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ON THE ORBITS IN THE LIE ALGEBRAS OF SOME (PSEUDO) ORTHOGONAL GROUPS

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A complete classification of the orbits in the Lie algebras of all the real orthogonal and pseudo-orthogonal groups of total dimension not exceeding five is presented. The classification is carried out using elementary geometrical methods, exhibiting in a clear way the relevance of the results for a lower dimensional group in obtaining the results for a higher dimensional one. For each orbit the values of the algebraic invariants are calculated, a representative element is displayed, and the geometric nature of the latter is described by listing a complete set of independent vectors invariant under it. While the orbit structure for the orthogonal groups turns out to be relatively simple, that for the Lorentz type and the de Sitter type pseudo-orthogonal groups become progressively complex. Particular care has been taken, in view of the intricacy of many of the results, to develop a suggestive and systematic notation. The orbits are classified and tabulated in a form that makes it particularly easy to apply them in practical physical problems. Examples of such problems are pointed out.

1. INTRODUCTION

The real orthogonal and pseudo-orthogonal groups of low dimensions play an important role in a variety of problems of physical interest. Thus, for instance, problems possesing spherial spmmetry in three dimensions involve the group SO (3) in their analysis[1,2]. Physical systems subject to the requirements of special relativity similarly involve the (homogeneous orthochronous) Lorentz group SO (3, 1), and in suitable kinematical situations also the important subgroups SO (3) and SO (2.1)[1,2]. The latter subgroup, SO (2,1), is closely related to the two-dimensional real unimodular group SL (2,R) which is the same as the real symplectic two-dimensional group Sp (2, R) relevant in Hamiltonian mechanics[1,2]. Thus SL (2, R) is the group of linear canonical transformations on one canonical pair of Hamiltonian variables; and as is well-known, there is a two-to-one homomorphism from SL (2, R) to SO (2,1). Similarly, when one considers problems involving two canonical pairs of variables on a four dimensional phase space, the group of linear canonical transformations is the symplectic group Sp (4, R), which is a double covering of the real pseudo-orthogonal de Sitter group SO (3, 2) in five dimensions. Physical problems in which SO (2, 1) and SO(3, 2) play a significant role on account of their relation to canonical transformations are many, among which we mention here the following as examples : Fourier optics in the paraxial limit and the related study of ideal optical systems[3-8]; description and propagation of optical Gaussian Schell-model beams[9,10]; squeezed coherent states[11-15] and two-photon coherent states[16-18] in quantum optics; the representation theory of para-Bose operator algebras[19-28] and studies of particles with internal structure[29] based on the new Dirac equation[30,31]. Of course the relevance of the de Sitter groups SO (3,2) and SO (4, 1) in the context of certain linear relativistic quantum mechanical wave equations has been long appreciated; thus the tensor and vector matrices associated with the original Dirac equation[32] generate a de Sitter algebra, and a corresponding statement is true in the case of the infinite component Majorana equations [33-35] as well as with the well-known Bhabha equations[36].

For most practical physical applications, it is adequate and convenient to work initially with the elements of the Lie algebras of these groups, and later by a process of integration or exponentiation to arrive at finite group elements. This is particularly true in dealing with the linear (unitary or nonunitary) representations of these groups. In the case of the simplest group SO (3), it is a well-known and geometrically evident fact that all (infinitesimal) generators are basically alike, differing from one another only in orientation and overall magnitude. This is the essential content of Euler's theorem[37] which states that every rotation in three dimensions leaves one direction invariant, and so is a rotation through some angle about that direction as axis. However, when one goes to higher dimensions, or alters the signature of the metric, or both, the elements of the Lie algebra separate into many essentially distinct types, with quite different geometrical properties. This situation can be expressed in the following way: The Lie algebra **G** of any one of the groups G under consideration, viewed as a linear vector space, carries a particular representation of G, namely the adjoint representation. Two vectors in the Lie algebra, i. e. two infinitesimal generators, which are connected by some transformation in the adjoint representation may be regarded as being essentially equivalent and not differing from one another in any intrinsic manner. Starting with any element in the Lie algebra and subjecting it to all the transformations of the adjoint representation, one builds up the orbit on which the starting element lies. The entire Lie algebra **G** thus splits into distinct and mutually disjoint orbits under the adjoint action. (Of course all the preceding statements are valid for any Lie group, not just the ones we are concerned with here).. While for the group SO(3) all orbits in its Lie algebra **SO (3)** (except the trivial one) are basically similar in structure, this is not so in the other cases, and one does find significantly different kinds of elements and therefore of orbits in the Lie algebra. Examples of this situation are of course familiar in the context of special relativity, where generators of spatial rotations and of pure Lorentz transformations are the opposite ends of a spectrum of possibilities.

It is the purpose of this paper to provide a complete classification of the orbits in the Lie algebras of all the real orthogonal and pseudo-orthogonal groups of total dimension not exceeding five. Our aim is to make use of elementary geometrical methods in obiaining this classification, and also to exhibit in the clearest possible way the relevance of the the results for a lower dimensional groups in obtaining the results for a higher dimensional one. We endeavour to derive and present our results in a manner that makes it particularly easy to apply them in practical physical problems. We shall be concerned with the orthogonal groups SO (n) for $n = 3, 4, 5$; the Lorentz pseudo-orthogonal groups SO (n, 1) for $n = 2, 3, 4$; and the "de Sitter" type pseudo-othogonal groups SO (n,2) for $n = 2, 3$. Thus there are eight groups included in our study, divided in the above manner into three distinct sets. In some cases, such as SO(3) and possibly also SO (2, 1), the classification of and the geometrical nature of the elements on each orbit is well known. Nevertheless, for the sake of completeness and the setting up of uniform notations, we shall include all cases in the analysis, the familiar ones being dealt with only briefly.

The material of this paper is organised as follows. In section 2 the three groups SO (n) are taken up, in the sequence $n = 3, 4,5$. For the treatment of **SO (4)**, the decompositions SO (4) $\simeq$ SO (3) $\otimes$ SO (3) (locally) and **SO (4)** = **SO (3)** $\oplus$ **SO (3)** are exploited. Section 3 treats the Lorentz type grous SO(n, 1) for $n = 2,3$ and 4. In the last of these, namely in classifying the orbits in **SO (4,1)**, it is necessary in one case to deal with an E (3) subgroup of SO (4, 1), and its Lie algebra. Section 4 classifies orbits in the two "de Sitter" type algebras **SO** (2,2), **SO (3,2)**. For the former, the decomposition **SO (2.2)** = **SO (2,1)** $\otimes$ **SO (2,1)** in exploited. Since the number of different types is quite large in these two cases, the results are presented in two separate tables (corresponding to ranks 2 and 4 respectively) in each case. The **SO(3,2)** analysis involves, in a particular situation, use of an **E (2,1)** subalgebra. The paper concludes in section 5 with some general comments.

As mentioned earlier, in order to make the results more transparent and useful and to clarify the relationships between the structures for different groups, we will express the orbit classifications for the various groups in a mutually compatible manner. This means that the notation, in particular the choices of indices labelling components of vectors, tensors, .. and their ranges, must be chosen judiciously. We now explain the choices which we shall adhere to throughout the paper. For the two groups SO(3), SO (2,1) operating on three-dimensional spaces, we use lower case Latin letters a, b, c, ... as indices for components of vectors, tensors, etc. For the three groups SO(4), SO (3,1), SO (2.2) operating on four-dimensional spaces the lower case Greek letters λ, μ, ν, ... will be used. For the three groups SO (5), SO (4,1), (SO (3,2) acting on five-dimensional spaces, the capitan Latin letters A, B, C, ... will be used. Turning to the ranges of indices for the orthogonal groups SO(n) the dimensions will be numbered 1, 2, ... 3. Thus for SO (3) the indices a, b, ... run over 1, 2, 3; for SO (4) the indices λ, μ, ... go from 1 to 4; and for SO (5), A, B, ... run over 1, 2, ..., 5. For these three groups, the metric tensor is just the kronecker symbol δ_{ab}, $\delta_{\mu\nu}$ or δ_{AB}. For the Lorentz type groups SO (n, 1) the dimensions will be numbered 0, 1, 2, 3, 4. The metric tensor $g_{..}$ will be diagonal and "space like": $g_{00} = -1$, $g_{11} = \ldots g_{44} = 1$. In the discussion of SO (2,1), we will let a, b, .. run over 0, 1, 2; for SO (3,1) we shall have μ, ν,... going over 0, 1, 2, 3; and for SO (4, 1), A, B, ... will span the full range 0, 1, ..., 4. For the "de Sitter" type groups SO (n, 2) we number the dimensions as 0, 1, 2, 3, 5 omitting the numeral 4. This is in fact the convention often used in physical problems where SO (3,2) in relevant. The metric tensor will again be diagonal and "space like": $g_{00} = g_{55} = -1$, $g_{11} = g_{22} = g_{33} = 1$. (There is a minor mismatch here in that the dimension 5 carries a positive signature in the context of SO (5) but a negative signature in the case of SO (n, 2); however this will not cause any serious problem). For SO (2,2) we let μ, ν, ... go over 0, 1, 2, 5; and for SO (3,2) we have A, B, ... going over the full range 0, 1, 2, 3, 5. With these conventions, the appearance af indices a, b, ... will immediately signify that we are dealing with "three dimensional quantities"; whether the relevant group is SO (3) or SO (2, 1) will be clear from the context. Similarly the appearance of indices μ, ν, ... will signify that "four-dimensional objects" are involved, and so on. The generic symbol $J_{..}$ will be a basis element for any one of the Lie algebras : thus J_{ab} for SO (3) and SO (2,1); $J_{\mu\nu}$ for SO (4), SO (3,1) and SO (2,2); and J_{AB} for SO (5), SO (4,1) and SO (3,2). In all cases we have antisymmetry in the subscripts. The components of a general element in the Lie algebra will be $\xi^{..}$, with antisymmetry in the superscripts. Thus the Lie algebra element will be

$$J(\xi) = \tfrac{1}{2}\,\xi^{ab} J_{ab} \text{ or } \tfrac{1}{2}\,\xi^{\mu\nu} J_{\mu\nu} \text{ or } \tfrac{1}{2}\xi^{AB} J_{AB} \qquad \ldots(1.1)$$

with all indices being lowered in the case of SO (n). The quadratic invariant will uniformly be denoted by $\mathscr{C}_1(\xi)$

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi^{ab}\,\xi_{ab} \text{ or } \tfrac{1}{2}\,\xi^{\mu\nu}\,\xi_{\mu\nu} \text{ or } \tfrac{1}{2}\xi^{AB}\,\xi_{AB}. \qquad \ldots(1.2)$$

For the cases of groups in four or five dimensions, there is a second algebraic invariant $\mathscr{C}_2(\xi)$ which will be defined at the appropriate places. It is easily constructed once one realises that, in all cases, the adjoint representation transforms $\xi^{..}$ as a second rank antisymmetric tensor under the appropriate orthogonal or pseudo-orthogonal rotation group. Finally the symbols e_a e_μ, e_A will be used for a basic set of mutually orthogonal unit vectors in three, four or five dimensions respectively; while the letters t, l and s (relevant only for the SO (n, 1) and SO (n, 2) analyses) will stand for general "timelike" "lightlike" and "spacelike" vectors.

