{real } 2n \times 2n \text{ matrix} \mid S\beta S^T = \beta\}. \tag{2.6}$$

The matrix β is real, even-dimensional, antisymmetric and nonsingular. It is a "symplectic metric matrix". As we see explicitly later, $\mathrm{Sp}(2n, \mathscr{R})$ transformations preserve symplectic scalar products and the symplectic metric.

In quantum mechanics the Hilbert space $\mathscr{H}$ on which the $\hat{\xi}_a$ act irreducibly can be described in many ways. The most familiar is the Schrödinger description using wave functions on $\mathscr{R}^n$, that is, elements of $L^2(\mathscr{R}^n)$. The $\hat{q}_r$ act multiplicatively while the $\hat{p}_r$ are differential operators

$$
\begin{aligned}
\mathscr{H} &= \{\psi(\mathbf{q}) | \int_{\mathscr{R}^n} d^n q |\psi(\mathbf{q})|^2 < \infty\};\\
(\hat{q}_r\psi)(\mathbf{q}) &= q_r \psi(\mathbf{q})\\
(\hat{p}_r\psi)(\mathbf{q}) &= -i\hbar \frac{\partial}{\partial q_r}\psi(\mathbf{q}),\\
\mathbf{q} &= (q_1, \ldots, q_n) \in \mathscr{R}^n.
\end{aligned} \tag{2.7}
$$

Since the $\hat{\xi}_a$ are hermitian and irreducible, and since for any $S \in Sp(2n, \mathscr{R})$ the transformed $\hat{\xi}'_a$ are also hermitian and irreducible and obey the same CR's, by the Stone-von Neumann theorem [5] the change $\hat{\xi} \to \hat{\xi}'$ is unitarily implementable. Thus for each $S \in Sp(2n, \mathscr{R})$ it is definitely possible to construct a unitary operator $\mathscr{U}(S)$ acting on $\mathscr{H}$ such that

$$
\begin{aligned}
\hat{\xi}'_a &= S_{ab}\hat{\xi}_b = \mathscr{U}(S)^{-1}\hat{\xi}_a \mathscr{U}(S),\\
\mathscr{U}(S)^\dagger \mathscr{U}(S) &= 1 \text{ on } \mathscr{H}.
\end{aligned} \tag{2.8}
$$

This $\mathscr{U}(S)$ is arbitrary up to an S-dependent phase factor. The general composition law that follows from the irreducibility of the $\hat{\xi}_a$ is

$$
S', S \in Sp(2n, \mathscr{R}): \mathscr{U}(S')\mathscr{U}(S) = (\text{phase factor dependent on } S', S)\mathscr{U}(S'S). \tag{2.9}
$$

We shall discuss in § 6 the maximum simplification that can be achieved in this phase factor by exploiting the phase freedom in each $\mathscr{U}(S)$.

3. Aspects of symplectic geometry

In this Section we develop a few basic concepts related to symplectic vector spaces, so that the contrast with Euclidean and unitary geometries can be clearly seen [4].

Let V be a real $2n$-dimensional vector space, with vectors $x, y, \ldots$. Suppose a non-degenerate bilinear antisymmetric form $(\cdot,\cdot)$ – a "scalar product" – is given on V. Thus for any vectors $x, y \in V$, (x, y) is a real number separately linear in x and y; and in addition the following hold:

$$
\begin{aligned}
&\text{antisymmetry: } (x, y) = -(y, x),\\
&\text{nondegeneracy: } (x, y) = 0 \quad \text{for all} \quad y \Leftrightarrow x = 0.
\end{aligned} \tag{3.1}
$$

Then V is a symplectic vector space, and $(.;.)$ is a symplectic scalar product.

We have stated the properties of the bilinear form in a basis independent way. It can be shown that if in a general basis we express (x, y) in terms of components of x and y in the form

$$
(x, y) = x^T \eta y, \tag{3.2}
$$

involving an antisymmetric nonsingular matrix η, we can always change to more convenient bases in which η takes on particularly simple canonical, or normal, forms. Two such forms are worth mentioning. In one, η becomes the matrix β of (2.2)

$$\eta = \beta\colon \quad (x, y) = x_1 y_{n+1} + x_2 y_{n+2} + \cdots + x_n y_{2n} - x_{n+1} y_1 - x_{n+2} y_2 - \cdots - x_{2n} y_n. \tag{3.3}$$

Here the first and $(n+1)$th components belong to one canonical pair; the second and $(n+2)$th to the second pair; and so on. Another normal form disposes the canonical pairs one at a time

$$\eta = \text{block-diag}(i\sigma_2, i\sigma_2, \ldots, i\sigma_2)\colon$$
$$(x, y) = x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3 + \cdots \tag{3.4}$$

With the normal form (3.3) the meaning of the defining condition (2.5) for symplectic matrices becomes geometrically clear

$$S \in \mathrm{Sp}(2n, \mathscr{R}),\ x' = Sx, \quad y' = Sy \Rightarrow (x', y') = (x, y). \tag{3.5}$$

In this sense the symplectic scalar product is preserved. Of course this means that we could have replaced the condition (2.5) by another entirely equivalent one, using in place of β the matrix in (3.4).

Now let us look at linear subspaces $V_1 \subseteq V$. In both Euclidean and unitary geometries, it is well known that all subspaces of the same dimension are basically similar, and cannot be distinguished from each other in any intrinsic sense. With symplectic geometry there is a difference, as a new concept comes in. Given a subspace V_1, we consider the bilinear form (x, y) defined over V, restrict both arguments to V_1, and regard the result as a bilinear form on V_1. Now the nondegeneracy property may well fail! Thus, there may exist a vector $x \in V_1$ such that $(x, y) = 0$ for all $y \in V_1$. So we define the rank of this restricted form as the symplectic rank of V_1

$$\text{Symp. rk.} \quad V_1 = \text{rank}\,(x, y), \quad x \text{ and } y \in V_1. \tag{3.6}$$

Thus, if x_r, $r = 1, \ldots, k$ is a basis for V_1, where k is the dimension of V_1, then Symp. rk. V_1 is the rank of the $k \times k$ antisymmetric matrix $((x_r, x_s))$. The symplectic rank is necessarily an even integer. We have the obvious limits

$$0 \leqslant \text{symp. rk.} \quad V_1 \leqslant k. \tag{3.7}$$

But nondegeneracy of $(.,.)$ over V leads to another nontrivial lower bound, which is effective if $k > n$

$$(0, 2(k-n))_> \leqslant \text{symp. rk.} \quad V_1 \leqslant k. \tag{3.8}$$

Basically we can say that the symplectic rank of a subspace V_1 is twice the number of complete canonical pairs contained in V_1. Clearly this concept is symplectic invariant. In particular two subspaces V_1 and V_1' of the same dimension cannot be mapped on to one another by any $\mathrm{Sp}(2n, \mathscr{R})$ element if they have unequal symplectic ranks.

As in the Euclidean case, we can pass from V_1 to its complement written for convenience as $V_1^{\perp}$. But the geometrical significance is quite different. We call $V_1^{\perp}$ the symplectic complement of V_1 and define it as a subspace of V by

$$V_1^\perp = \{x \in V | (x, y) = 0 \quad \text{for all} \quad y \in V_1\}. \tag{3.9}$$

Taking the complement twice gives back V_1

$$(V_1^\perp)^\perp = V_1. \tag{3.10}$$

This is as in the Euclidean case. Even the dimensions follow the same rule

$$\text{Dim } V_1^\perp = 2n - k. \tag{3.11}$$

This can be shown by using the nondegeneracy of $(.,.)$. But there the similarity ends. It can well happen that V_1 and $V_1^\perp$ have nontrivial intersection, and to that extent their sum does not give back all of V. In general,

$$V_1 \cap V_1^\perp \neq 0,$$
$$V \neq V_1 \oplus V_1^\perp. \tag{3.12}$$

The two symplectic ranks can be related:

$$\text{Symp. rk. } V_1^\perp = 2(n-k) + \text{symp. rk. } V_1. \tag{3.13}$$

The extreme case of (3.12), which is very nonintuitive on the basis of Euclidean geometric notions, is when symp. rk. V_1 vanishes. For this case we give a special name and find

$$\begin{aligned}\text{Symp. rk. } V_1 = 0 \Leftrightarrow V_1 \text{ is an isotropic subspace of } V \Leftrightarrow (x,y) = 0 \\ \text{for all } x, y \in V_1 \Leftrightarrow V_1 \subseteq V_1^\perp \Rightarrow k \leqslant n.\end{aligned} \tag{3.14}$$

So in this case V_1 is contained in $V_1^\perp$. The opposite can also happen and then we call V_1 a co-isotropic subspace

$$V_1 \text{ is a co-isotropic subspace of } V \Leftrightarrow V_1^\perp \text{ is isotropic} \Leftrightarrow V_1 \supseteq V_1^\perp \Rightarrow k \geqslant n. \tag{3.15}$$

So if one of the pair V_1, $V_1^\perp$ is isotropic, the other is co-isotropic.

An isotropic subspace has dimension $k \leqslant n$, while a co-isotropic one has dimension $k \geqslant n$. When they coincide, we have a special situation and name. A n-dimensional subspace V_1 of V which has vanishing symplectic rank is both isotropic and co-isotropic, and coincides with $V_1^\perp$. It is called a Lagrangian subspace. This notion is important in Hamilton-Jacobi theory in classical dynamics; it is also relevant in the choice of complete commuting sets of operators in quantum mechanics.

These properties and notions give a feeling for symplectic geometry, and for the ways in which it differs from orthogonal and unitary geometries. In particular the notions of length, angle and perpendicularity are no longer available.

4. Properties of Sp$(2n, \mathscr{R})$ matrices, complex form, subgroups, decompositions

The matrices in the defining representation of Sp$(2n, \mathscr{R})$ obey (2.5). From here many useful consequences follow, and we list them:

(i) Sp$(2n, \mathscr{R})$ is of dimension $n(2n+1)$.
(ii) $\beta \in \text{Sp}(2n, \mathscr{R})$.

(iii) $S\in \mathrm{Sp}(2n,\mathscr{R})\Rightarrow -S, S^{-1}, S^T\in \mathrm{Sp}(2n,\mathscr{R})$,
$S^T=\beta S^{-1}\beta^{-1}, (S^{-1})^T=\beta S\beta^{-1}, S^{-1}=\beta S^T\beta^{-1}$.
(iv) $\det S=+1$.
(v) $S\in \mathrm{Sp}(2n,\mathscr{R})\Rightarrow$ eigenvalue spectrum of S is invariant under reflection about the real axis, and through unit circle ($re^{i\theta}\to \frac{1}{r}e^{i\theta}$); eigenvalues ± 1 have even multiplicities. (4.1)

Property (i) can be seen from the number of conditions contained in (2.5), and will be confirmed at the Lie algebra level. While properties (ii) and (iii) are easily checked, (iv) is rather subtle; an indication will be given later to obtain it. Property (v) is a consequence of S and $(S^{-1})^T$ being real and related by a similarity transformation.

Sometimes it is convenient to write S in $n\times n$ block form, and then (2.5) becomes a set of conditions on the blocks

$$S=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}(2n,\mathscr{R}):$$

$$\begin{aligned} S\beta S^T=\beta &\Leftrightarrow AB^T, CD^T \quad \text{symmetric}, \quad AD^T-BC^T=1_{n\times n}\\ S^T\beta S=\beta &\Leftrightarrow A^TC, B^TD \quad \text{symmetric}, \quad A^TD-C^TB=1_{n\times n}. \end{aligned} \tag{4.2}$$

While it is easy to check (as mentioned above) that $S\in \mathrm{Sp}(2n,\mathscr{R})$ implies $S^T\in \mathrm{Sp}(2n,\mathscr{R})$ as well, it is not so easy to pass directly from the first set of conditions above to the second set, aside from reconstituting A, B, C, D into S and then passing to S^T!

Complex form of $\mathrm{Sp}(2n,\mathscr{R})$

The β matrix reflects the precise way in which the real q's and p's have been put together in (2.1) to form the $2n$ component object ξ with real entries. Sometimes it is convenient, for instance in dealing with modes of the radiation field, to work with complex combinations of the $\hat{q}$'s and $\hat{p}$'s-mode annihilation and creation operators defined in this way

$$\hat{a}_j=\frac{1}{\sqrt{2}}(\hat{q}_j+i\hat{p}_j), \quad \hat{a}_j^\dagger=\frac{1}{\sqrt{2}}(\hat{q}_j-i\hat{p}_j), \quad j=1,\dots,n. \tag{4.3}$$

It is useful to arrange these into a new column vector $\hat{\xi}^{(c)}$ with non-hermitian entries,

$$\begin{aligned} &\hat{\xi}^{(c)}=(\hat{\xi}_a^{(c)})=(\hat{a}_1,\dots,\hat{a}_n \quad \hat{a}_1^\dagger,\dots,\hat{a}_n^\dagger)^T=\Omega\hat{\xi},\\ &\hat{\xi}=\Omega^\dagger\hat{\xi}^{(c)},\\ &\Omega=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i1\\ 1 & -i1\end{pmatrix}, \quad \Omega^{-1}=\Omega^\dagger=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1\\ -i1 & i1\end{pmatrix}. \end{aligned} \tag{4.4}$$

Then the basic commutation relations in (2.2) can be written in two equivalent ways

$$\begin{aligned} &[\hat{\xi}_a^{(c)},\hat{\xi}_b^{(c)}]=\beta_{ab}.\\ &[\hat{\xi}_a^{(c)},\hat{\xi}_b^{(c)\dagger}]=(\Sigma_3)_{ab},\\ &\Sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \end{aligned} \tag{4.5}$$

Now when we subject $\hat{\xi}$ to the real transformation $S\in\mathrm{Sp}(2n,\mathscr{R})$, $\hat{\xi}^{(c)}$ experiences an equivalent complex transformation

$$\hat{\xi}' = S\hat{\xi} \Leftrightarrow \hat{\xi}'^{(c)} = S^{(c)}\hat{\xi}^{(c)},$$
$$S^{(c)} = \Omega S \Omega^{-1}$$
$$= \frac{1}{2}\begin{pmatrix} A+D+i(C-B) & A-D+i(B+C) \\ A-D-i(B+C) & A+D+i(B-C) \end{pmatrix}. \tag{4.6}$$

Thus $S^{(c)}$ is just a convenient complex form of the real transformation S, much like the passage from Cartesian to spherical components of spherical tensors.

Some subgroups of $\mathrm{Sp}(2n,\mathscr{R})$

We shall describe here some useful subgroups of $\mathrm{Sp}(2n,\mathscr{R})$. Their dimensions will be given, and where it is useful their complex forms exhibited.

(a) $\mathrm{GL}(n,\mathscr{R})$: This is the n^2-dimensional general real linear group; in terms of the block matrices A, B, C, D it is given thus

$$A\in\mathrm{GL}(n,\mathscr{R}), \quad B=C=0, \quad D=(A^{-1})^T. \tag{4.7}$$

Here the $\hat{q}$'s are subject to a general real linear nonsingular transformation among themselves, and then the $\hat{p}$'s change in a compensating contragradient manner.

(b) $\mathrm{O}(n,\mathscr{R})$: This is the orthogonal subgroup of $\mathrm{GL}(n,\mathscr{R})$, of dimension $\frac{1}{2}n(n-1)$. It is that part of $\mathrm{GL}(n,\mathscr{R})$ under which the $\hat{q}$'s and the $\hat{p}$'s change in the same way

$$A=D\in\mathrm{O}(n,\mathscr{R}), \quad B=C=0. \tag{4.8}$$

If we impose the condition $\det A = +1$, we get the subgroup $\mathrm{SO}(n,\mathscr{R})$ of proper orthogonal transformations.

(c) $\mathrm{U}(n)$: Now we come to the n-dimensional unitary group, of dimension n^2, a maximal compact subgroup within the noncompact $\mathrm{Sp}(2n,\mathscr{R})$. We shall sometimes write $\mathrm{K}(n)$, or simply K, for it. The corresponding symplectic matrices S are identified as follows. If we split any $\mathrm{U}\in\mathrm{U}(n)$ into real and imaginary parts we find the properties

$$\mathrm{U}=X-iY\in\mathrm{U}(n), \quad \mathrm{U}^\dagger\mathrm{U}=\mathrm{U}\mathrm{U}^\dagger=1 \Leftrightarrow X^TX+Y^TY=XX^T+YY^T=1,$$
$$X^TY, XY^T \text{ symmetric.} \tag{4.9}$$

We can then produce a solution to the matrix condition (4.2). We find

$$A=D=X, \quad B=-C=Y$$
$$S(X,Y)=\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}\in\mathrm{Sp}(2n,\mathscr{R}). \tag{4.10}$$

It is an interesting and easy exercise to check the following: If a $2n\times 2n$ real matrix is both orthogonal and symplectic, then it is unimodular as well and has to have the form $S(X,Y)$ for some $\mathrm{U}=X-iY\in\mathrm{U}(n)$

$$\mathrm{O}(2n,\mathscr{R})\cap\mathrm{Sp}(2n,\mathscr{R})=\mathrm{SO}(2n,\mathscr{R})\cap\mathrm{Sp}(2n,\mathscr{R})=\mathrm{K}(n)$$
$$=\{S(X,Y)\,|\,X-iY\in\mathrm{U}(n)\}. \tag{4.11}$$

The complex form of these matrices is very revealing

$$S^{(c)}(X, Y) = \Omega S(X, Y)\Omega^{-1} = S^{(c)}(\mathrm{U}) = \begin{pmatrix} \mathrm{U} & 0 \\ 0 & \mathrm{U}^* \end{pmatrix}. \tag{4.12}$$

So the $\hat{a}$'s and the $\hat{a}^\dagger$'s undergo separate unitary rotations, not mixing with one another

$$\mathrm{U} \in \mathrm{U}(n): \quad \hat{a} \to \mathrm{U}\hat{a}, \quad \hat{a}^\dagger \to \mathrm{U}^*\hat{a}^\dagger. \tag{4.13}$$

Indeed, $\mathrm{K}(n)$ is the maximal subgroup of $\mathrm{Sp}(2n, \mathscr{R})$ such that $\hat{a}$'s and $\hat{a}^\dagger$'s transform independently. We also have the expected relation between the subgroups $\mathrm{O}(n, \mathscr{R})$, $\mathrm{GL}(n, \mathscr{R})$, and $\mathrm{U}(n)$ exhibited above

$$\mathrm{O}(n, \mathscr{R}) = \mathrm{GL}(n, \mathscr{R}) \cap \mathrm{U}(n). \tag{4.14}$$

Finally we turn to some Abelian subgroups [6].

(d) T^f: This is a subgroup of dimension $\frac{1}{2}n(n+1)$ and may be called the "free propagation" subgroup

$$A = D = 1, \quad B = B^T, \quad C = 0. \tag{4.15}$$

The name comes from the actions on $\hat{q}$ and on $\hat{p}$

$$\hat{q}' = \hat{q} + B\hat{p}, \quad \hat{p}' = \hat{p}. \tag{4.16}$$

Group composition corresponds to adding the B matrices, which explains the Abelian nature.

(e) $T^{(l)}$: This is the result of conjugating elements of $T^{(f)}$ by β. We call it the "lens" subgroup, on account of the action on $\hat{q}$'s and $\hat{p}$'s

$$A = D = 1, \quad B = 0, \quad C = C^T;$$
$$\hat{q}' = \hat{q}, \quad \hat{p}' = \hat{p} + C\hat{q}. \tag{4.17}$$

The dimension is again $\frac{1}{2}n(n+1)$, and group composition amounts to adding the C matrices.

Some other subgroups of $\mathrm{Sp}(2n, \mathscr{R})$ will appear in connection with global decomposition theorems.

Global decomposition theorems

Now we describe four useful ways of expressing any $S \in \mathrm{Sp}(2n, \mathscr{R})$ as a product of specially chosen factors, either two or three in number.