2. Orbits in the Lie Algebras SO (n) n, = 3, 4, 5

The generators J_{ab} of SO (3) obey the familiar Lie bracket relations

$$[J_{ab}, J_{cd}] = \delta_{ac} J_{bd} - \delta_{bc} J_{ad} + \delta_{ad} J_{cb} - \delta_{cd} J_{ca},\ a, b, \ldots = 1, 2, 3. \qquad \ldots(2.1)$$

The Lie relations for SO (4) and SO (5) (indeed, for any SO (n)) are similar, with a, b, c, d replaced by μ, ν, ρ, σ or A, B, C, D and the ranges of the indices suitably extended:

they need not be written down explicitly. We now briefly review the orbit structure in SO (3), then take up the cases of SO (4) and SO (5).

SO (3) A general element in the Lie algebra is as in eqn. (1.1)

$$J(\xi) = \tfrac{1}{2}\,\xi_{ab}\,J_{ab}. \qquad \text{...(2.2)}$$

With the use of the three-index antisymmetric symbol, both J_{ab} and ξ_{ab} can be replaced by single index "vector" quantities

$$J_a = \tfrac{1}{2}\,\epsilon_{abc}\,J_{bc},\ \ \xi_a = \tfrac{1}{2}\,\epsilon_{abc}\,\xi_{bc};$$

$$J_{ab} = \epsilon_{abc}\,J_c,\ \ \xi_{ab} = \epsilon_{abc}\,\xi_c;$$

$$J(\xi) = \xi_a\,J_a. \qquad \text{...(2.3)}$$

We treat ξ_a as the components of a three-vector ξ. In terms of J_a, eqn. (2.1) takes the familar from

$$[J_a, J_b] = \epsilon_{abc}\,J_c. \qquad \text{...(2.4)}$$

The quadratic invariant $\mathscr{C}_1(\xi)$, the only one in the case of SO (3), is the squared length of ξ:

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi_{ab}\,\xi_{ab} = \xi_a\,\xi_a = |\,\xi\,|^2 \qquad \text{...(2.5)}$$

If $\delta\theta$ is a small parameter, the effect on a general vector z_a of an infinitesimal transformation generated by $J(\xi)$ is

$$\delta z_a = \delta\theta\,\xi_{ab}\,z_b = -\,\delta\theta\,(\xi_\wedge z)a. \qquad \text{...(2.6)}$$

Thus ξ itself is invariant under the rotations generated by $J(\xi)$. This can be understood as a matrix property which we later generalise to higher dimensional groups. The three-dimensional real antisymmetric matrix (ξ_{ab}) is necessarily of rank 2, since we exclude $\xi = 0$; it therefore has exactly one null eigenvector, namely ξ itself, which is therefore invariant under the rotations generated by $J(\xi)$[38].

The adjoint action of SO (3) on ξ, as is well known, amounts to subjecting ξ to a three-dimensional rotation. It is conveniently represented via the spin 1/2 representation of SO (3), which also leads to the defining representation of SU (2). In it, the generators J_a are the Pauli matrices :

$$J_a \to \frac{-i}{2}\,\sigma_a,\quad a = 1,2,3. \qquad \text{...(2.7)}$$

$J(\xi)$ is a general traceless antihermitian 2×2 matrix :

$$J(\xi) = \frac{-i}{2}\,\xi.\,\sigma. \qquad \text{...(2.8)}$$

For any $U \in$ SU (2), the adjoint action changes ξ to ξ' in this way :

$$U\,J(\xi)\,U^{-1} = J(\xi'),$$

$$\xi'_a = R_{ab}(U)\,\xi_b. \qquad \text{...(2.9)}$$

Here R(U) $\in$ SO(3) is the image of U $\in$ SU (2) under the homomorphism SU (2) $\rightarrow$ SU (3). Thus the orbit of ξ consists of all ξ' with the same (squared) length as ξ. We can therefore label obits in SO(3) with a positive nonzero parameter u: the orbit $\vartheta_3(u)$ consists of all $J(\xi)$ for which

$$\mathscr{C}_1(\xi) = |\xi|^2 = u^2. \qquad \ldots(2.10)$$

This is a sphere S^2 in the 3-dimsnsional ξ space. A convenient representative element on $\vartheta_3(u)$ is the positive multiple uJ_{12} of J_{12}. This element can be characterized geometrically by the statement that under the rotatious generated by it, the single vector e_3 is invariant. This reflects the fact that the rank of the matrix (ξ) is constant over an orbit, and so is 2 at the representative point uJ_{12}. All these properties for SO (3) can be summarised in a table which sets the pattern for presentation of results in other cases:

SO(3) Orbit structure :

Rank (ξ)	Orbit	Parameter range	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vector
2	$\vartheta_3(u)$	$u > 0$	u^2	uJ_{12}	e_3

SO (4)

With the index conventions explained in the Introduction, a general element of SO (4) is written as

$$J(\xi) = \tfrac{1}{2}\xi_{\mu\nu} J_{\mu\nu} \qquad \ldots(2.11)$$

the subscripts taking the values 1,..., 4. It is convenient to define three-component objects and quantities in the following manner :

$$J_a = \tfrac{1}{2}\epsilon_{abc} J_{bc}, \quad K_a = J_{4a};$$

$$\xi_a = \tfrac{1}{2}\epsilon_{abc}\xi_{bc}, \quad \eta_a = \xi_{4a}. \qquad \ldots(2.12)$$

(Of course, the latin subcripts here go over 1, 2, 3). Then the basic Lie relations of SO(4) are :

$$[J_a, J_b] = [K_a, K_b] = \epsilon_{abc} J_c,$$

$$[J_a, K_b] = \epsilon_{abc} K_c. \qquad \ldots(2.13)$$

If now we define the combinations

$$M_a = \tfrac{1}{2}(J_a + K_a)$$

$$N_a = \tfrac{1}{2}(J_a - K_a) \qquad \ldots(2.14)$$

the familiar SO (3) $\oplus$ SO (3) structure of SO (4) emerges :

$$[M_a, M_b] = \epsilon_{abc} M_c,$$

$$[N_a, N_b] = \epsilon_{abc} N_c,$$

$$[M_a, N_b] = 0. \qquad \ldots(2.15)$$

Therefore the results of the orbit classificatian for **SO (3)** can be used to tackle the **SO (4)** problem.

The general element $J(\xi) \in$ **SO** (4) can be written in the forms

$$J(\xi) = \boldsymbol{\xi}.\mathbf{J} + \boldsymbol{\eta}.\mathbf{K}$$

$$= \boldsymbol{\alpha}.\mathbf{M} + \boldsymbol{\beta}.\mathbf{N},$$

$$\alpha_a = \xi_a + \eta_a, \quad \beta_a = \xi_a - \eta_a. \qquad \ldots(2.16)$$

The invariant $\mathscr{C}_1(\xi)$ has the value

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\xi_{\mu\nu}\xi_{\mu\nu} = |\boldsymbol{\xi}|^2 + |\boldsymbol{\eta}|^2 = \tfrac{1}{2}(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2). \quad \ldots(2.17)$$

Since SO (4) is a group of rank two (as is SO (5)), there is a second algebraic invariant which can be obtained with the use of the four index antisymmetric symbol :

$$\mathscr{C}_2(\xi) = -\tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}\xi_{\mu\nu}\xi_{\rho\sigma}$$

$$= \boldsymbol{\xi}.\boldsymbol{\eta} = \tfrac{1}{4}(|\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2). \qquad \ldots(2.18)$$

This invariant is related to the rank of the 4×4 antisymmetric matrix $(\xi_{\mu\nu})$: in fact one finds

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\boldsymbol{\xi}.\boldsymbol{\eta})^2$$

$$= (\mathscr{C}_2(\xi))^2. \qquad \ldots(2.19)$$

Now the rank of $(\xi_{\mu\nu})$ is either 2 or 4,since we exclude $\xi_{\mu\nu} = 0$. Therefore the vanishing of $\mathscr{C}_2(\xi)$ corresponds to rank $(\xi_{\mu\nu}) = 2$, and a nonvanishing $\mathscr{C}_2(\xi)$ implies rank $(\xi_{\mu\nu}) = 4$. The rank in turn determines the number of independent vectors in four-space invariant under the infinitesimal rotations generated by $J(\xi)$. On a general four-vector z_μ, this rotation acts as

$$\delta z_\mu = \delta\theta\, \xi_{\mu\nu}\, z_\nu. \qquad \ldots(2.20)$$

We see: if $\mathscr{C}_2 = 0$, there are two independent vectors which are both invariant under the rotations (2.20), and without loss of generality they may be assumed to be orthonormal ; if $\mathscr{C}_2 \neq 0$, there are no such vectors.

On account of the local SO (3) $\otimes$ SO (3) structure of SO (4), the effect of the adjoint action is to subject the two three-component quantities α_a, β_a to independent SO (3) rotations. With this remark and the results of the **SO (3)** analysis, we can immediately classify the **SO (4)** orbits. Given an element $J(\xi) \in$ **SO (4)**, there is a unique element on its orbit having the form $|\boldsymbol{\alpha}| M_3 + |\boldsymbol{\beta}| N_3$. Here $|\boldsymbol{\alpha}|$ and $|\boldsymbol{\beta}|$ cannot both vanish. $J(\xi)$, and with it all the elements on its orbit, can be characterised as being of rank 2 if $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$; otherwise they are all of rank 4. Rewritten in terms of the original $J_{\mu\nu}$, the above mentioned representative element is[39].

$$|\boldsymbol{\alpha}| M_3 + |\boldsymbol{\beta}| N_3 = u\, J_{12} + u'\, J_{43},$$

$$u = \frac{1}{2}(|\alpha| + |\beta|) > 0,$$

$$u' = \frac{1}{2}(|\alpha| - |\beta|),\ u \geqslant |u'|. \qquad \ldots(2.21)$$

Now, the rank 2 case corresponds to vanishing u'. For such orbits we see that $u > 0$ is a single labelling parameter, and the choice of representative element is such that e_3 and e_4 are both invariant under the rotations generated by it. On the other hand, the rank 4 case corresponds to $u' \neq 0$, and the rotations generated by $uJ_{12} + u'J_{43}$ definitely alter every non-zero four-vector. These results can be summarised as follows :

SO (4) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_4(u)$	$u > 0$	u^2	0	uJ_{12}	e_3, e_4
4	$\vartheta_4(u, u')$	$u \geqslant \vert u' \vert > 0$	$u^2 + u'^2$	uu'	$uJ_{12}+u'J_{43}$	—

The following remarks can now be made concerning these results. The problem of classifying the rank 2 orbits of **SO (4)** reduces easily to a problem at the level of the **SO** (3) which generates the canonical SO (3) subgroup in SO (4) acting on the dimensions 1, 2 and 3. This is because on any such orbit one can always find representative elements for which one of the invariant vectors is e_4. On restricting oneself to this part of the orbit, the further classification depends only on the already available **SO** (3) results. On the other hand, the rank 4 orbits in **SO (4)** are quite new in the sense that they cannot be reduced to a problem within the **SO (3)** algebra corresponding to the canonical SO(3) subgroup of SO (4); they may be thought of as characteristic of SO (4), notwithstanding the fact that the local SO (3) ⊗ SO (3) structure of SO (4) simplified matters. The table of results for **SO (4)** also shows that the values of the algebraic invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ together determine the orbit to which $J(\xi)$ belongs. (Note that they are restricted by $\mathscr{C}_1 \geqslant 2|\mathscr{C}_2|$). In the rank 2 case, when $\mathscr{C}_2 = 0$, $\mathscr{C}_1$ determines u and the statement follows. In the rank 4 case, $\mathscr{C}_1$ and $\mathscr{C}_2$ do determine u and u' individually because of the restriction $u \geqslant |u'|$, and the statement again follows.

SO(5)

A general element of **SO (5)** is

$$J(\xi) = \tfrac{1}{2}\xi_{AB} J_{AB}, \qquad \ldots(2.22)$$

with the indices going over 1, 2, ..., 5. The infinitesimal SO (5) rotations produced by this generator alter a general vector z_A in the standard manner:

$$\delta z_A = \delta\theta\, \xi_{AB}\, z_B. \qquad \ldots(2.23)$$

The first algebraic invariant is

$$\mathscr{C}_1(\xi) = \tfrac{1}{4}\xi_{AB}\xi_{AB}. \qquad \ldots(2.24)$$

The rank of the matrix (ξ_{AB}) is either 2 or 4; correspondingly the number of independent null vectors of this matrix, equivalently vectors invariant under (2.23), is either 3 or 1. Thus for any $J(\xi)$ there is at least one invariant vector. In constructing the second algebraic invariant $\mathscr{C}_2(\xi)$ we are led to a possible invariant vector defined in terms of ξ_{AB} itself. Since with SO (5) we have a five-index antisymmetric symbol, we define the five-vector

$$\zeta_A = \tfrac{1}{8}\,\epsilon_{ABCDE}\,\xi_{BC}\,\xi_{DE}. \qquad \ldots(2.25)$$

It is an easily checked fact that

$$\xi_{AB}\,\zeta_B = 0. \qquad \ldots(2.26)$$

Therefore, whenever ζ_A does not vanish identically, it provides us with one vector invariant under (2.23). The second algebraic invariant is the squared length of ζ :

$$\mathscr{C}_2(\xi) = \zeta_A\,\zeta_A. \qquad \ldots(2.27)$$

In the case when rank $(\xi_{AB}) = 2$, let us denote a preliminary choice of three independent null vectors of (ξ_{AB}) by $e^{(m)}_A$ $m = 1, 2, 3$, and set up the matrix of inner products

$$(M_{mm'}) = (e^{(m)}_A\ e^{(m')}_A) = (e^{(m)}, e^{(m')}). \qquad \ldots(2.28)$$

This three-dimensional real symmetric matrix is positive definite because we are dealing with SO (5) rather than SO (4, 1) or SO (3,2). Now by an SO (3) transformation acting on the indices m, m', and therefore amounting to a different choice of the $e^{(m)}$, we can diagonalize M, when its nonzero entries become all strictly positive. By a further renormalization of its eigenvetors, it can be seen that M becomes the unit matrix. This argument shows that without loss of generality the three invariant vectors under the transformation (2.23) can be chosen to be othonormal[40].

With the help of the above result, the analysis of rank 2 orbits in SO (5) becomes quite easy. Let some $J(\xi) \in$ SO(5) of rank 2 be given. By means of suitable SO (5) transformations we can pass to those elements on the orbit of $J(\xi)$ for which the three invariant vectors are e_3, e_4 and e_5. Such elements must be of the form uJ_{12}, $u \neq 0$, where $|u|$ is fixed by the value of $\mathscr{C}_1(\xi)$. This is similar to the SO (3) situation. Since the elements uJ_{12} and $-uJ_{12}$ can be connected to one another even within SO (3), it follows that we can restrict u to be strictly positive in choosing uJ_{12} as an orbit representative for rank 2 orbits of SO(5). On the other hand, no further reduction in distinct orbit representatives is possible even with the greater freedom of transformation available with SO(5) as compared with SO (3); i. e. as one can quite easily convince oneself, two elements uJ_{12} and $u'J_{12}$ with $u, u' > 0$, $u \neq u'$, cannot be connected to one another by any SO (5) transformation. At the representative point uJ_{12}, the only nonzero component of ξ_{AB} is $\xi_{12} = u$; so $\zeta_A = 0$ identically at this point and consequently also at every point on every rank 2 orbit. This result may in a sense have been anticipated : if ζ_A were not identically zero, it "would not know which of the three independent invariant vectors" it should be. As with SO (3) and SO (4) the present rank 2 orbits will be denoted as $\vartheta_5(u)$.

The classification of rank 4 orbits is slightly more intricate. Let an element $J(\xi) \in$ SO (5) of rank 4 be given. The matrix (ξ_{AB}) has just one nonzero null vector. Using suitable SO (5) transformations we can pass to elements on the orbit of $J(\xi)$ for which the invariant vector is e_5. Such elements therefore belong to SO(4), the Lie algebra of the canonical SO (4) subgroup of SO (5) acting on the dimensions 1, 2, 3, 4; and moreover they are of rank 4 within SO (4), meaning that there is nonontrivial combination e_1, e_2, e_3, e_4 annihilated by any of them. Now from the SO (4) results we know that with the help of transformations within SO (4) we can find among the above elements on the orbit of $J(\xi)$ some of the form $uJ_{12} + u'J_{43}$ with $u \geqslant |u'| > 0$. However, since an SO (5) orbit could be larger than an SO (4) orbit, two distinct elements of the above form which cannot be connected within SO (4) may possibly be connected by some SO (5) transformation. If this happens, the concerned SO (5) transformation must take e_5 into $-e_5$. The point being made is that while in the first instance, by arranging the invariant vector to be e_5, the problem is brought down to the level of rank 4 SO (4) orbits, in thereafter using the SO (4) results one must take account of the fact that SO (5) is larger than SO(4). In this way one sees that, say by a rotation of amount π in the 4-5 plane, the elements $uJ_{12} + u J_{43}$ and $uJ_{12} - u'J_{43}$ are on the same SO (5) orbit. Thus unlike the SO (4) case, we may here restrict the parameters by $u \geqslant u' > 0$; the corresponding orbit in SO (5), denoted $\vartheta_5(u, u')$, is of rank 4 with unique representative element $uJ_{12} + u'J_{43}$. At this point on the orbit, only ξ_{12} and ξ_{43} out of ξ_{AB} are non-zero; correspondingly, $\zeta_5 = -uu'$ is non zero, while the other components $\zeta_\mu = 0$. This is consistent with our arranging the choice of orbit representative so that the invariant vector is e_5. The general conclusion to be drawn is that at every point $J(\xi)$ on a rank 4 orbit, the five-vector ζ_A does not vanish, and is the single vector annihilated by $J(\xi)$. The final results for SO (5) are thus as follows :

SO (5) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_5(u)$	$u > 0$	u^2	0	uJ_{12}	e_3, e_4, e_5
4	$\vartheta_5(u,u')$	$u \geqslant u' > 0$	$u^2 + u'^2$	$u^2 u'^2$	$uJ_{12} + u'J_{43}$	e_5

Several points are worth noting. All results for SO (5) are obtainable on making use of the previously obtained results for SO (3) and SO (4), provided one pays attention to the extra freedom of transformation available within SO (5) as compared to SO (4).

As with SO (4), the vanishing here of the algebraic invariant $\mathscr{C}_2(\xi)$ unambiguously signifies that the rank is 2. The restriction on $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ in the **SO (5)** case are:

$$\mathscr{C}_1 > 0,\ \mathscr{C}_2 \geqslant 0,\ \mathscr{C}_1 \geqslant 2\sqrt{\mathscr{C}_2}. \qquad ...(2.29)$$

We see that $\mathscr{C}_1(\xi)$ determines u in the rank 2 case, while $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ determine u, u' unambiguously in the rank 4 case. Thus the values of the algebriac invariants fix the orbit to which $J(\xi)$ belongs. The manner in which the SO (4) results helped simplify the **SO** (5) problem sets the pattern for similar simplifications in the **SO** (n, 1) and **SO** (n, 2) analyses.

3. Orbits in the Lie Algebras SO (n, 1), $n = 2, 3, 4$

A basis for **SO (2,1)** is given by three elements $J_{ab} = -J_{ba}$, $a, b = 0, 1, 2,$ obeying the bracket relations

$$[J_{ab}, J_{cd}] = g_{ac} J_{bd} - g_{bc} J_{ad} + g_{ad} J_{cb} - g_{bd} J_{ca}. \qquad ...(3.1)$$

The diagonal metric is $g_{00} = -1$, $g_{11} = g_{22} = 1$. For SO (3, 1) and SO (4,1) we replace a, b, c, d by μ, ν, ρ, σ and A, B, C, D respectively, and extend the metric tensor with $g_{33} = g_{44} = 1$. For the pseudo-orthogonal groups indices must be raised and lowered using the appropriate metric tensor, and the antisymmetric symbol is defined by

$$\epsilon_{012} = \epsilon_{0123} = \epsilon_{01234} = 1. \qquad ...(3.2)$$

SO (2,1)

A general element of **SO (2,1)** is

$$J(\xi) = \tfrac{1}{2}\xi^{ab} J_{ab}. \qquad (3.3)$$

Since the dimension of the space is three, as with **SO** (3) we can use ϵ_{abc} to replace J_{ab} and ξ_{ab} by single index objects:

$$J_a = \tfrac{1}{2}\epsilon_{abc} J^{bc}, \quad \xi_a = -\tfrac{1}{2}\epsilon_{abc}\xi^{cb};$$

$$J_{ab} = -\epsilon_{abc} J^c, \quad \xi_{ab} = \epsilon_{abc}\xi^c. \qquad ...(3.4)$$

The relative signs are adjusted so that $J(\xi)$ has a neat form :

$$J(\xi) = \xi^a J_a. \qquad ...(3.5)$$

In terms of J_a, the bracket relations (3.1) are

$$[J_a, J_b] = \epsilon_{abc} J_c. \qquad ...(3.6)$$

Under the adjoint action by SO (2,1), when ξ^{ab} transforms as a second rank antisymmetric tensor, ξ^a transforms as a three-vector because ϵ_{abc} is an invariant tensor. The quadratic invariant $\mathscr{C}_1(\xi)$, the only algebraic invariant in the **SO(2,1)** case, is

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\xi^{ab}\xi_{ab} = -\xi^a\xi_a. \qquad ...(3.7)$$

The rank of the matrix (ξ_{ab}) must be 2, which means that under infinitesimal Lorentz transformations

$$\delta z^a = \delta\theta \xi^a_b z^b \qquad ...(3.8)$$

there is just one invariant vector. This is ξ^a itself since

$$\xi^a_b \; \xi^b = 0. \qquad ...(3.9)$$

As long as $J(\xi)$ is non-zero, so is ξ^a; and $J(\xi)$ generates Lorentz transformations "about ξ^a as axis". The adjoint action can be explicitly realised via the 2×2 matrix representation of SL (2, R), as in the SO (3) case. In this representation we have, for instance.

$$J_0 \rightarrow \frac{i}{2}\sigma_2, \; J_1 \rightarrow \tfrac{1}{2}\sigma_1, \; J_2 \rightarrow \tfrac{1}{2}\sigma_3 \qquad ...(3.10)$$

so $J(\xi)$ is a general real traceless 2×2 matrix :

$$J(\xi) = \tfrac{1}{2}(\xi^0 . \, i\sigma_2 + \xi^1 \sigma_1 + \xi^2 \sigma_3). \qquad ...(3.11)$$

Then for $S \in SL(2, R)$ we have

$$SJ(\xi) S^{-1} = J(\xi')$$
$$\xi'^a = \Lambda^a{}_b(S) \xi^b, \qquad ...(3.12)$$

where $\Lambda(S) \in$ SO (2, 1) is the image of $S \in$ SL (2, R) under the homomorphism SL(2, R) $\rightarrow$ SO (2,1).