(a) *Polar decomposition* [7]: This says that any $S \in \mathrm{Sp}(2n, \mathscr{R})$ can be written uniquely as the product of two factors, one belonging to the maximal compact subgroup $\mathrm{K}(n)$, the other to an important subset $\Pi(n)$ in $\mathrm{Sp}(2n, \mathscr{R})$. This subset is defined by

$$\Pi(n) = \{S \in \mathrm{Sp}(2n, \mathscr{R}) \mid S^T = S, S \text{ positive definite}\} \subset \mathrm{Sp}(2n, \mathscr{R}). \tag{4.18}$$

and it is definitely not a subgroup. The decomposition reads

$$S \in \mathrm{Sp}(2n, \mathscr{R}): \quad S = S(X, Y)P \text{ uniquely},$$
$$S(X, Y) \in K(n), \quad P \in \Pi(n). \tag{4.19}$$

Of course by conjugating P with $S(X, Y)$ one could have written the two factors in the opposite sequence. The important points here are the global nature of this result, and the uniqueness of the factors. From this decomposition one can see that of the two possibilities det $S = \pm 1$ allowed by (2.5), the choice det $S = +1$ is the only one allowed.

(b) *Euler decomposition*: Next we turn to a decomposition which involves three factors, each drawn from a subgroup of $\mathrm{Sp}(2n, \mathscr{R})$, but which is nonunique. Two of the factors are from $K(n)$, the third from those elements of $\Pi(n)$ which are diagonal and do form a subgroup

$$S \in \mathrm{Sp}(2n, \mathscr{R}) \quad S = S(X_1, Y_1)\, D(\boldsymbol{\kappa})\, S(X_2, Y_2),$$
$$S(X_1, Y_1), S(X_2, Y_2) \in \mathrm{K}(n),$$
$$D(\boldsymbol{\kappa}) = \mathrm{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1^{-1}, \ldots, \kappa_n^{-1}) \in \Pi(n),$$
$$\kappa_r > 0, \quad r = 1, \ldots, n. \tag{4.20}$$

If one adds the number of free parameters in the three factors, one gets the sum $n(2n+1)$ which is just the dimension of $\mathrm{Sp}(2n, \mathscr{R})$. Thus the nonuniqueness of this decomposition is of a discrete, not a continuous, nature. It stems essentially from the freedom to order the first n diagonal elements of $D(\boldsymbol{\kappa})$ in any way we like.

(c) *Pre-Iwasawa decomposition*: Here we have a three-factor unique decomposition which one encounters on the way to establishing the (next) Iwasawa decomposition but the factors do not all belong to subgroups of $\mathrm{Sp}(2n, \mathscr{R})$. The present decomposition results from attempting to reduce the off diagonal block B in a general $S \in \mathrm{Sp}(2n, \mathscr{R})$ to zero, by using an element of $\mathrm{K}(n)$ on the right. The result reads

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C_0 A_0^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_0 & 0 \\ 0 & A_0^{-1} \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix},$$
$$A_0 = (AA^T + BB^T)^{1/2},$$
$$X - iY = A_0^{-1}(A - iB),$$
$$C_0 = (CA^T + DB^T)A_0^{-1}. \tag{4.21}$$

Here the matrix A_0 is to be chosen symmetric positive definite, and all factors are unique. The symmetry of $C_0 A_0^{-1}$ can be checked, so the first factor lies in the lens subgroup T^l of (4.17). The middle factor belongs to the intersection $\mathrm{GL}(n, \mathscr{R}) \cap \Pi(n)$, which is not a subgroup. And the third factor is from $\mathrm{K}(n)$. This particular decomposition is of importance in obtaining the generalised Huyghens kernel, which we describe in § 6.

(d) *Iwasawa decomposition* [8]: The polar and pre-Iwasawa decompositions are similar since they involve unique factors, but each factor is not taken from a subgroup. The Euler decomposition solves the latter problem, but in the process uniqueness is lost. The fourth and last Iwasawa decomposition retains both virtues:

it is global, has unique factors, and each is taken from a characteristic subgroup of $Sp(2n, \mathscr{R})$. It is a result of fundamental group theoretical significance, valid for all simple non-compact Lie groups. The three subgroups involved are the maximal compact $K(n)$, a certain maximal Abelian subgroup $\mathscr{A}$, and a certain nilpotent subgroup $\mathscr{N}$. Therefore this decomposition is often called the $\mathscr{K}\mathscr{A}\mathscr{N}$ decomposition. We first display it for $Sp(2, \mathscr{R})$

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathscr{R}), \quad ad - bc = 1:$$

$$S = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix},$$

$$\xi = (ac + bd)/(a^2 + b^2) \in (-\infty, \infty),$$

$$\eta = \ln(a^2 + b^2) \in (-\infty, \infty),$$

$$\varphi = 2\arg(a - ib) \in (-2\pi, 2\pi]. \tag{4.22}$$

Here the first $-\xi-$ factor belongs to the subgroup $\mathscr{N}$, coinciding for $n = 1$ with the lens subgroup T^l; the second $-\eta-$ factor belongs to the subgroup $\mathscr{A}$; and the third $-\varphi-$ factor is from $K(1) = SO(2)$.

For general $Sp(2n, \mathscr{R})$, the situation is more involved. The subgroups $\mathscr{A}$ and $\mathscr{N}$ are

$$\mathscr{A} = \{D(\boldsymbol{\kappa}) = \mathrm{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1^{-1}, \ldots, \kappa_n^{-1}) \,|\, \kappa_r > 0\} \subset \Pi(n);$$

$$\mathscr{N} = \left\{ \begin{pmatrix} A & 0 \\ C & (A^{-1})^T \end{pmatrix} \middle| A = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & 1 & \ddots \\ & & 1 \end{pmatrix}, A^T C \text{ symmetric} \right\} \subset Sp(2n, \mathscr{R}). \tag{4.23}$$

The abelian subgroup $\mathscr{A}$ consists of just the elements $D(\boldsymbol{\kappa})$ that were used in the Euler decomposition (4.20). The Iwasawa decomposition for $Sp(2n, \mathscr{R})$ then states that any $S \in Sp(2n, \mathscr{R})$ can be uniquely expressed as the product of three factors,

$$S = \begin{pmatrix} A & 0 \\ C & (A^{-1})^T \end{pmatrix} D(\boldsymbol{\kappa})\, S(X, Y). \tag{4.24}$$

taken respectively from $\mathscr{N}$, $\mathscr{A}$ and $K(n)$. The dimensionalities of these subgroups, respectively n^2, n and n^2, add up correctly to $n(2n + 1)$.

5. The Lie algebra of Sp(2n, $\mathscr{R}$)

We first study the Lie algebra $\underline{Sp(2n, \mathscr{R})}$ in the defining representation, and then generalise to any other representation. In keeping with quantum mechanical convention, we shall retain a factor of i in the definition, even though this might seem unnecessary in dealing with a group of real matrices.

We examine the form of matrices $S \in Sp(2n, \mathscr{R})$ close to the identity

$$S = \exp(-i\varepsilon J) \simeq 1 - i\varepsilon J. \quad |\varepsilon| \ll 1:$$

$$S\beta S^T = \beta \Rightarrow (\beta J)^T = \beta J, \quad (J\beta)^T = J\beta.$$
$$J^* = -J. \tag{5.1}$$

Thus the generator matrix J is pure imaginary, and both βJ and $J\beta$ are symmetric. In other words in the defining representation we get all possible J's by pre- or post-multiplying all possible pure imaginary symmetric $2n \times 2n$ matrices by β. Taking the former alternative and choosing the simplest possible basis for symmetric $2n \times 2n$ matrices, we obtain the following basis for Sp$(2n, \mathscr{R})$:

$$X^{(0)}_{ab} = X^{(0)}_{ba}, \quad a, b = 1, \ldots, 2n;$$
$$(X^{(0)}_{ab})_{cd} = i(\delta_{ad}\beta_{cb} + \delta_{bd}\beta_{ca}). \tag{5.2}$$

These matrices can be easily seen to obey the commutation relations

$$[X^{(0)}_{ab}, X^{(0)}_{cd}] = i(\beta_{ac}X^{(0)}_{bd} + \beta_{bc}X^{(0)}_{ad} + \beta_{ad}X^{(0)}_{cb} + \beta_{bd}X^{(0)}_{ca}). \tag{5.3}$$

The structure of Sp$(2n, \mathscr{R})$ is determined by these relations. In a general representation of Sp$(2n, \mathscr{R})$ we have generators $X_{ab} = X_{ba}$ obeying

$$[X_{ab}, X_{cd}] = i(\beta_{ac}X_{bd} + \beta_{bc}X_{ad} + \beta_{ad}X_{cb} + \beta_{bd}X_{ca}). \tag{5.4}$$

Finite dimensional representations of Sp$(2n, \mathscr{R})$ are necessarily nonunitary, hence in them the X_{ab} cannot all be hermitian. This is because of the noncompactness of Sp$(2n, \mathscr{R})$. On the other hand, in a unitary representation which is necessarily infinite dimensional, we have $X^\dagger_{ab} = X_{ab}$.

To help identify the subsets of generators for various subgroups it is useful to use split index notation. We use $r, s, \ldots = 1, \ldots, n$ to label the various canonical pairs; and $\alpha, \beta, \ldots = 1, 2$ to pick out the q and the p in each pair

$$a, b, \ldots = 1, \ldots, 2n: \quad a \to r\alpha, \quad b \to s\beta;$$
$$\beta_{ab} = \beta_{r\alpha, s\beta} = \delta_{rs}\varepsilon_{\alpha\beta},$$
$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.5}$$

Then the various components of $X_{ab} = X_{r\alpha, s\beta}$ are handled thus:

$$X_{r1,s1} = V_{rs} = V_{sr};$$
$$X_{r1,s2} = W_{rs},$$
$$X_{r2,s2} = Z_{rs} = Z_{sr}. \tag{5.6}$$

There are $\frac{1}{2}n(n+1)$ V's, a similar number of Z's, and n^2 W's; in a unitary representation, each of them is hermitian. In this split form the commutation relations (5.4) read

$$[W_{rs}, W_{uv}] = i(\delta_{rv}W_{us} - \delta_{us}W_{rv}).$$
$$[W_{rs}, V_{uv}] = -i(\delta_{us}V_{rv} + \delta_{vs}V_{ru}).$$
$$[W_{rs}, Z_{uv}] = i(\delta_{ru}Z_{sv} + \delta_{rv}Z_{su}).$$

$$[V_{rs}, Z_{uv}] = i(\delta_{ru} W_{sv} + \delta_{su} W_{rv} + \delta_{rv} W_{su} + \delta_{sv} W_{ru}).$$
$$[V, V] = [Z, Z] = 0. \tag{5.7}$$

Now one can pick out the subsets of generators for various subgroups of $Sp(2n, \mathscr{R})$; we give the results in the form of a table.