We see from this discussion that, since there are three qualitatively different kinds of vectors in a 2+1 space, there is a similar number of qualitatively different orbit types. If $J(\xi)$ with a timelike ξ^a is given, the orbit of $J(\xi)$ consists of all $J(\xi')$ with ξ'^a having the same Lorentz square as ξ^a, and ξ'^0 the same sign as ξ^0. Similar statements can be made for the cases when ξ^a is lightlike (positive or negative). When ξ^a is spacelike only the same Lorentz square is required. Thus each nontrivial orbit can be distinguished by a symbol t, l or s in these three cases. Further, in the t case, the representative point on the orbit can be chosen as $\xi'^a = (u, 0, 0)$, $u \neq 0$; in the l case we can arrange $\xi'^a = \epsilon(1, 0, 1)$, $\epsilon = \pm 1$; and in the s case, $\xi'^a = (0, 0, v)$, $v > 0$. At these representative points the invariant vectors are respectively e_0, $e_0 + e_2$ and e_2.

A table presenting all the distinct orbits in **SO (2,1)** can be drawn up based on these results :

SO (2.1) orbit structure ;

Rank (ξ)	Orbit	Parameter range	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vector
2	$\vartheta_{2,1}(t; u)$	$u \neq 0$	u^2	uJ_{12}	e_0
2	$\vartheta_{2,1}(l; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{12} + J_{10})$	$e_0 + e_2$
2	$\vartheta_2(s; v)$	$v > 0$	$-v^2$	vJ_{10}	e_2

The "arguments" of $\vartheta_{2,1}$ (...) in the various cases are self-explanatory. In comparison with the **SO (3)** table, there is an expected increase in complexity. In particular we see that only when $\mathscr{C}_1(\xi)$ is negative does it uniquely determine the orbit to which $J(\xi)$ belongs. In case $\mathscr{C}_1(\xi) > 0$, this algebraic invariant cannot determine the sign of u or ϵ, as the case may be.

SO (3,1)

In handling this case we can follow the SO (4) pattern to the extent possible and deal with three-dimensional quantities by breaking up $\xi_{\mu\nu}$ and $J_{\mu\nu}$; we can also exploit the simplicity of the defining two-dimensional spinor represention of SL (2,C), as was done with SO (3) and SO (2,1). In the sequel both approaches will be used and related to each other. Concerning the index conventions, since in this Section dealing with the SO (n, 1) groups the Latin indices a, b, ...run over 0, 1, 2, we shall use indices j, k, ... to go over the range 1, 2, 3 covering the "space" dimensions

In terms of $\xi_{\mu\nu}$ and $J_{\mu\nu}$ we define

$$J_j = \tfrac{1}{2}\,\epsilon_{jkl}\,J_{kl},\; K_j = J_{0j};$$

$$\xi_j = \tfrac{1}{2}\,\epsilon_{jkl}\;\xi_{kl},\; \eta_j = \xi_{j0}. \qquad ..(3.13)$$

Then the general element $J(\xi) \in$ **SO (3,1)** and the Lie brackets are

$$J(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,J_{\mu\nu} = \boldsymbol{\xi}.\mathbf{J} + \boldsymbol{\eta}.\mathbf{K};$$

$$[J_j,\, J_k] = -\,[K_j,\, K_k] = \epsilon_{jkl}\,J_l,$$

$$[J_j,\, K_k] = \epsilon_{jkl}\,K_l. \qquad ...(3.14)$$

The two algebraic invariants are

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,\xi_{\mu\nu} = |\,\boldsymbol{\xi}\,|^2 - |\,\boldsymbol{\eta}\,|^2,$$

$$\mathscr{C}_2(\xi) = \tfrac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}\,\xi^{\mu\nu}\,\xi^{\rho\sigma} = \boldsymbol{\xi}.\,\boldsymbol{\eta}. \qquad ...(3.15)$$

The infinitesimal Lorentz transformation generated by $J(\xi)$:

$$\delta z^{\mu} = \delta\theta.\xi^{\mu}{}_{\nu}.z^{\nu} \qquad ...(3.16)$$

is characterised by the 4 × 4 antisymmetric matrix $(\xi_{\mu\nu})$ whose rank is either 2 or 4. The rank can be related to $\mathscr{C}_2(\xi)$ since

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\boldsymbol{\xi}.\boldsymbol{\eta})^2 = (\mathscr{C}_2(\xi))^2. \qquad ...(3.17)$$

Thus vanishing $\mathscr{C}_2(\xi)$ means rank $(\xi_{\mu\nu}) = 2$, and there are then two independent vectors invariant under (3.16): nonvanishing $\mathscr{C}_2(\xi)$ means rank $(\xi_{\mu\nu}) = 4$, hence no invariant vectors.

At this point we introduce the two-dimensional spinor representation of $J_{\mu\nu}$:

$$J_j = \frac{-i}{2}\,\sigma_j,\; K_j = -\,\frac{1}{2}\,\sigma_j;$$

(equation continued on p. **104)**

$$J(\xi) = -\frac{i}{2}(\xi - i\eta).\sigma. \qquad ..(3.18)$$

So $J(\xi)$ is a general complex traceless 2×2 matrix. Adjoint action by SL (2,C), which is the same as by SO (3, 1), amounts to subjecting $J(\xi)$ to a similarity transformation: for any $S \in$ SL (2, C),

$$S(\xi - i\eta).\sigma S^{-1} = (\xi' - i\eta').\sigma \qquad ...(3.19)$$

In searching for the "most natural" form into which $J(\xi)$ can be put via the adjoint representation, we can therefore use the results of the theory of the Jordan canonical form of a matrix. Remembering the tracelessness property, this allows for two possibilities: (i) $J(\xi)$ can be diagonalised, with nonzero equal and opposite generally complex eigenvalues; (ii) $J(\xi)$ cannot be diagonalised, but can be put into the upper triangular Jordan from $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. These two distinct possibilites can be partially correlated with the classification based on the rank of $(\xi_{\mu\nu})$, since in the spinor representation

$$\det J(\xi) = -\frac{1}{4}\det(\xi - i\eta).\sigma = \frac{i}{2}\mathscr{C}_2(\xi) - \frac{1}{4}\mathscr{C}_1(\xi). \qquad ...(3.20)$$

Therefore possibility (i) with diagonalisable $J(\xi)$ must correspond to at least one of $\mathscr{C}_1$ and $\mathscr{C}_2$ being nonzero; while possibility (ii) implies $\mathscr{C}_1 = \mathscr{C}_2 = 0$. Only in the latter case can the definite statement be made that rank $(\xi_{\mu\nu}) = 2$;in the former case, we can have both values 2 and 4 for the rank, according as $\mathscr{C}_2 = 0$ or $\neq 0$.

With this preparation, we can proceed to analyse and classify first the rank 2 orbits in **SO (3,1)**. Let $J(\xi)$ be given with $\mathscr{C}_2(\xi) = 0$. By adapting the argument given in Section 2 in connection with SO (5), we can assume without loss of generality that the two independent null-vectors of $(\xi_{\mu\nu})$ are mutually orthogonal, and each of them is normalised to ± 1 if it is not a light-like vector. In a $1 + 3$ space with signature $-+++$ there are three distinct possibilities for this pair of vectors : is *ts*, *ls* or *ss*. Throughout an orbit one and the same possibility is realised. There is therefore at least one spacelike vector invariant under $J(\xi)$. Using suitable SO (3,1) transformations, we can pass to those elements on the orbit of $J(\xi)$, each of which leaves e_3 invariant. Such elements therefore are linear combinations of J_{ab}, $a, b = 0, 1, 2$, and to further analyse them the **SO (2,1)** results can be used. These now tell us that one of the following three mutually exclusive possibilities must occur : (i) there are elements uJ_{12}, $u \neq 0$, on the orbit of $J(\xi)$, for which e_0, e_3 are the two invariant vectors, realising the configuration *ts*; (ii) there are elements $\epsilon(J_{12} + J_{10})$, $\epsilon = \pm 1$, on the orbit of $J(\xi)$, for which $e_0 + e_2$, e_3 are two invariant vectors, corresponding to the configuration *ls*; (iii) there is an element vJ_{10}, $v > 0$, on the orbit of $J(\xi)$, which leaves e_2, e_3 invariant and realises the configuration *ss*. Obviously in case (i) the value of $\mathscr{C}_1(\xi)$ fixes $|u|$, and we can use the extra freedom available in SO (3,1) as compared to SO (2, 1) to achieve $u > 0$; in case (ii) both $\mathscr{C}_1$ and $\mathscr{C}_2$ vanish, and in case $\epsilon = -1$ it can be converted to $+1$ by a rotation of amount π in the $1 - 3$ plane; in case (iii) $\mathscr{C}_1(\xi)$ determines $v > 0$ unambiguously. Of these three cases, we can recognise that case (ii) is precisely that possibility encountered among the Jordan canonical forms when $J(\xi)$ could not be diagonalised. Since in cases (i) and (iii) $\mathscr{C}_1(\xi)$ is nonzero, these are included among those Jordan canonical forms wherein $J(\xi)$ could be disgonalised. (Note that through $J_{10} = \frac{1}{2}\sigma_1$ is not diagonal, it can be diagonalised). This completes the classification of and choice of representative elements from the rank 2 orbits of **SO (3,1)**.

Now we consider the rank 4 orbits consisting of $J(\xi)$ with $\mathscr{C}_2(\xi) \neq 0$. Here we can immediately use the results of the Jordan canonical forms : since by eqn. (3.20) det $J(\xi)$ is nonzero, $J(\xi)$ can be diagonalised. Therefore on the orbit of $J(\xi)$ there certainly are elements which in the spinor representation are complex multiples of σ_3. One can easily see then that on a given orbit there is a unique element of the form $uJ_{12}+vJ_{03}$ with $u > 0$ and $v \neq 0$. The values of u and v are unambiguously determined by $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$.

Putting together the two sets of results for both ranks. We get the following table :

SO (3,1) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_{3,1}(ts;u)$	$u > 0$	u^2	0	$u\,J_{12}$	e_0, e_3
2	$\vartheta_{3,1}(ls)$	—	0	0	$J_{12} + J_{10}$	e_0+e_2, e_3
2	$\vartheta_{3,1}(ss;\,v)$	$v > 0$	$-v^2$	0	$v\,J_{10}$	e_2, e_3
4	$\vartheta_{3,1}(u,v)$	$u>0$, $v \neq 0$	$u^2 - v^2$	uv	$u\,J_{12} + uJ_{03}$	—

It is clear that by a combination of geometrical arguments and matrix-theoretical arguments, the complete results for **SO (3,1)** emerge relatively easily. It is also clear that in all cases, the values of $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ determine the orbit to which $J(\xi)$ belongs; this contrasts with the situation in **SO (2,1)**.

SO (4,1)

Now we turn to the third and last of the Lorentz type groups to be studied. As in the case of SO (5), here too since the total dimension of the space is five which is odd, every generator $J(\xi)$ has an associated matrix (ξ_{AB}) which has at least one nontrivial null vector. However whereas with **SO (4)** such a result essentially reduced the problen to **SO (4)** (and in suitable circumstances further down to **SO (5)**), in the present case we have a grater variety of possible configurations to consider, due to the changed metric.

Let us begin by listing those expressions for SO (4,1) which are similar to corresponding ones for SO (5) :

$$J(\xi) = \tfrac{1}{2}\,\xi^{AB}\,J_{AB}; \qquad \text{(a)}$$

$$\mathcal{C}_1(\xi) = \tfrac{1}{2}\,\xi^{AB}\,\xi_{AB}; \qquad \text{(b)}$$

$$\zeta_A = \tfrac{1}{8}\,\epsilon_{ABCDE}\,\xi^{BC}\,\xi^{DE}; \qquad \text{(c)}$$

$$\mathcal{C}_2(\xi) = \zeta^A\,\zeta_A. \qquad \text{(d)} \quad (3.21)$$

These are to be supplemented by the identity

$$\xi_{AB}\,\zeta^B = 0. \qquad \ldots(3.22)$$

Depending on whether the rank of (ξ_{AB}) is 2 or 4, we have three or one independent invariant vectors. In the former case, a choice can be made such that they are mutually orthogonal, and if nonlightlike are normalised to ± 1. We will find that $\zeta^A = 0$ when rank (ξ_{AB}) is 2; and that when rank $(\xi_{AB}) = 4$, ζ^A is the sole nonvanishing invariant vector.

We classify first the rank 2 orbits. The three independent and mutually orthogonal invariant vectors could in principle be of any one of the ten types (listed in dictionary order) *ttt*, *ttl*, *tts*; *tll*, *tls*; *tss*; *lll*, *lls*; *lss*; *sss*. However, in a space with signature $-++++$, the configurations *tt*, *tl*, *ll* cannot occur; i.e., we cannot find two mutually orthogonal time-like vectors etc. Therefore the only possible configurations for the three invariant vectors are three in numbers *tss*, *lss*, *sss*. In every case, there are two mutually orthogonl (normalised) space-like vectors. We may conclude: if an element $J(\xi) \in$ **SO (4.1)** with rank 2 is given, there definitely are elements on its orbit which leave e_3 and e_4 invariant. Such elements belong to the **SO (2,1)** generating the SO (2,1) operat ing on indices 012, and so are linear combinations of J_{ab}. Analysis of the rank 2 orbits is thus related to the **SO (2,1)** problem, so there are precisely three mutually exclusive possibilities characterising the orbit of $J(\xi)$: (i) there is an element leaving e_0 invariant, realising the situation *tss*; (ii) there is an element leaving $e_0 + e_2$ invariant, corresponding to *lss*. (iii) there is an element leaving e_2 invariant, corresponding to *lss*. In case (i) there is a unique element uJ_{12}, $u > 0$, on the orbit of $J(\xi)$; the possibility $u < 0$ can be changed to $u > 0$ by a rotation in the 1-4 plane. In case (ii) the element $J_{12} + J_{10}$ is on the orbit; again the element $-(J_{12} + J_{10})$ goes into $J_{12} + J_{10}$ by a suitable 1-4 rotation. Lastly in case (iii) a unique element vJ_{10}, $v > 0$, lies on the orbit. This completes the catalogue of rank 2 orbits. In all cases the vanishing of ζ^A is obvious.

The discussion of the rank 4 orbits brings in a group not encounted in the treatment so far. If a $J(\xi) \in$ **SO (4,1)** of rank 4 be given, there is just one vector which it leaves invariant, which is of type *t*, *l* or *s*. In the *l* and *s* cases, the problem reduces to the subgroup SO (4) or SO (3,1) respectively. However the *t* case involves an E(3) subgroup of SO (4,1), which is an inhomogeneous real orthogonal group. We dispose of the *t* and *s* cases first, then take up the *l* case.

In case a rank 4 orbit in **SO** (4,1) is of type t, there definitely are elements on it which leave e_0, and no other vector, invariant. Such elements belong to **SO** (4) generating the SO (4) group acting on the "space" dimensions 1, ..., 4; and within **SO** (4) they must be of rank 4. One can then check that out of them, a unique representative element $u\, J_{12} + u' J_{43}$ with $u \geqslant | u' | > 0$ can be picked and can serve as a representative element for the **SO** (4,1) orbit itself. At the representative point the only nonzero component of ζ_A is $\zeta_0 = - uu'$, so ζ_A is in the direction $\pm\, e_0$. If the rank 4 orbit is of type s, then e_0, **SO** (4) and SO (4) are replaced by e_4, **SO** (3,1) and SO (3,1) acting on 0, ..., 3 respectively. So elements $uJ_{12} + vJ_{03}$ with $u > 0$, $v \neq 0$ exist on the orbit. But since the freedom of rotations is the 3-4 plane in available in SO (4,1), we can find a unique point on the orbit by requiring both $u > 0$ and $v > 0$, Now, the only nonzero component of ζ_A is $\zeta_4 = uv$, which confirms our expectations.

The last case to be examined in **SO** (4,1) is a rank 4 orbit of type 1. By suitable SO (4,1) transformations, starting with any element on such an orbit, we can pass to those elements for which the (single) invariant vector is $e_0 + e_4$. We must now determine the general form of such elements $J\,(\xi)$, using the property that the matrix (ξ_{AB}) is of rank 4 and annihilates only $z^A = (1, 0, 0, 0, 1)$. Among the equations $\zeta_{AB}\, z^B = 0$ there are four independent ones :

$$\xi_{04} = 0$$
$$\xi_{j4} = - \xi_{j0}, \quad j = 1, 2, 3. \qquad \ldots(3.23)$$

So we can write

$$J\,(\xi) = \tfrac{1}{2}\, \xi^{AB}\, J_{AB}$$
$$= \tfrac{1}{2}\, \xi_{jk}\, J_{jk} + \xi_{j0}\, (J_{0j} + J_{4j}). \qquad \ldots(3.24)$$

Here, as in the treatment of SO (3) within SO (3,1), the indices j, k, ... range over 1,2, 3. The generators **J**, **P** $\in$ **SO** (4,1) defined as

$$J_j = \tfrac{1}{2}\, \epsilon_{jkl}\, J_{kl},$$
$$P_j = J_{0j} + J_{4j} \qquad \ldots(3.25)$$

obey the brackets relations of E (3), the Lie algebra of the Euclidean group in three dimensions :

$$[J_j, J_k] = \epsilon_{jkl}\, J_l$$
$$[J_j, P_k] = \epsilon_{jkl}\, P_l,$$
$$[P_j, P_k] = 0. \qquad \ldots(3.26)$$

This is expected, as the stability group of a light like vector in 1 + 4 space is E (3). Now we must search for the necessary and sufficient conditions on (ξ_{AB}) apart from (3.23) which ensure that $e_0 + e_4$ is the 'only' vector invariant under $J\,(\xi)$. Let us write the $J(\xi)$ in eqn. (3.24) as

$$J(\xi) = \boldsymbol{\xi}.\mathbf{J} + \boldsymbol{\alpha}.\mathrm{P},$$

$$\xi_j = \tfrac{1}{2}\,\epsilon_{jkl}\,\xi_{kl},$$

$$\alpha_j = \xi_{j0}. \qquad \ldots(3.27)$$

The condition that this (ξ_{AB}) annihilate a general z^A is

$$\boldsymbol{\alpha}, \mathbf{z} = 0,$$

$$\boldsymbol{\xi}_\wedge\, \mathbf{z} + (z^0 - z^4)\,\boldsymbol{\alpha} = 0. \qquad \ldots(3.28)$$

It is an easy consequence of these equations that

$$\boldsymbol{\xi}.\boldsymbol{\alpha}\,\mathbf{z} = 0,$$

$$\boldsymbol{\xi}.\boldsymbol{\alpha}\,(z^0 - z^4) = 0. \qquad \ldots(3.29)$$

Now from eqn. (3.28) we see that if either $\boldsymbol{\xi}$ or $\boldsymbol{\alpha}$ were to vanish, the desired conclusion on z^A, namely $\mathbf{z} = 0$ and $z^0 = z^4$, would not follow; hence we must insist that both $\boldsymbol{\xi}$ and $\boldsymbol{\alpha}$ be nonzero. If $\boldsymbol{\xi}.\boldsymbol{\alpha} = 0$, then from eqn. (3.29) we see that the desired conclusion on z^A does follow. On the other hand, if $\boldsymbol{\xi}.\boldsymbol{\alpha} \neq 0$ with neither $\boldsymbol{\xi}$ nor $\boldsymbol{\alpha}$ vanishing individually, we can find a solution to eqn. (3.28) setting $z^0 = z^4$ and $\mathbf{z}$ proportional to $\boldsymbol{\xi}$. Therfore the necessary and sufficient condition we are seeking on $J(\xi) \in$ E (3) is $\boldsymbol{\xi}.\boldsymbol{\alpha} \neq 0$.

We have in this way found the general form of elements on the given 1-type orbit, where the invariant vector is $e_0 + e_4$. These elements can be denoted by pairs $(\boldsymbol{\xi}, \boldsymbol{\alpha})$. Now we must exploit the adjoint action by E (3) to try and put such a pair into some simple and natural from. The adjoint action by the SO (3) part of E (3) is given by

$$(\boldsymbol{\xi}, \boldsymbol{\alpha}) \rightarrow (R\,\boldsymbol{\xi}, R\,\boldsymbol{\alpha}), \qquad \ldots(3.30)$$

where $R \in$ SO (3). On the other hand the translation by an amount $\mathbf{b}$ acts in the adjoint representation in this way :

$$(\boldsymbol{\xi}, \boldsymbol{\alpha}) \rightarrow (\boldsymbol{\xi}, \boldsymbol{\alpha} + \mathbf{b} \wedge \boldsymbol{\xi}). \qquad \ldots(3.31)$$

Under these changes (3.30, 31), $\boldsymbol{\xi}.\,\boldsymbol{\alpha}$ is invariant. Now given a pair $(\boldsymbol{\xi}, \boldsymbol{\alpha})$ with $\boldsymbol{\xi}.\boldsymbol{\alpha} \neq 0$, we can first use the freedom (3.30) to put $\boldsymbol{\xi}$ into the form

$$\boldsymbol{\alpha} \rightarrow (0, 0, u),\; u > 0. \qquad \ldots(3.32)$$

After this, the translational freedom (3.31) can be used to reduce the first two components of $\boldsymbol{\alpha}$ to zero:

$$\boldsymbol{\alpha} \rightarrow (0, 0, \alpha)\; \alpha \neq 0. \qquad \ldots(3.33)$$

Having achieved this, we see that $J(\xi)$ has been carried by adjoint action using E (3) alone to the form

$$J(\xi) \rightarrow u\, J_{12} + \alpha\, P_3$$
$$= u\, J_{12} + \alpha\, (J_{03} + J_{43}). \qquad \ldots(3.34)$$

At this point we have to ask if the further freedom of transformation remaining in SO (4.1) after all of E (3) has been used up will help simplify the expression (3.34) still further. It turns out that this is possible: the generator $J_{04} \in$ SO (4.1) has the effect of changing the scale of P_j:

$$[J_{04}, J_j] = 0,$$
$$[J_{04}, P_j] = -P_j. \qquad \ldots(3.35)$$

Therefore by using finite transformations generated by J_{04}, we can reduce the parameter α in (3.44) to $\epsilon = \pm 1$, depending on the sign of ξ. α in the original pair (ξ, α).