Subgroup	Generators
$GL(n, \mathscr{R})$	W_{rs}
$SO(n, \mathscr{R})$	$J_{rs} = W_{sr} - W_{rs}$
$U(n) = K(n)$	$J_{rs}, Q_{rs} = V_{rs} + Z_{rs}$ $A_{rs} = \frac{1}{2}(Q_{rs} - iJ_{rs})$
$T^{(\mathscr{J})}$	Z_{rs}
$T^{(I)}$	V_{rs}
$\mathscr{A}$	$W_{rr}, r = 1, \ldots, n$
$\mathscr{N}$	W_{rs} for $r < s$, and all V_{rs}.

We have mentioned that any nontrivial finite dimensional representation of $Sp(2n, \mathscr{R})$ is necessarily nonunitary. It turns out that, with no loss of generality, we may assume that the "compact" generators of $K(n)$ are hermitian, while a balance of "noncompact" generators are antihermitian. That is, in any finite dimensional representation we can assume the following

$$\begin{aligned}\text{Generators of } K(n) &= \text{compact generators}\\ &= W_{rs} - W_{sr}, V_{rs} + Z_{rs} = \text{hermitian:}\\ \text{Balance of generators} &= \text{noncompact generators}\\ &= W_{rs} + W_{sr}, V_{rs} - Z_{rs} = \text{antihermitian.}\end{aligned} \tag{5.8}$$

The noncompact generators can be arranged into complex combinations with definite tensor behaviour under $U(n)$. These combinations are

$$T_{rs} = T_{sr} = V_{rs} - Z_{rs} - i(W_{rs} + W_{sr}).$$
$$\bar{T}_{rs} = \bar{T}_{sr} = V_{rs} - Z_{rs} + i(W_{rs} + W_{sr}). \tag{5.9}$$

Then the complete set of $Sp(2n, \mathscr{R})$ commutation relations (5.4) appears in a $U(n)$ adapted form [9]

$$[A_{rs}, A_{uv}] = \delta_{su} A_{rv} - \delta_{rv} A_{us};$$
$$[A_{rs}, T_{uv}] = \delta_{su} T_{rv} + \delta_{sv} T_{ru};$$
$$[A_{rs}, \bar{T}_{uv}] = -\delta_{ru} \bar{T}_{sv} - \delta_{rv} \bar{T}_{su};$$
$$[T_{rs}, \bar{T}_{uv}] = -4(\delta_{ru} A_{sv} + \delta_{rv} A_{su} + \delta_{su} A_{rv} + \delta_{sv} A_{ru});$$
$$[T, T] = [\bar{T}, \bar{T}] = 0. \tag{5.10}$$

We see that T_{rs} and $\bar{T}_{rs}$ are second rank symmetric tensors under $U(n)$, of contravariant and covariant types respectively. While in any representation we can assume $A_{rs}^{\dagger} = A_{sr}$, only in unitary representations do we have $T_{rs}^{\dagger} = \bar{T}_{rs}$ as well.

6. The metaplectic unitary representation and generalised Huyghens kernel

We saw in §2 that for each $S \in Sp(2n, \mathscr{R})$, on account of the Stone-von Neumann Theorem, we can construct a unitary operator $\mathscr{U}(S)$ such that (2.8) holds. Clearly the phase of $\mathscr{U}(S)$ is free. We can ask if this S-dependent phase can be chosen so as to make the composition law (2.9) of the $\mathscr{U}$'s as simple as possible. The answer is that this can be done, and upon maximum simplification we can achieve

$$S_1, S_2 \in Sp(2n, \mathscr{R}) \quad \mathscr{U}(S_1)\mathscr{U}(S_2) = \pm \mathscr{U}(S_1 S_2). \tag{6.1}$$

This sign ambiguity cannot be eliminated. So we say that we have here a two-valued unitary representation of $Sp(2n, \mathscr{R})$. A more correct or useful statement is that the operators involved provide a faithful unitary representation of the metaplectic group $Mp(2n)$, which is a two-fold covering of $Sp(2n, \mathscr{R})$ [10]. Strictly speaking this means that the argument of $\mathscr{U}(\cdot)$ should be an element of $Mp(2n)$, not $S \in Sp(2n, \mathscr{R})$. However, having made this point, we shall continue to write $\mathscr{U}(S)$ as in (2.8), (2.9) and (6.1).

The generators of this metaplectic representation of $Sp(2n, \mathscr{R})$ are all hermitian; in terms of $\hat{q}$'s and $\hat{p}$'s they are the quadratic expressions [11]

$$\hat{W}_{rs} = \tfrac{1}{2}\{\hat{q}_r, \hat{p}_s\},$$
$$\hat{V}_{rs} = \hat{q}_r \hat{q}_s, \; \hat{Z}_{rs} = \hat{p}_r \hat{p}_s. \tag{6.2}$$

The characteristic differences between the compact and the noncompact combinations become clear when expressed in terms of $\hat{a}$'s and $\hat{a}^\dagger$'s

$$\text{Compact generators: } \hat{W}_{rs} - \hat{W}_{sr} = i(\hat{a}_s^\dagger \hat{a}_r - \hat{a}_r^\dagger \hat{a}_s),$$
$$\hat{V}_{rs} + \hat{Z}_{rs} = (\hat{a}_r^\dagger \hat{a}_s + \hat{a}_s^\dagger \hat{a}_r + \delta_{rs}), \tag{6.3a}$$
$$\text{Non compact generators: } \hat{W}_{rs} + \hat{W}_{sr} = i(\hat{a}_r^\dagger \hat{a}_s^\dagger - \hat{a}_r \hat{a}_s),$$
$$\hat{V}_{rs} - \hat{Z}_{rs} = \hat{a}_r^\dagger \hat{a}_s^\dagger + \hat{a}_r \hat{a}_s. \tag{6.3b}$$

We see that the compact generators of $U(n)$ conserve "total photon number", thus this subgroup of $Sp(2n, \mathscr{R})$ consists of "passive" transformations. The noncompact generators on the other hand do not conserve "photon number", so we may call them "active" generators. These properties are expressed thus

$$\hat{N} = \hat{a}_r^\dagger \hat{a}_r:$$
$$[\hat{W}_{rs} - \hat{W}_{sr} \quad \text{or} \quad \hat{V}_{rs} + \hat{Z}_{rs}, \hat{N}] = 0,$$
$$[\hat{W}_{rs} + \hat{W}_{sr} \quad \text{or} \quad \hat{V}_{rs} - \hat{Z}_{rs}, \hat{N}] \neq 0. \tag{6.4}$$

In fact, single exponentials of i times real linear combinations of the compact generators give us operators of the form $\mathscr{U}(S(X, Y))$; while single exponentials of i times real linear combinations of the noncompact generators give us operators of the form $\mathscr{U}(P)$, $P \in \Pi(n)$. For this reason the latter may be called "squeezing transformations" [12], [13]; and the polar decomposition (4.19) may be read as stating that any metaplectic unitary transformation is uniquely the product of a compact passive factor and a noncompact active squeeze factor. The definition and production of squeezed states are taken up in more detail in §10.

The Schrödinger description of the Hilbert space $\mathscr{H}$ on which the metaplectic representation acts has been given in (2.7). In this description, the eigenvectors $|\mathbf{q}\rangle$ of the commuting position operators $\hat{q}_r$ appear as a basis

$$
\begin{aligned}
|\psi\rangle\in\mathscr{H}:\quad & \psi(\mathbf{q})=\langle\mathbf{q}|\psi\rangle,\\
& \hat{q}_r|\mathbf{q}\rangle=q_r|\mathbf{q}\rangle,\\
& \langle\mathbf{q}'|\mathbf{q}\rangle=\delta^{(n)}(\mathbf{q}'-\mathbf{q}),\\
& \langle\mathbf{q}|\hat{p}_r=-i\hbar\frac{\partial}{\partial q_r}\langle\mathbf{q}|.
\end{aligned}
\tag{6.5}
$$

It is useful to know that certain operators $\mathscr{U}(S)$ have very simple actions on these basis vectors. We list them below

$$
\begin{aligned}
& S(A)=\begin{pmatrix}A & 0\\ 0 & (A^{-1})^T\end{pmatrix},\quad A\in\mathrm{GL}(n,\mathscr{R}):\\
& \mathscr{U}(S(A))|\mathbf{q}\rangle=|\det A|^{1/2}|A\mathbf{q}\rangle.\\
& \langle\mathbf{q}|\mathscr{U}(S(A))=|\det A|^{-1/2}\langle A^{-1}\mathbf{q}|;
\end{aligned}
\tag{6.6a}
$$

$$
\begin{aligned}
& D(\boldsymbol{\kappa})=\mathrm{diag}(\kappa_1,\ldots,\kappa_n,\kappa_1^{-1},\ldots,\kappa_n^{-1}),\ \kappa_r>0:\\
& \mathscr{U}(D(\boldsymbol{\kappa}))=\exp\left(-i\sum_{r=1}^{n}\ln(\kappa_r)\widehat{W}_{rr}\right),\\
& \mathscr{U}(D(\boldsymbol{\kappa}))|\mathbf{q}\rangle=\left(\prod_{r=1}^{n}\kappa_r\right)^{1/2}|\kappa_1 q_1,\ldots,\kappa_n q_n\rangle,\\
& \langle\mathbf{q}|\mathscr{U}(D(\boldsymbol{\kappa}))=\left(\prod_{r=1}^{n}\kappa_r\right)^{-1/2}\langle\kappa_1^{-1}q_1,\ldots,\kappa_n^{-1}q_n|;
\end{aligned}
\tag{6.6b}
$$

$$
\begin{aligned}
& L(g)=\begin{pmatrix}1 & 0\\ -g & 1\end{pmatrix}\in T^{(l)},\quad g^T=g:\\
& \mathscr{U}(L(g))=\exp\left(-\frac{i}{2}g_{rs}\hat{V}_{rs}\right),\\
& \mathscr{U}(L(g))|\mathbf{q}\rangle=\exp\left(-\frac{i}{2}q^T g q\right)|\mathbf{q}\rangle.
\end{aligned}
\tag{6.6c}
$$

With the help of these results, and the pre-Iwasawa decomposition for elements of $\mathrm{Sp}(2n,\mathscr{R})$ described in §4, it turns out to be possible to calculate the generalized Huyghen's kernel in n-dimensions without too much effort. This kernel is the configuration space matrix element $\langle\mathbf{q}|\mathscr{U}(S)|\mathbf{q}'\rangle$ of the metaplectic unitary operator $\mathscr{U}(S)$. We recall that in the case of one degree of freedom, this kernel has the following form [11]

$$
S=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{Sp}(2,\mathscr{R})
$$

$$\langle q|\mathscr{U}(S)|q'\rangle = \begin{cases} \dfrac{e^{-i\pi/4}}{\sqrt{h|b|}}\exp[i(dq^2 - 2qq' + aq'^2)/2\hbar b], & b \neq 0; \\ \exp(icq^2/2a)\delta\left(\dfrac{q}{a} - q'\right)\Big/|a|^{1/2}, & b = 0. \end{cases} \tag{6.7}$$

(These results are, strictly speaking, valid only if S is sufficiently close to the identity, the point being that $\mathscr{U}(\cdot)$ is actually a representation of Mp(2) and so carries as argument an element of this group. Therefore the generalised Huyghens kernel is not expressible totally in terms of $S \in \mathrm{Sp}(2, \mathscr{R})$.) We may regard the case $b \neq 0$ as generic. This generalizes nicely to any number of dimensions and we find

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathscr{R}), \quad \det B \neq 0:$$

$$\langle \mathbf{q}|\mathscr{U}(S)|\mathbf{q}'\rangle = \frac{e^{-in\pi/4}}{h^{n/2}} \cdot \frac{1}{\sqrt{|\det B|}} \times \exp\left[\frac{i}{2\hbar}\{q^T DB^{-1}q - 2q'^T B^{-1}q + q'^T B^{-1}Aq'\}\right]. \tag{6.8}$$

The nongeneric case when $\det B = 0$ has to be handled carefully – then the kernel collapses to a lower dimensional expression with a certain number of delta function factors. However, having given an indication of the structure involved, we will not go into any further details.

7. Sp(2n, $\mathscr{R}$) actions on Wigner and diagonal coherent state representations

Let $\hat{\Gamma}$ be any quantum mechanical operator, specified in the Schrödinger representation by its configuration space kernel $\langle \mathbf{q}|\hat{\Gamma}|\mathbf{q}'\rangle$. We can see that if $\hat{\Gamma}$ is conjugated by $\mathscr{U}(S)$ for general $S \in \mathrm{Sp}(2n, \mathscr{R})$, the change in the kernel involves an integral transformation in which the generalised Huygens kernel (6.8) and its complex conjugate both appear. We can ask whether there is any other way of specifying or describing $\hat{\Gamma}$ such that this change takes a simpler form, not requiring any integrations at all. Indeed there is, and it is given by the use of the techniques due to Weyl, Wigner and Moyal (WWM) [14]. We describe this aspect, and then go on to another practically important way of describing operators, namely via the diagonal coherent state representation, and its behaviour under Sp(2n, $\mathscr{R}$).

Let us hereafter set $\hbar = 1$. From the configuration space kernel $\langle \mathbf{q}|\hat{\Gamma}|\mathbf{q}'\rangle$ of $\hat{\Gamma}$ we obtain its Wigner distribution or WWM representative by a partial Fourier transformation

$$\hat{\Gamma} \to W(\xi) = (2\pi)^{-n}\int d^n q' \langle \mathbf{q} - \tfrac{1}{2}\mathbf{q}'|\hat{\Gamma}|\mathbf{q} + \tfrac{1}{2}\mathbf{q}'\rangle \exp(i\mathbf{q}'\cdot\mathbf{p}),$$

$$\langle \mathbf{q}|\hat{\Gamma}|\mathbf{q}'\rangle = \int d^n p\, W(\tfrac{1}{2}(\mathbf{q} + \mathbf{q}'), \mathbf{p})\exp(-i\mathbf{p}\cdot(\mathbf{q} - \mathbf{q}')). \tag{7.1}$$

Here $W(\xi)$ is a function on the classical phase space corresponding to the quantum system, with arguments which are $2n$ classical c-number q's and p's. As seen above, one can recover the operator $\hat{\Gamma}$ from its WWM representative $W(\xi)$ unambiguously. Then one finds that under conjugation by the metaplectic operators $\mathscr{U}(S)$, the changes in $\hat{\Gamma}$ are very simply expressed in terms of $W(\xi)$ [15]

$$\hat{\Gamma}' = \mathscr{U}(S)^{-1}\hat{\Gamma}\mathscr{U}(S) \Leftrightarrow W'(\xi) = W(S\xi). \tag{7.2}$$

This behaviour of $W(\xi)$ may in fact be regarded as the key or characteristic property of the WWM method in quantum mechanics; we may say that this description of operators is covariant under the full symplectic group $\mathrm{Sp}(2n, \mathscr{R})$.

Next we turn to the diagonal coherent state description of operators $\hat{\Gamma}$ [16]. For n degrees of freedom the coherent states are defined as usual by

$$\begin{aligned} |\mathbf{z}\rangle &= \exp\left\{ -\frac{1}{2}\sum_{r=1}^{n} |z_r|^2 + \sum_{r=1}^{n} z_r \hat{a}_r^\dagger \right\} |0\rangle, \\ \hat{a}_r|0\rangle &= 0, \\ \hat{a}_r|\mathbf{z}\rangle &= z_r|\mathbf{z}\rangle, \quad z_r \in C, \quad \mathbf{z} = (z_1, \ldots, z_n). \end{aligned} \tag{7.3}$$

These are normalised states, no two being orthogonal, and can be written also as the result of phase space displacement operators acting on the ground state $|0\rangle$:

$$\begin{aligned} |\mathbf{z}\rangle &= \exp\left\{ \sum_{r=1}^{n} (z_r \hat{a}_r^\dagger - z_r^* \hat{a}_r) \right\} |0\rangle. \\ \langle \mathbf{z}'|\mathbf{z}\rangle &= \exp\{-\tfrac{1}{2}z'^\dagger z' - \tfrac{1}{2}z^\dagger z + z'^\dagger z\}. \end{aligned} \tag{7.4}$$

The coherent states form an (over) complete set; the resolution of the identity

$$1 = \int \prod_{r=1}^{n} \frac{\mathrm{d}^2 z_r}{\pi} |\mathbf{z}\rangle\langle\mathbf{z}| \tag{7.5}$$

shows that any vector $|\psi\rangle$ can certainly be expanded using them

$$\begin{aligned} |\psi\rangle &= \int \prod_{r=1}^{n} \frac{\mathrm{d}^2 z_r}{\pi} \psi(\mathbf{z}^*)|\mathbf{z}\rangle, \\ \psi(\mathbf{z}) &= \langle \mathbf{z}^*|\psi\rangle = \exp(-\tfrac{1}{2}z^\dagger z) \text{ (entire analytic function of } \mathbf{z}). \end{aligned} \tag{7.6}$$

Moreover the overcompleteness allows expansion of any operator $\hat{\Gamma}$ in the form of an integral over projections to these states

$$\hat{\Gamma} = \int \prod_{r=1}^{n} \frac{\mathrm{d}^2 z_r}{\pi} \phi(\mathbf{z})|\mathbf{z}\rangle\langle\mathbf{z}|. \tag{7.7}$$

For given $\hat{\Gamma}$, this expansion and the weight function $\phi(\mathbf{z})$ are unique, however the latter could in general be a distribution of quite a singular kind.

Now we look at the behaviour under $\mathrm{Sp}(2n, \mathscr{R})$. It turns out that the states $|\mathbf{z}\rangle$ have a simple behaviour only under the "passive" maximal compact subgroup $\mathrm{K}(n) \subset \mathrm{Sp}(2n, \mathscr{R})$

$$\mathrm{U} = X - iY \in \mathrm{U}(n): \mathscr{U}(S(X, Y))|\mathbf{z}\rangle = |\mathrm{U}\mathbf{z}\rangle. \tag{7.8}$$

This can be traced to the fact that the generators of $\mathrm{K}(n)$ involve only terms of the form $\hat{a}^\dagger \hat{a}$, as we see in (6.3a). On the other hand, the effect of $\mathscr{U}(P)$, for any $P \in \Pi(n)$, on $|\mathbf{z}\rangle$

involves an integration using a suitable kernel, namely the generalised Huyghens kernel expressed in the coherent state language [17], which for the $Sp(2,\mathscr{R})$ case can be written in a simple form in terms of complex SU(1,1) parameters $\lambda\mu$

$$\mathscr{K}(z,z';S^c) = \langle z'|\mathscr{U}(S)|z\rangle$$

$$= \zeta(\varphi/2)^* \exp\left\{-\frac{z^2}{2} - \frac{z'^{*2}}{2} - |z'|^2\right\}$$

$$\times \begin{cases} \dfrac{1}{(2|\mathrm{Im}(\mu-\lambda)|)^{1/2}}\left(1-\dfrac{\lambda+\mu^*}{\lambda^*+\mu}\right)^{1/2}\left(1+\dfrac{\mu}{\lambda^*}\right)^{1/2} \\ \times \exp\left[\dfrac{1}{2\lambda^*}\{(\lambda^*+\mu)z^2 + zz'^* + (\lambda^*-\mu^*)z'^{*2}\}\right], \\ \mathrm{Im}(\mu-\lambda)\neq 0, \quad \text{i.e.,} \quad \varphi\neq 2n\pi; \\ \left(\dfrac{2|\mathrm{Re}(\lambda+\mu)|}{1+(\lambda+\mu)\mathrm{Re}(\lambda+\mu)}\right)^{1/2}\exp\left[\dfrac{(z\,\mathrm{Re}(\lambda+\mu)+z'^*)^2}{1+(\lambda+\mu)\mathrm{Re}(\lambda+\mu)}\right], \\ \mathrm{Im}(\mu-\lambda)=0, \quad \text{i.e.,} \quad \varphi = 2n\pi, \quad n=-1,0,1, \text{ or } 2, \end{cases}$$

$$S=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in Sp(2,\mathscr{R}), \quad S^c=\begin{pmatrix}\lambda & \mu \\ \mu^* & \lambda^*\end{pmatrix}, \quad \begin{array}{l}\lambda=\frac{1}{2}(a+d+ic-ib)\\ \mu=\frac{1}{2}(a-d+ib+ic).\end{array} \tag{7.9}$$

(Here each individual square root is defined to have a positive real part.)