Thus we have succeeded in showing that on a rank 4 orbit in **SO (4,1)** of type 1, there is a unique representative element $uJ_{12} + \epsilon\,(J_{03} + J_{43})$ for some $u > 0$ and some $\epsilon = \pm 1$. We can then calculate the five-vector ζ^A at this representative point and find $\zeta^A = \epsilon\,(1, 0, 0, 0, 1)$, as expected.

The complete table of results for **SO (4,1)** is:

SO (4,1) orbit structure:

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_{t,1}\,(tss;\, u)$	$u>0$	u^2	0	uJ_{12}	e_0, e_3 e_4
2	$\vartheta_{4,1}\,(lss)$	–	0	0	$J_{12}+J_{10}$	e_0+e_2, e_3 e_4
2	$\vartheta_{4,1}\,(sss;\, v)$	$v > 0$	$-v^2$	0	vJ_{10}	e_2, e_3, e_4
4	$\vartheta_{4,1}\,(t;u,u')$	$u \geqslant \lvert u' \rvert > 0$	$u^2+u'^2$	$-u^2u'^2$	$uJ_{12}+u'J_{43}$	e^0
4	$\vartheta_{4,1}\,(l;\, u,t)$	$u>0;\ \epsilon=\pm 1$	u^2	0	$uJ_{12}+\epsilon(J_{03}+J_{43})$	$e_0 + e_4$
4	$\vartheta_{4,1}\,(s;u,v)$	$u>0;\ v>0$	$u^2 - v^2$	u^2v^2	$uJ_{12}+vJ_{03}$	e_4

It is obvious from this table that the invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ no longer suffice to fix the orbit to which $J(\xi)$ belongs.

4. Orbits in the Lie Algebras SO (n,2), $n = 2$ and 3

The two "de Sitter" type groups SO (n,2) are the last ones we analyse in this paper. The relevant dimensions are numbered 0, 1, 2, 3, 5 with signature $-+++-$. For SO (2,2) we have indices μ, ν, ... and the dimension 3 is omitted. The generators $J_{\mu\nu}$ obey

$$[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}\, J_{\nu\sigma} - g_{\nu\rho}\, J_{\mu\sigma} + g_{\mu\sigma}\, J_{\rho\nu} - g_{\nu\sigma}\, J_{\rho\mu}. \qquad \ldots(4.1)$$

For SO (3,2) we replace $\mu\nu\rho\sigma$ by $ABCD$ going over the full range. For these two groups the antisymmetric symbols are normalised by

$$\epsilon_{01235} = \epsilon_{0125} = 1 \qquad \ldots(4.2)$$

SO (2,2)

Just as with **SO (4)** in Section 2 where the orbit classification was simplified because of the local decomposition SO (4) $\simeq$ SO (3) $\otimes$ SO (3), here we can use the local decomposition SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1) and the results pertaining to **SO (2,1)**. However since **SO (2,1)** has a much richer orbit structure than SO (3), there being the various types tu, $l\epsilon$ and sv, many more possibilities arise with SO (2,2) than arose with SO (4). More over we must remember that the t, l and s classification of orbits within each **SO (2,1)** factor in SO (2,2) has no such geometrical interpretation in four dimensions[18].

Adapting eqns. (2.12, 14) to the present situation we define :

$$J_a = \tfrac{1}{2}\,\epsilon_{abc}\,J^{bc}, \quad K_a = J_{5a};$$

$$M_a = \tfrac{1}{2}\,(J_a + K_a), \quad N_a = \tfrac{1}{2}\,(J_a - K_a);\ a, b = 0, 1, 2. \qquad \ldots(4.3)$$

Then eqns. (4.1) can be expressed in two ways:

$$[J_a, J_b] = [K_a, K_b] = \epsilon^{\,c}_{ab}\,J_c,$$

$$[J_a, K_b] = \epsilon^{\,c}_{ab}\,K_c; \qquad \text{(a)}$$

$$[M_a, M_b] = \epsilon^{\,c}_{ab}\,M_c; \quad [N_a, N_b] = \epsilon^{\,c}_{ab}\,N_c;$$

$$[M_a, N_b] = 0. \qquad \text{(b)}\ldots(4.4)$$

These show the decomposition **SO (2,2) = SO (2,1) $\oplus$ SO (2,1)**. For the components $\xi^{\mu\nu}$ of a general element $J(\xi) \in$ SO (2,2) we define :

$$\xi^a = -\tfrac{1}{2}\,\epsilon^{abc}\,\xi_{bc}, \; \eta^a = \xi^{5a};$$

$$\alpha^a = \xi^a + \eta^a, \; \beta^a = \xi^a - \eta^a. \qquad \ldots(4.5)$$

Then $J(\xi)$ and the two algebraic invariants are :

$$J(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,J_{\mu\nu} = \xi^a\,J_a + \eta^a\,K_a = \alpha^a\,M_a + \beta^a\,N_a;$$

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,\xi_{\mu\nu} = -\,\xi^a\,\xi_a - \eta^a\eta_a = -\tfrac{1}{2}\,(\alpha^a\,\alpha_a + \beta^a\,\beta_a);$$

$$\mathscr{C}_2(\xi) = \tfrac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}\,\xi^{\mu\nu}\,\xi^{\rho\sigma} = \xi^a\,\eta_a = \tfrac{1}{4}\,(\alpha^a\,\alpha_a - \beta^a\beta_a). \qquad \ldots(4.6)$$

The invariant $\mathscr{C}_2(\xi)$ determines the rank of $(\xi_{\mu\nu})$ since

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\xi^a\eta_a)^2 = (\mathscr{C}_2(\xi))^2. \qquad \ldots(4.7)$$

Therefore $\mathscr{C}_2(\xi) = 0\ (\neq 0)$ corresponds to rank $(\xi_{\mu\nu}) = 2\ (4)$.

The adjoint action of SO (2,2) on $\xi^{\mu\nu}$ amounts to the following: we subject α^a and β^a to independent SO (2,1) transformations as 3-vectors, corresponding to the two factors in the product SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1). Therefore each orbit in SO (2,2) is the Cartesian product of two SO (2,1) orbits, with α^a lying on the first factor and β^a on the second. In looking for nontrivial SO (2,2) orbits, we can allow at most one factor in the product to be trivial. Thus to begin with, the distinct SO (2,2) orbits can be listed in this way: $(0;\ tu_2)$, $(0;\ l\,\epsilon_2)$, $(0;\ sv_2)$; $(tu_1;\ 0)$,...$(tu_1;\ sv_2)$; ..;$(sv_1;0)$...,$(sv_1;\ sv_2)$. Here u_1, u_2 $\neq 0$; ϵ_1, ϵ_2 + ± 1; v_1, $v_2 > 0$; and there are fifteen combinations. On any one of these **SO (2,2)** orbits, a unique representative element is obtained as the sum of representative elements from each factor. As examples we have:

$$(tu_1;\ 0) \rightarrow u_1\, M_0 = \tfrac{1}{2}\, u_1\, (J_{12} + J_{50});$$

$$(l\epsilon_1;\ sv_2) \rightarrow \epsilon_1\,(M_0 + M_2) + v_2 N_2$$

$$= \tfrac{1}{2}\, v_2\,(J_{10} - J_{52}) + \tfrac{1}{2}\,\epsilon_1\,(J_{10} + J_{12} + J_{50} + J_{52}). \qquad ...(4.8)$$

In each case, the values of $\xi_{\mu\nu}$ at the representative element can be read off, and then $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ for the entire orbit calculated. Towards classifying SO (2,2) orbits according to their ranks in the four-dimensional sense, we give in a table the value of $4\,\xi^a\eta_a = 4\,\mathscr{C}_2(\xi)$ in each cartesian product. The rows (columns) are labelled by the first (second) factor in the product.

Values of $4\,\mathscr{C}_2(\xi)$:

	0	tu_2	$l\epsilon_2$	sv_2
0	—	u_2^2	0	$-v_2^2$
tu_1	$-u_1^2$	$u_2^2 - u_1^2$	$-u_1^2$	$-u_1^2 - v_2^2$
$l\epsilon_1$	0	u_2^2	0	$-v_2^2$
sv_1	v_1^2	$v_1^2 + u_2^2$	v_1^2	$v_1^2 - v_2^2$

Since the total number of orbit types for **SO (2,2)** (and later also for **SO (3,2)**)) is quite large, we present the pattern of rank 2 orbits separately from that of rank 4 orbits. As a first step we read off from the above table all those cases when $\mathscr{C}_2(\xi) = 0$, corresponding to $J(\xi)$ being of rank 2, and also in each case write down a representative element, as was donn in the examples of (4.8) (the preliminary expressions in terms of M_a and N_a are omitted):

$$(tu_1;\ tu_1) \rightarrow u_1\ J_{12} \qquad \text{(i)}$$

$$(l\epsilon_1;\ l\epsilon_1) \rightarrow \epsilon_1\ (J_{10} + J_{12}) \qquad \text{(ii)}$$

$$(sv_1;\ sv_1) \rightarrow v_1\ J_{10} \qquad \text{(iii)}$$

$$(l\epsilon_1;\ 0) \rightarrow \tfrac{1}{2}\ \epsilon_1\ (J_{10} + J_{12} + J_{50} + J_{52}) \qquad \text{(iv)}$$

$$(0;\ l\epsilon_2) \rightarrow \tfrac{1}{2}\ \epsilon_2\ (J_{10} + J_{12} - J_{50} - J_{52}) \qquad \text{(v)}$$

$$(l\epsilon_1;\ l,\ -\epsilon_1) \rightarrow \epsilon_1\ (J_{50} + J_{52}) \qquad \text{(vi)}$$

$$(tu_1;\ t,\ -u_1) \rightarrow u_1\ J_{50}. \qquad \text{(vii)} \quad (4.9)$$

The reason for listing these seven cases in this particular order will become clear soon. Now in each of these cases we know, since $\mathscr{C}_2\ (\xi) = 0$, that there are two independent (mutually orthogonal) vectors in four dimensions which are annihilated by the matrix $(\xi_{\mu\nu})$ at the representative point. These pairs of vectors are easily calculated and for the seven situations listed in (4.9) they are, in the same order: e_0, e_5; $e_0 + e_2$, e_5; e_2, e_5; $e_0 + e_2$, $e_1 + e_5$; $e_0 + e_2$, $e_1 - e_5$; $e_0 + e_2$, e_1; e_1, e_2. We see that in case (i) the representative generator $J\ (\xi)$ leaves invariant two (mutually orthogonal) unit time like vectors, so it is a realisation of the possibility tt for the invariant vectors 'in the four-dimensional sense', i. e. in the 0125 space on which SO (2,2) acts. Similarly case (ii) corresponds to the configuration tl; and the remaining ones to ts, ll, ll, ls and ss in that order. (Now we see that the sequence in (4.9) corresponds to dictionary order in the symbols t, l, s interpreted in the four dimensional sense). The appearance of two inequivalent ll configurations, namely $e_0 + e_2$, $e_1 + e_5$ and $e_0 + e_2$, $e_1 - e_5$ is to be noted: they cannot be transformed into one another by any SO (2,2) transformation[42]. In all the other cases, namely tt, tl, ts, ls and ss, any configuration of the concerned type can be transformed via SO (2,2) into the given configuration. The complete list of rank 2 orbits in **SO (2,2)** can now be tabulated. We drop the subscripts 1,2 on the parameters in (4.9), and in cases (iv) and (v) we use the scaling freedom provided by the generators M_1 and N_1 respectively to replace $\frac{1}{2}\,\epsilon$ by ϵ. Thus we arrive at the following table, where in the first column the case number taken from (4.9) is given :