In contrast to the WWM result (7.2), we now have covariance under K(n) alone

$$\hat{\Gamma}' = \mathscr{U}(S(X,Y))^{-1}\hat{\Gamma}\mathscr{U}(S(X,Y)) \Leftrightarrow \phi'(\mathbf{z}) = \phi(\mathrm{U}\mathbf{z}) \tag{7.10}$$

Active elements of $Sp(2n,\mathscr{R})$ change ϕ in a manner involving a nontrivial integral transformation.

8. Quantum noise matrices and their $Sp(2n,\mathscr{R})$ behaviour

Let $\hat{\rho}$ be the density operator of any (pure or mixed) quantum state. For simplicity alone let us assume that the means of $\hat{\xi}_a$ vanish

$$\langle\hat{\xi}_a\rangle = \mathrm{Tr}(\hat{\rho}\hat{\xi}_a) = 0. \tag{8.1}$$

Nonzero values for these means can always be reinstated by a suitable phase space displacement. The variance or noise or second order moment matrix of the state $\hat{\rho}$ is then defined as follows [12]

$$V = (V_{ab}) = \begin{pmatrix} V_1 & V_2 \\ V_2^T & V_3\end{pmatrix}.$$

$$V_{ab} = V_{ba} = \tfrac{1}{2}\langle\{\hat{\xi}_a,\hat{\xi}_b\}\rangle$$

$$= \int d^{2n}\xi\,\xi_a\xi_b\,W(\xi);$$

$$(V_1)_{rs} = \langle \hat{q}_r \hat{q}_s \rangle.$$
$$(V_2)_{rs} = \tfrac{1}{2}\langle \{\hat{q}_r, \hat{p}_s\} \rangle.$$
$$(V_3)_{rs} = \langle \hat{p}_r \hat{p}_s \rangle. \tag{8.2}$$

This is a real symmetric $2n \times 2n$ positive definite matrix subject to further matrix inequalities which express the uncertainty principles (see below).

We must note the compact way in which we are able to express V_{ab} as a phase space integral involving the WWM representative $W(\xi)$ of the density operator $\hat{\rho}$. This is a consequence of the general rule [14]

$$\mathrm{Tr}(\hat{\rho}\exp\{i\xi_0^T \beta \hat{\xi}\}) = \int d^{2n}\xi \quad W(\xi)\exp\{i\xi_0^T \beta \xi\} \tag{8.3}$$

valid for any numerical ξ_0. Indeed, one can regard this property of $W(\xi)$ as being as basic as the symplectic transformation rule (7.2). A particular case of (8.3), when $\xi_0 = 0$, shows that $W(\xi)$ is normalised, because $\mathrm{Tr}\,\hat{\rho} = 1$; its phase space integral is unity. But we must remember that though for hermitian $\hat{\rho}$ the function $W(\xi)$ is real, it may not be non-negative everywhere.

The change in the noise matrix V when $\hat{\rho}$ is changed by a symplectic transformation is now easily obtained by exploiting (7.2) along with the above expression for V_{ab} in terms of $W(\xi)$. We have the extremely simple law, for any $S \in \mathrm{Sp}(2n, \mathscr{R})$

$$\hat{\rho}' = \mathscr{U}(S)\hat{\rho}\,\mathscr{U}(S)^{-1} \Rightarrow V' = SVS^T. \tag{8.4}$$

(Note that in comparison to the change $\hat{\Gamma} \to \hat{\Gamma}'$ in (7.2), S has been replaced by S^{-1} here). We may say that V undergoes a symmetric symplectic transformation, which preserves its symmetry and positive definiteness.

The information contained in V can also be given in complex form using second order moments of $\hat{a}$'s and $\hat{a}^\dagger$'s

$$V^{(c)} = (V^{(c)}_{ab}) = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{B}^* & \mathscr{A}^* \end{pmatrix},$$
$$V^{(c)}_{ab} = V^{(c)*}_{ba} = \tfrac{1}{2}\langle \{\hat{\xi}^{(c)}_a, \hat{\xi}^{(c)\dagger}_b\} \rangle;$$
$$\mathscr{A}_{rs} = \mathscr{A}^*_{sr} = \tfrac{1}{2}\langle \{\hat{a}_r, \hat{a}^\dagger_s\} \rangle,$$
$$\mathscr{B}_{rs} = \mathscr{B}_{sr} = \langle \hat{a}_r \hat{a}_s \rangle. \tag{8.5}$$

Thus $V^{(c)}$ is a hermitian $2n \times 2n$ positive definite matrix subject to further uncertainty inequalities given below.

The relations connecting these two forms of the noise matrix are

$$V^{(c)} = \Omega V \Omega^\dagger:$$
$$\mathscr{A} = \tfrac{1}{2}\{V_1 + V_3 + i(V_2^T - V_2)\},$$
$$\mathscr{B} = \tfrac{1}{2}\{V_1 - V_3 + i(V_2^T + V_2)\};$$
$$V_1 = \tfrac{1}{2}\{\mathscr{A} + \mathscr{A}^* + \mathscr{B} + \mathscr{B}^*\},$$

$$V_2 = \frac{i}{2}\{\mathscr{A} - \mathscr{A}^* - \mathscr{B} + \mathscr{B}^*\},$$

$$V_3 = \tfrac{1}{2}\{\mathscr{A} + \mathscr{A}^* - \mathscr{B} - \mathscr{B}^*\}. \tag{8.6}$$

Here the matrix Ω is as given in (4.4). And in place of (8.4) we have the equivalent transformation law

$$\hat{\rho}' = \mathscr{U}(S)\hat{\rho}\mathscr{U}(S)^{-1} \Leftrightarrow V^{(c)\prime} = S^{(c)} V^{(c)} S^{(c)\dagger}. \tag{8.7}$$

9. Williamson's theorem and uncertainty principles

For general $S \in \mathrm{Sp}(2n, \mathscr{R})$ the transformation law (8.4) for the noise matrix is *not* a similarity transformation, it is a similarity only if $S = S(X, Y) \in \mathrm{K}(n)$. Normally we expect that the diagonalization of V will require a matrix belonging to the group $\mathrm{SO}(2n, \mathscr{R})$. However, the fundamental Theorem of Williamson comes to our rescue [18]. This theorem is a complete answer to the question: Given a real symmetric $2n \times 2n$ matrix V, what is the maximum simplification we can achieve in the form of $V' = SVS^T$ by allowing S to vary all over $\mathrm{Sp}(2n, \mathscr{R})$? For general V, the normal or canonical form of V' is not a diagonal form; however the theorem shows that in case V is positive (or negative) definite to begin with, then we can certainly choose an S so that V' is diagonal. By suitable further rescaling and ordering of elements we can then achieve the following

V real symmetric positive definite:

$$SVS^T = \mathrm{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1, \ldots, \kappa_n),$$
$$\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n,$$
$$\text{suitable } S \in \mathrm{Sp}(2n, \mathscr{R}). \tag{9.1}$$

We shall call this the *Williamson normal form of V*. In general, the κ_r are *not* the eigenvalues of V at all. Also note that when V is in this form, then we have $V^{(c)} = V$.

In the Williamson normal form we see that for each canonical pair $\hat{q}_r, \hat{p}_r$ we have equal uncertainties: $\Delta q_r = \Delta p_r = \kappa_r^{1/2}$. Also *all* the off-diagonal variances vanish. Therefore the complete statement of the uncertainty principles for all degrees of freedom would be

$$\kappa_r \geqslant 1/2, \quad r = 1, \ldots, n. \tag{9.2}$$

A given $2n \times 2n$ real symmetric positive definite matrix V is quantum mechanically realizable as the noise matrix of some state $\hat{\rho}$ if and only if in its Williamson normal form (9.1) every diagonal entry is greater than or equal to one-half. This is an $\mathrm{Sp}(2n, \mathscr{R})$ invariant statement. It is naturally useful to express these uncertainty principles directly in terms of V without actually passing to its Williamson normal form [12]. There are several ways of doing this and for illustration we quote just one, expressed both in terms of V and $V^{(c)}$

$$V + \frac{i}{2}\beta = \text{hermitian positive semidefinite.}$$

$$V^{(c)} + \tfrac{1}{2}\Sigma_3 = \text{hermitian positive semidefinite.} \tag{9.3}$$

We emphasize that any $V, V^{(c)}$ obeying these conditions are quantum mechanically realizable – they are necessary and sufficient.

There is a subtle distinction between the matrix V being brought to diagonal form, and the matrix $V^{(c)}$ being diagonal. From the relations (8.6) among the two sets of submatrices we can see easily

$$\begin{aligned} V \text{ diagonal} &\Leftrightarrow V_1, V_3 \text{ diagonal}, V_2 = 0, \\ &\Leftrightarrow \mathscr{A}, \mathscr{B} \text{ real diagonal} \\ &\not\Rightarrow V^{(c)} \text{ diagonal:} \end{aligned} \tag{9.4a}$$

$$\begin{aligned} V^{(c)} \text{ diagonal} &\Leftrightarrow \mathscr{A} \text{ real diagonal}, \mathscr{B} = 0 \\ &\Rightarrow V_1, V_3 \text{ diagonal}, V_2 = 0 \\ &\Leftrightarrow V \text{ diagonal.} \end{aligned} \tag{9.4b}$$

This means that the set of states which have diagonal $V^{(c)}$ is a subset of the set of states having diagonal V: the former is a more restrictive condition, so fewer states obey it. In the particular situation when V is in Williamson normal form, it is not only diagonal but in addition $V_1 = V_3$; thus, in spite of the general statement (9.4a), we do find $V^{(c)}$ also diagonal in that case. In complex form, the relevant consequence of Williamson's Theorem for $V^{(c)}$ is that with the transformation law (8.7), $V^{(c)}$ can be "diagonalized".