Rank 2 orbit structure in **SO (2,2)** : $\mathscr{C}_2\ (\xi) = 0$:

Case In(4.9)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1\ (\xi)$	Representative Point	Invariant vectors
(i)	$\vartheta_{2,2}\ (tt;\ u)$	$u \neq 0$	u^2	uJ_{12}	e_0, e_5
(ii)	$\vartheta_{22}\ (tl;\ \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon\ (J_{10} + J_{12})$	$e_0 + e_2$, e_5
(iii)	$\vartheta_{2,2}\ (ts;\ v)$	$v > 0$	$-v^2$	vJ_{10}	e_2, e_5

(*table continued on p.* 113)

Case	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vectors
(iv)	$\vartheta_{2,2}(ll;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{10}+J_{12}+J_{50}+J_{52})$	e_0+e_2, e_1+e_5
(v)	$\vartheta'_{22}(ll;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{10}+J_{12}-J_{50}-J_{52})$	e_0+e_2, e_1-e_5
(vi)	$\vartheta_{22}(ls;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{50}+J_{50})$	e_0+e_2, e_1
(vii)	$\vartheta_{22}(ss;u)$	$u\neq 0$	u^2	uJ_{50}	e_1, e_2.

One can see already at the level of rank 2 orbits the extent to which the values of the algebraic invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ fail to fix the orbit to which $J(\xi)$ belongs. If one also compares the patterns of rank 2 orbits in **SO (4), SO (3,1)** and **SO (2,2)** with one another, all of the corresponding groups being defined on a four-dimensional space, one can see a gradual increase in complexity as the metric changes from Euclidean to Lorentzian to de Sitter.

When we turn next to cataloguing the rank 4 orbits in **SO (2,2)** their variety is again vastly greater than with either SO(4) or **SO (3,1)**. Recall that in the latter cases, these orbits can be compactly denoted as $\vartheta_4(u, u')$, $\vartheta_{3,1}(u, v)$ respectively, with uniformly valid expressions for the invariants and representative elements. With **SO(2,2)** the situation will turn out to be very different. One of the aspeets requiring specific attention will be that of finding suitable symbols for the various distinct families of orbit since the labels t,l,s are no longer available, there being no nullvectors for a (ξ_μ) of rank 4.

Going through the table of values of $4\mathscr{C}_2(\xi)$ row by row, we find in the first instance twelve types of SO (2,2) orbits over which $\mathscr{C}_2(\xi)$ does not vanish. As in (4.9), we list these cases now, in the sequence in which they occur in the $\mathscr{C}_2(\xi)$ table, giving in each case the corresponding representative element as a liaear combination of $J_{\mu\nu}$:

$$(0{:}\ tu_2) \rightarrow \tfrac{1}{2}u_2(J_{12}-J_{50}) \tag{i}$$

$$(0;\ sv_2) \rightarrow \tfrac{1}{2}v_2(J_{10}-J_{52}) \tag{ii}$$

$$(tu_1;\ 0) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50}) \tag{iii}$$

$$(tu_1{:}\ tu_2),\ u_1\neq\pm u_2 \rightarrow \tfrac{1}{2}(u_1+u_2)J_{12}+\tfrac{1}{2}(u_1-u_2)J_{50} \tag{iv}$$

$$(tu_1;\ l\epsilon_2) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50})+\tfrac{1}{2}\epsilon_2(J_{10}+J_{12}-J_{50}-J_{52}) \tag{v}$$

$$(tu_1;\ sv_2) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50})+\tfrac{1}{2}v_2(J_{10}-J_{50}-J_{52}) \tag{vi}$$

$$(l\epsilon_1;\ tu_2) \rightarrow \tfrac{1}{2}u_2(J_{12}-J_{50})+\tfrac{1}{2}\epsilon_1(J_{10}+J_{12}+J_{50}+J_{52}) \tag{vii}$$

$$(l\epsilon_1;\ sv_2) \rightarrow \tfrac{1}{2}v_2(J_{10}-J_{52})+\tfrac{1}{2}\epsilon_1(J_{10}+J_{12}+J_{50}+J_{52}) \tag{viii}$$

$$(sv_1; 0) \rightarrow \tfrac{1}{2}v_1 (J_{10} + J_{52}) \qquad \text{(ix)}$$

$$(sv_1; tu_2) \rightarrow \tfrac{1}{2} v_1 (J_{10} + J_{52}) + \tfrac{1}{2} u_2 (J_{12} - J_{50}) \qquad \text{(x)}$$

$$(sv_1 ; l\epsilon_2) \rightarrow \tfrac{1}{2} v_1 (J_{10} + J_{52}) + \tfrac{1}{2} \epsilon_2 (J_{10} + J_{12} - J_{50} - J_{52}) \qquad \text{(xi)}$$

$$(sv_1 ; sv_2),\ v_1 \neq v_2 \rightarrow \tfrac{1}{2} (v_1 + v_2) J_{10} + \tfrac{1}{2} (v_1 - v_2) J_{52}. \qquad \text{(xii)} \ldots(4.10)$$

(We remind ourselves that every $u_i \neq 0$, every $v_i > 0$ and every $\epsilon_i = \pm 1$). Naturally none of these representative elements has any invariant vector. Now our task is to combine sets of these families of orbits judiciously to form larger coherent families, based on the forms of the representative elements. One can see for example that the representative elements in cases (i), (iii) and (iv) combine neatly into the two-parameter family $uJ_{12}+u'J_{50}$ subject to the restrictions $u, u' \neq 0$; here we have identified $\frac{1}{2}(u_1 \pm u_2)$ with u, u' respectively. Let us call this family of orbits as famiiy A, so an individual member of it is written $\vartheta_{2,2}(A; u, u')$. Similarly, cases (ii), (ix) and (xii) in (4.10) have representative elements combining neatly into the expression $vJ_{10} + v'J_{52}$ subject to $v \geqslant | v' | > 0$. These restrictions on v,v' result from identifying them with $\frac{1}{2}(v_1 \pm v_2)$ respectively. This family of orbits will be labelled by the letter B, leading to $\vartheta_{2,2}(B; v, v')$. Thus six of the twelve cases listed in (4.10) are taken care of, leaving six more to be handled, namely (v), (vi), (vii), (viii), (x) and (xi).

Now these six cases split naturally into three pairs: (v) and (vii), (vi) and (x), (viii) and (xi). Within each pair, the relationship is that the representative elements get interchanged by the reversal of the sign of the dimension 5 (and accompanying relabelling of parameters). This discrete operation is an outer automorphism on SO (2,2), not an element in the identity component of SO (2,2) and it amounts to interchanging the SO (2,1) factors in the (local) product SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1). Thus within SO (2,2), it is the interchange $M_a \leftrightarrow N_a$. Taking up fiirst the pair (vi), (x) in (4.10): we replace $u_{1,2} \rightarrow 2u$, $v_{1,2} \rightarrow 2v$, and introduce a sign parameter $\epsilon' = \pm 1$ to distinguish cases (vi) and (x) respectively. Thus we get a combined expression $u(J_{12} + \epsilon' J_{50}) + v (J_{10} - \epsilon' J_{52})$ for the representative element on an orbit of family C, with the parameter conditions $u \neq 0$, $v > 0$, $\epsilon' = \pm 1$. For the pair (v), (vii): we use the scaling freedom via transformations generated by M_1, N_1 to alter $\frac{1}{2}\epsilon_{1,2}$ to $\epsilon_{1,2}$; then with the changes $\epsilon_{1,2} \rightarrow \epsilon$, $u_{1,2} \rightarrow 2u$, and introduction of $\epsilon' = \pm 1$ to distinguish between cases (v) and (vii), we get the representative element $u(J_{12} + \epsilon' J_{50}) + \epsilon (J_{10} + J_{12} - \epsilon' J_{50} - \epsilon' J_{52})$ on an orbit of family D, subject to $u \neq 0$, $\epsilon, \epsilon' = \pm 1$. For the last pair (viii), (xi) in (4.10): by simillar steps we get an element $v (J_{10} - \epsilon' J_{52}) + \epsilon (J_{10} + J_{12} + \epsilon' J_{50} + \epsilon' J_{52})$, $v > 0$, $\epsilon, \epsilon' = \pm 1$, representing an orbit in the family E.

By this reorganisation of the entries in (4.10), the rank 4 orbits in SO (2,2) fall into five major families. Calculation of $\mathscr{C}_1(\xi)$, $\mathscr{C}_2(\xi)$ in each case is straightforward, and the final results are presented in the table on p 115.

Rank 4 orbit structure in SO (2,2):

Cases in (4.10)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative point
(i), (iii), (iv)	$\vartheta_{2,2}(A; u, u')$	$u, u' \neq 0$	$u^2 + u'^2$	$-uu'$	$uJ_{12} + u' J_{50}$
(ii), (ix), (xii)	$\vartheta_{2,2}(B; v, v')$	$v \geqslant \mid v' \mid > 0$	$-v^2 - v'^2$	vv'	$vJ_{10} + v' J_{52}$
(vi), (x)	$\vartheta_{2,2}(C, \epsilon'; u, v)$	$u \neq 0, v > 0,$ $\epsilon' = \pm 1$	$2(u^2 - v^2)$	$-\epsilon'(u^2 + v^2)$	$u(J_{12} + \epsilon' J_{50}) + v(J_{10} - \epsilon' J_{52})$
(v), (vii)	$\vartheta_{2,2}(D, \epsilon'; u, \epsilon)$	$u \neq 0, \epsilon' = \pm 1,$ $\epsilon = \pm 1$	$2u^2$	$-\epsilon' u^2$	$u(J_{12} + \epsilon' J_{50})$ $+ \epsilon(J_{10} + J_{12} - \epsilon' J_{50} + \epsilon' J_{52})$
(viii), (xi)	$\vartheta_{2,2}(E, \epsilon'; v, \epsilon)$	$v > 0, \epsilon' = \pm 1,$ $\epsilon = \pm 1$	$-2v^2$	$-\epsilon' v^2$	$v(J_{10} - \epsilon' J_{52})$ $+ \epsilon(J_{10} + J_{12} + \epsilon' J_{50} + \epsilon' J_{52})$

The complete picture of all **SO (2,2)** orbits is obtained by combining the two tables refering to rank 2 and rank 4 orbits respectively. It may be useful to mention here that in any practical application of these results, the identification of the orbit to which a given $J(\xi) \in$ **SO (2,2)** belongs is best done by splitting it into its M_a and N_a components, checking with the cases in (4.9) or (4.10) as appropriate, and then assingning it to the correct $\vartheta_{2,2}(\ldots)$. The algebraic invariants $\mathscr{C}_1(\xi)$, $\mathscr{C}_2(\xi)$ do not by themselves determine the orbit.