10. A U(n)-invariant multimode squeezing criterion

The noise matrix V for any state of a single mode system is two dimensional and has the form

$$V = \begin{pmatrix} (\Delta q)^2 & \Delta(q,p) \\ \Delta(q,p) & (\Delta p)^2 \end{pmatrix} \tag{10.1}$$

with obvious meanings for the various matrix elements. The usual Heisenberg uncertainty relation (with $\hbar = 1$) which reads [19]

$$\Delta q \Delta p \geqslant \tfrac{1}{2} \tag{10.2}$$

can, as is well known, be strengthened to the statement [20]

$$\det V \equiv (\Delta q)^2 (\Delta p)^2 - (\Delta(q,p))^2 \geqslant \tfrac{1}{4}. \tag{10.3}$$

This is in fact the content of the conditions (9.3) in this case, and this is $Sp(2, \mathscr{R})$ invariant.

The state $\hat{\rho}$ with variance matrix V is usually said to be a squeezed state if either one of the two diagonal elements of V (but of course not both) is less than one half. However this definition possesses no useful or interesting continuous invariance at all. A definition of squeezing possessing invariance under the maximal compact U(1) or SO(2) subgroup of $Sp(2, \mathscr{R})$ is this: the state $\hat{\rho}$ is squeezed if and only if the lesser of the two eigenvalues of V is less than one half. A state squeezed in the former sense is squeezed also in the latter sense but not conversely, so the former is more restrictive

$$\Delta q \text{ or } \Delta p < \frac{1}{\sqrt{2}} \Rightarrow \text{lesser eigenvalue of } V < \tfrac{1}{2}. \tag{10.4}$$

The U(1)-invariant squeezing criterion has been used in several studies. It is of course clear that we cannot ask for any more invariance in the squeezing criterion, for example it would be meaningless to think of an $\mathrm{Sp}(2,\mathscr{R})$ invariant squeezing criterion.

Motivated by the above, we now describe a criterion for squeezing for states of n mode systems [12]. Suppose that for a given state $\hat{\rho}$ the noise matrix V already has some diagonal element less than one half. Then we say that the state is manifestly squeezed. However it may happen that every diagonal element V_{aa} of V exceeds or equals one half, yet squeezing is buried and not manifest. We have divided elements of $\mathrm{Sp}(2n,\mathscr{R})$ into passive $\mathrm{U}(n)$ elements and active $\Pi(n)$ elements. We would like to have a definition of squeezing invariant under passive $\mathrm{U}(n)$ transformations. Such a definition is the following

$$\hat{\rho} \text{ is a squeezed state} \Leftrightarrow (S(X,Y)\,V S(X,Y)^T)_{aa} < \tfrac{1}{2},\quad \text{some } X - iY \in \mathrm{U}(n),\quad \text{some } a = 1,2,\ldots,2n. \tag{10.5}$$

Thus, $\hat{\rho}$ is squeezed if V shows it to be manifestly so, or if this happens after a suitable passive transformation.

We know that in general V cannot be diagonalized by similarity transformations within $\mathrm{K}(n)$ – we expect to have to use matrices from the much larger group $\mathrm{SO}(2n,\mathscr{R})$, which may be noncanonical. Nevertheless it is remarkable that the above $\mathrm{U}(n)$-invariant squeezing criterion can be expressed in terms of the eigenvalue spectrum of V

$$\hat{\rho} \text{ is a squeezed state} \Leftrightarrow \quad l(V) = \text{minimum eigenvalue of } V < \tfrac{1}{2}. \tag{10.6}$$

This is precisely equivalent to the definition (10.5).

Obviously the squeezed or nonsqueezed nature of a state $\hat{\rho}$ is unchanged by passive elements of $\mathrm{Sp}(2n,\mathscr{R})$ lying within $\mathrm{K}(n)$, since then V undergoes a similarity transformation which leaves $l(V)$ unaltered. To change $l(V)$, and so the status of a state, in either direction, we *must* use active noncompact elements lying in $\Pi(n)$, if we wish to do so within the framework of $\mathrm{Sp}(2n,\mathscr{R})$ transformations. Thus justifies our using the term "squeezing transformation" for the metaplectic unitary operator $\mathscr{U}(P)$ for elements $P \in \Pi(n)$.

11. Some interesting families of variance matrices

Let us denote by s the set of all allowed noise matrices V, i.e., all physically realizable ones obeying the uncertainty principles

$$s = \left\{ V = 2n \times 2n \text{ real symmetric positive definite} \,\middle|\, V + \frac{i}{2}\beta \text{ positive semidefinite} \right\}. \tag{11.1}$$

This is an $n(2n+1)$ parameter family. We have seen that a general $V \in s$ is not diagonalizable by elements within $\mathrm{K}(n)$. We ask: can we characterize the subset $s_K \subset s$ consisting of all noise matrices which *are* diagonalizable using elements of $\mathrm{K}(n)$? The answer is that this can be done rather elegantly, and for this the complex form $V^{(c)}$ of V is more convenient [12]

$$\begin{aligned} s_K &= \{V \in s \mid \mathscr{A}\mathscr{B} = \text{symmetric}\} \subset s: \\ V \in s_K &\Rightarrow S(X, Y)\, V S(X, Y)^T = \text{diagonal} \\ \text{some } \mathrm{U} &= X - iY \in \mathrm{U}(n). \end{aligned} \tag{11.2}$$

The subset s_K is an $n(n+2)$ parameter family.

There are two further subsets of s_K which are interesting [12]. We call them the "hermitian" family s_H and the "Gaussian" family s_G – they are of dimensions n^2 and $n(n+1)$ respectively. The definition of s_H is motivated by the fact that the general transformation rule (8.7) for $V^{(c)}$ becomes very simple if we have a $\mathrm{K}(n)$ element

$$\begin{aligned} \mathrm{U} \in \mathrm{U}(n)\!: V^{(c)} &\to S^{(c)}(\mathrm{U})\, V^{(c)} S^{(c)}(\mathrm{U})^\dagger. \\ \mathscr{A} &\to \mathrm{U}\mathscr{A}\mathrm{U}^{-1}. \\ \mathscr{B} &\to \mathrm{U}\mathscr{B}\mathrm{U}^T. \end{aligned} \tag{11.3}$$

So if $\mathscr{B} = 0$ to begin with, it remains zero; and $\mathscr{A}$ being hermitian can be diagonalized by some U. This would then result in both $V^{(c)}$ and V becoming diagonal. Thus, including the uncertainty conditions (9.3), the definition of s_H is

$$s_H = \{V \in s_K \mid \mathscr{B} = 0, \mathscr{A} - \tfrac{1}{2}\cdot 1 \text{ positive semidefinite}\} \subset s_K \subset s. \tag{11.4}$$

One can verify that *no* member of this family s_H is squeezed.

Next to the family s_G. Here the definition involves the "noncompact" subset $\Pi(n)$ of $Sp(2n, \mathscr{R})$

$$s_G = \{V \in s_K \mid 2V \in \Pi(n) \subset Sp(2n, \mathscr{R})\} \subset s_K \subset s. \tag{11.5}$$

It is a fact that such noise matrices are physically realizable, i.e., they obey the uncertainty conditions, and are diagonalizable within $\mathrm{K}(n)$. Further, except for $V = \frac{1}{2}\cdot 1$, *every other* $V \in s_G$ *is squeezed*. So we see incidentally that this is the only common element in s_H and s_G. All in all, we have

$$\begin{aligned} s_H, s_G &\subset s_K \subset s \\ s_H \cap s_G &= \{\tfrac{1}{2}\cdot 1\}. \end{aligned} \tag{11.6}$$

12. Gaussian pure states, Gaussian Wigner distributions

The generators of the metaplectic unitary representation of $Sp(2n, \mathscr{R})$ are quadratics in $\hat{q}$'s and $\hat{p}$'s. The most general centred n-mode Gaussian wave function involves a quadratic in the q's in the exponent. It turns out that the former act in very nice and compact ways on the latter [21]. We describe the main features briefly in this Section.

A general centred and normalized n-mode Gaussian pure state can be parameterized by two real symmetric $n \times n$ matrices u and v, of which the former is positive definite

$$\begin{aligned} \psi_{(u,v)}(\mathbf{q}) &= \pi^{-n/4} (\det u)^{1/4} \exp\{-\tfrac{1}{2} q^T (u + iv) q\}. \\ \int d^n q\, |\psi_{(u,v)}(\mathbf{q})|^2 &= 1. \end{aligned} \tag{12.1}$$

For $(u,v)=(1,0)$ we get the ground state of the isotropic oscillator in n-dimensions

$$\psi_{(1.0)}(\mathbf{q})=\pi^{-n/4}\exp(-\tfrac{1}{2}q^Tq),$$
$$\hat{a}_r\psi_{(1,0)}=0. \tag{12.2}$$

The calculation of the WWM representative for $\psi_{(u,v)}(\mathbf{q})$, and of the noise matrix, are easy since only Gaussian integrals and moments are involved. The results are

$$\begin{aligned}W_{(u,v)}(\xi)&=(2\pi)^{-n}\int d^nq'\psi_{(u,v)}(\mathbf{q}-\tfrac{1}{2}\mathbf{q}')\psi_{(u,v)}(\mathbf{q}+\tfrac{1}{2}\mathbf{q}')^*\exp(i\mathbf{q}'\cdot\mathbf{p})\\&=\pi^{-n}\exp\{-\xi^TG(u,v)\xi\}.\\G(u,v)&=\begin{pmatrix}u+vu^{-1}v & vu^{-1}\\ u^{-1}v & u^{-1}\end{pmatrix}\\&=(S(u,v)^{-1})^TS(u,v)^{-1}\\S(u,v)&=\begin{pmatrix}u^{-1/2} & 0\\ -vu^{-1/2} & u^{1/2}\end{pmatrix}\in\mathrm{Sp}(2n,\mathscr{R});\\V(u,v)&=\tfrac{1}{2}G(u,v)^{-1}=\tfrac{1}{2}S(u,v)S(u,v)^T\in s_G.\end{aligned} \tag{12.3}$$

While the calculations leading to these results are elementary, it is worth paying attention to the structures involved. The WWM representative $W_{(u,v)}(\xi)$ is expected to be a Gaussian, with a positive definite matrix $G(u,v)$ in the exponent. What is interesting is the factorization of this matrix in terms of an $\mathrm{Sp}(2n,\mathscr{R})$-matrix $S(u,v)$; an added feature is that this $S(u,v)$ is an example of the product of the first two factors in the general pre-Iwasawa decomposition (4.21) for any $S\in\mathrm{Sp}(2n,\mathscr{R})$! That the noise matrix $V(u,v)$ should be essentially the inverse of $G(u,v)$ is clear from (8.2); it is then the product structure for $G(u,v)$ that results in $V(u,v)$ being an element of the family s_G defined in the previous Section. Incidentally we also see that $\Pi(n)\subset\mathrm{Sp}(2n,\mathscr{R})$ defined in (4.18) and used in defining the family s_G in (11.4) can be described more explicitly using $S(u,v)$

$$\begin{aligned}\Pi(n)=\{&S(u,v)S(u,v)^T\,|\,S(u,v)\in\mathrm{Sp}(2n,\mathscr{R})\\&u\text{ and }v\text{ real symmetric }n\times n\text{ matrices, }u\text{ positive definite}\}.\end{aligned} \tag{12.4}$$

We can argue further along similar lines, concerning the form to be expected for the action of an operator $\mathscr{U}(S)$ on any $\psi_{(u,v)}(\mathbf{q})$. From (7.2) it is clear that $W_{(u,v)}(\xi)$ must get mapped onto another Gaussian; the fact that it arises from a pure state wave function must also be retained; and the transformation rule (8.4) plus the explicit form of $V(u,v)$ in (12.3) means that any $\psi_{(u,v)}(\mathbf{q})$ can be mapped onto $\psi_{(1,0)}(\mathbf{q})$ by a suitable $\mathscr{U}(S)$. All this is indeed true. We find that

$$\begin{aligned}\psi_{(u,v)}&=(\text{phase factor})\mathscr{U}(S(u,v))\psi_{(1,0)},\\V(u,v)&=S(u,v)\,V(1,0)S(u,v)^T.\\V(1,0)&=\tfrac{1}{2}\cdot 1.\end{aligned} \tag{12.5}$$

So $\mathrm{Sp}(2n, \mathscr{R})$ acts *transitively* on the set of Gaussian pure states. The subgroup of $\mathrm{Sp}(2n, \mathscr{R})$ leaving the particular state $\psi_{(1,0)}$ invariant (apart from phases) is just $\mathrm{K}(n) - \hat{a}_r$ annihilates $\psi_{(1,0)}$, and the generators of $\mathrm{K}(n)$ in the metaplectic representation are of the form $\hat{a}_r^\dagger \hat{a}_s$. More precisely, the situation may be described as follows. One can easily establish that the behavior of $\psi_{(1,0)}$ under $\mathrm{K}(n)$ is given by

$$\mathrm{U} = X - iY \in \mathrm{U}(n):$$

$$\mathscr{U}(S(X, Y))\psi_{(1,0)} = \sqrt{\det \mathrm{U}}\,\psi_{(1,0)}. \tag{12.6}$$

The sign ambiguity here explains the appearance of the metaplectic group $\mathrm{Mp}(2n)$. The stability group of $\psi_{(1,0)}$ – the subgroup of $\mathrm{Sp}(2n, \mathscr{R})$ leaving this vector strictly invariant is thus the $n^2 - 1$ parameter subgroup $\mathrm{SU}(n) \in U(n)$. Correspondingly the orbit of $\psi_{(1,0)}$ under $\mathrm{Sp}(2n, \mathscr{R})$, made up of the vectors $\mathscr{U}(S)\psi_{(1,0)}$ for all $S \in \mathrm{Sp}(2n, \mathscr{R})$, is an $(n(n+1)+1)$ parameter family. It consists of the vectors $e^{i\alpha}\psi_{(u,v)}$ for $0 \leqslant \alpha < 2\pi$ and all allowed (u, v). This orbit is essentially the coset space $\mathrm{Sp}(2n, \mathscr{R})/\mathrm{SU}(n)$; there is a one-to-one correspondence between vectors $e^{i\alpha}\psi_{(u,v)}$ and points in this space. Suppressing the phase we can next say that the set of density matrices $\psi_{(u,v)}\psi^\dagger_{(u,v)}$, or equally well the set of representative vectors $\psi_{(u,v)}$, is essentially the coset space $\mathrm{Sp}(2n, \mathscr{R})/\mathrm{U}(n)$.

Turning next to the effect of $\mathscr{U}(S)$ on $\psi_{(u,v)}$, we see that apart from a phase factor it has to result in $\psi_{(u',v')}$ for suitable u' and v'. The formula for this change is a beautiful one

$$\begin{aligned} \mathscr{U}(S)\psi_{(u,v)} &= (\text{phase factor})\psi_{(u',v')}; \\ \wedge &= (iu - v)^{-1} \to \wedge' = (iu' - v')^{-1} \\ &= (A\wedge + B)(C\wedge + D)^{-1}. \end{aligned} \tag{12.7}$$

The Gaussian WWM function in (12.3) arose from a pure state. Suppose now we consider a general centred normalized Gaussian phase space distribution with a general parameter matrix G

$$\begin{aligned} W_G(\xi) &= \pi^{-n}(\det G)^{1/2}\exp(-\xi^T G \xi), \\ G &= \text{real symmetric } 2n \times 2n \text{ positive definite matrix}. \end{aligned} \tag{12.8}$$

The question is: When is this a WWM function corresponding to some pure or mixed quantum state? Thanks to Williamson's theorem, the answer is elementary [15]. The noise matrix would clearly be

$$V = \tfrac{1}{2}G^{-1}. \tag{12.9}$$

So we start from the given Gaussian $W_G(\xi)$, pass to its V, then go to the Williamson normal form (9.1) of V

$$V \to \mathrm{diag}(\kappa_1, \ldots, \kappa_n, \kappa_1, \ldots, \kappa_n). \tag{12.10}$$

and then demand that each κ_r be greater than or equal to one half. This is a complete necessary and sufficient condition for $W_G(\xi)$ to be a bonafide WWM phase space distribution; as we have seen, however, this condition can be stated directly without going to the normal form

$$W_G(\xi) \text{ is a WWM distribution} \Leftrightarrow G^{-1} + i\beta = \text{hermitian positive semidefinite.} \quad (12.11)$$

13. Concluding remarks

In this review we have tried to convey the main features of the family of real symplectic groups $\mathrm{Sp}(2n, \mathscr{R})$, and have outlined some problems in optics and quantum mechanics where they are useful. Our account has been descriptive and suggestive, omitting detailed proofs of various statements made. We believe that any interested reader wishing to apply symplectic techniques to any concrete problem would be well equipped for the purpose, and able to supply necessary details.

Some general remarks – partly to counter apparently common misconceptions – may be useful at this stage. The group $\mathrm{Sp}(2n, \mathscr{R})$ comes in when we define *linear* canonical transformations on given canonical variables. It is the fact that the commutation relations and hermiticity are maintained that is responsible for the existence of unitary operators $\mathscr{U}(S)$ implementing these transformations. In particular, $\mathscr{U}(S)$ is unitary whether $S \in K(n)$, when S itself is unitary, or $S \notin K(n)$, for example $S \in \Pi(n)$ in which case S is hermitian rather than unitary. Correspondingly, the "Hamiltonians" generating these unitary operators $\mathscr{U}(S)$ are hermitian quadratics in the $\hat{q}$'s and $\hat{p}$'s. For the case of the most general unitary evolution via a general hermitian Hamiltonian, the action on $\hat{q}$'s and $\hat{p}$'s (equivalently on $\hat{a}$'s and $\hat{a}^{\dagger}$'s) definitely preserves the canonical commutation relations and hermiticity properties, but the transformed operators may not be linear combinations of the original ones.

The maximal compact subgroup $\mathrm{U}(n)$ of $\mathrm{Sp}(2n, \mathscr{R})$ has naturally played an important role in our considerations. The n-mode squeezing criterion described in § 10 has a built-in $\mathrm{U}(n)$ invariance. As a result. for a state with a given variance matrix V, squeezing if present may be manifest (one of the diagonal elements of V is less than 1/2) or may be hidden (this happens only after a suitable $\mathrm{U}(n)$ transformation). This makes it clear that our squeezing criterion is weaker than the usual one stated directly in terms of the diagonal elements of V, since these would rule out the hidden case. Correspondingly there are more states which are squeezed by our criterion than by the usual one, which in any case has much less invariance built in.

The metaplectic group is essential to describe properly the structure of the unitary operators implementing linear canonical transformations on the canonical variables (in this respect the notation $\mathscr{U}(S)$ with $S \in \mathrm{Sp}(2n, \mathscr{R})$ is inadequate – the argument in $\mathscr{U}(S)$ should be an element of $\mathrm{Mp}(2n)$). The importance of this group is seen in, for example, the calculation of geometric phases for cyclic evolution of squeezed states, the interpretation of the Guoy phase, etc [22]. It has also been shown elsewhere that the metaplectic group is relevant in setting up operator Mobius transformations for one degree of freedom [17].

Work on extending $\mathrm{U}(n)$ invariant notions beyond quadrature squeezing is in progress and will be reported elsewhere. Thus, for two-mode systems, one can classify squeezing transformations into $\mathrm{U}(2)$ invariant equivalence classes, study bunching and antibunching with such invariance, etc. [13]. These notions can be generalized to n mode systems as well.

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