SO (3,2)

For this group all of eqns. (3.21) giving expressions for $J(\xi)$, $\mathscr{C}_1(\xi)\,\zeta_A$ and $\mathscr{C}_2(\xi)$ in the case of **SO (4,1)** can be taken over as they stand, with the understanding that A, B, ... now go aver 0 1 2 3 5 with signature $-+++-$. Moreover eqn. (3.22) also remains valid. The rank of (ξ_{AB}) is either 2 or 4, leading to the existence of 3 or 1 independent invariant vectors. The former corresponds to rank 2 orbits which we take up first.

As with **SO (4,1)**, we first list all ten conceivable configurations of three (mutually orthogonal) invariant vectors in dictionary order: *ttt*, *ttl*, *tts*; *tll*, *tls*; *tss*; *lll*, *lls*; *lss*; *sss*. While with the signature of a $4+1$ space only three of these actually exist, in a $3+2$ space six configurations survive and these are: *tts*, *tls*, *tss*; *lls*, *lss*, *sss*. These have been split into two sets of three configurations each because, as we shall soon see, the first set can be handled at the **SO (2,1)** level, while the second set in reducible to a problem within **SO (2,2)**.

Let $J(\xi) \in$ **SO (3,2)** be of rank 2, and let the null vectors of (ξ_{AB}) be invariantly characterised as being of one of the types *tts*, *tls* or *tss*. In every case we have one time like and one space like vector, mutually orthogonnl and normalised, included in the triad. One can therefore always pass via suitable SO(3,2) transformations to element(s) on the orbit of $J(\xi)$ which leave e_5 and e_3 invariant. Such element(s) then belong to the **SO (2,1)** algebra associated with the dimensions 0, 1, 2. The further separation into three mutually exclusive possibilities corresponds to whether the third invariant vector is of type t, l or s in the 0 1 2 subspace. Therefore the orbit of $J(\xi)$ definitely contains element(s) of one of the following three types: uJ_{12} with $u \neq 0$ or $\epsilon(J_{12}+J_{10})$ with $\epsilon = \pm 1$ or vJ_{10} with $v > 0$. In the first two cases, the sign of u or ϵ can be arranged to be positive, if necessary by making a suitable rotation in the 1-3 plane. This settles the quesition of finding suitable representative elements for rank 2 orbits of types *tts*, *tls* and *tss*.

Turning to the three remaining rank 2 orbits of types *lls*, *lss* and *sss*, we see that in every case there is at least one unit space like vector in the invariant triad. On all such orbits there are then elements which leave e_3 and two other vectors invariant. Such elements therefore lie in **SO (2,2)** associated with the dimensions 0 1 2 5, which has been analysed earlier in this Section; further they are of rank 2 within this **SO(2,2)**.

The classification of such SO (2,2) orbits shows that with the help of SO (2,2) transformations we can pass to elements which, in addition to e_3, leave invariant one of the following four pairs of vectors, depending on the invariant configuration associated with $J(\xi)$:

$$lls \to e_0 + e_2, e_1 + e_5 \text{ or } e_0 + e_2, e_1 - e_5;$$

$$lss \to e_0 + e_2, e_1;$$

$$sss \to e_1, e_2. \qquad \ldots(4.11)$$

However, while the two possible pairs associated with *lls* are inequivalent at the SO(2,2) level, they are transformable into one another by a suitable 1-3 rotation within SO(3,2). Therefore, depending on whether the orbit of $J(\xi)$ is of type *lls*, *lss*, or *sss*, there is a 'unique' element on it leaving $e_0 + e_2, e_1 + e_5, e_3$, or $e_0 + e_2, e_1, e_3$ or e_1, e_2, e_3 respectively invariant; and the form of this element can be taken from the table of rank 2 orbits in SO (2,2), namely case (iv), (vi) or (vii) respectively in that table. Now this table shows that in cases (iv) and (vi) there is a parameter ϵ which can take values ± 1, and these are distinct possibilities within SO (2,2). Similarly in case (vii) of that table the parameter u can be positive or negative. One must naturally examine whether, in view of the greater freedom of transformation available in SO (3,2), the parameter ϵ could be restricted to $+1$, and u to positive values alone. This however cannot be achieved, since it requires (among other things) swiching the sign of J_{50}.

Taking into account the results of the two preceding paragraphs, we can construct a catalogue of all the rank 2 orbits in SO (3,2). It is easily seen that on all such orbits, $\zeta_A = 0$ identically, so $\mathscr{C}_2(\xi) = 0$ as well. These facts are explicity indicated in the following tables :

Rank 2 orbit structure in SO (3,2) : $\zeta_A = \mathscr{C}_2(\xi) = 0$:

Orbit	Parameter range	invariant $\mathscr{C}_1(\xi)$	Representative point	Invariant vectors
$\vartheta_{3,2}(tts; u)$	$u > 0$	u^2	uJ_{12}	e_0, e_3, e_5
$\vartheta_{3,2}(tls)$	—	0	$J_{10} + J_{12}$	$e_0 + e_2, e_3, e_5$
$\vartheta_{3,2}(tss; v)$	$v > 0$	$-v^2$	vJ_{10}	e_2, e_3, e_5
$\vartheta_{3,2}(lls; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{10} + J_{12} + J_{50} + J_{52})$	e_0+e_2, e_1+e_5, e_3
$\vartheta_{3,2}(lss; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{50} + J_{52})$	e_0+e_2, e_1, e_3
$\vartheta_{3,2}(sss; u)$	$u \neq 0$	u^2	uJ_{50}	e_1, e_2, e_3

Let us now turn to our last topic, the analysis of orbits of rank 4 in SO (3,2). Following what is by now a familiar pattern, these orbits in every case can be related to some suitable six-dimensional subalgebra in **SO** (3,2). As we would expect, it will happen that for any rank 4 $J(\xi) \in$ **SO** (3,2), ζ_A is nonvanishing and is the sole nullvector of (ξ_{AB}). The possible orbits therefore split initially into three types, corresponding to ζ_A being of type t, l or s. We can pass via suitable SO (3,2) transformations to elements on the orbit of $J(\xi)$ for which the invariant vector ζ_A is proportional to e_5, $e_5 + e_3$ or e_3 respectively. Such elements must then belong to the **SO** (3,1) subalgebra associated with the dimensions 0 1 2 3, an **E** (2,1) subalgebra (as we shall see), or the **SO** (2,2) subalgebra associated with the dimensions 0 1 2 5. In all cases, we have to deal with rank 4 elements in these subalgebras. Except for the **E** (2,1) case, then, we can draw on previously derived results.

The case when ζ_A is of type t is easiest to handle. Then, among the elements on the orbit of $J(\xi)$ which leave e_5 invariant are some of the form $uJ_{12} + vJ_{03}$ for some $u > 0$, $v \neq 0$. This much follows from the nature of rank 4 orbits in **SO** (3,1). But going beyond this, the freedom to perform rotations in the 0-5 plane shows that we can arrange for v also to be positive. The result is that on any rank 4 orbit in **SO** (3,2) of type t, there is a unique element $uJ_{12} + vJ_{03}$ with both u, $v > 0$.

Next we consider the case when ζ_A is of type l, and ask for the most general $J(\xi)$ for which the only invariant vector is $e_3 + e_5$. Thus (ξ_{AB}) must annihilate only $z^A = (0, 0, 0, 1, 1)$. The equations $\xi_{AB} z^B = 0$ give the following conditions:

$$\xi_{35} = 0,$$

$$\xi_{a3} = -\xi_{a5}, \quad a = 0, 1, 2. \qquad \ldots(4.12)$$

This allows ξ_{ab} and ξ_{a3} to be independent. Using a notation patterned after that of section 3 in dealing with **E** (3), we write $J(\xi)$ as

$$J(\xi) = \xi^a J_a + \alpha^a P_a,$$

$$J_a = \tfrac{1}{2} \epsilon_{abc} J^{bc},$$

$$P_a = J_{a3} + J_{a5},$$

$$\xi^a = -\tfrac{1}{2} \epsilon^{abc} \xi_{bc}, \quad \alpha a = \xi^{a3}. \qquad \ldots(4.13)$$

The latin indices here and handled exactly as in the treatment of SO (2,1) in section 3. J_a and P_a span an **E** (2,1) sub-algebra within **SO** (3,2), i. e. a Poincaré algebra in a $2 + 1$ space, which is the stability group of a light like vector in de Sitter space

$$[J_a, J_b] = \epsilon_{ab}^{\;\;c} J_c,$$

$$[J_a, P_b] = \epsilon_{ab}^{\;\;c} P_c,$$

$$[P_a, P_b] = 0. \qquad \ldots(4.14)$$

Now we seek necessary and sufficient conditions on ξ^c, α^a to ensure that $e_3 + e_5$ is the only null vector for (ξ_{AB}). The condition that (ξ_{AB}) annihilate a general z^A is :

$$\alpha^a z_a = 0,$$

$$\epsilon_{abc} \xi^b z^c - \alpha_a (z^3 - z^5) = 0. \qquad \ldots(4.15)$$

A consequence of these equations is the pair

$$\alpha^a \xi_a z_b = 0,$$

$$\alpha^b \xi_a (z^3 - z^5) = 0. \qquad \ldots(4.16)$$

If $\alpha^a \xi_a \neq 0$, that is sufficient to lead to the desired results on z^A, namely, $z^a = 0$ and $z^3 = z^5$. The necessity of this condition is also eassy to prove. Therefore, on a given rank 4 orbit in **SO (3,2)** of type l. those elements $J(\xi)$ for which the invariant vector is $e_3 + e_5$ are of the form (4.13) with nonvanishing $\alpha^a \xi_a$, and therefore also with nonvanishing α^a and ξ^a.

In finding a natural representative element on such an orbit, we first use the E(2,1) adjoint action to simplify the pair (ξ^a, α^a) as much as possible, then search for further simplification using elements of SO (3,2) outside E (2,1). The adjoint actions by the homogeneous SO (2,1) part of E (2,1) and by the translations in E (2,1) are :

$$(\xi^a, \alpha^a) \rightarrow (\Lambda^a_b \xi^b, \Lambda^a_b \alpha^b),$$

$$\Lambda \in SO(2,1); \qquad \text{(a)}$$

$$(\xi^a,\alpha^a) \rightarrow (\xi^a, \alpha^a + \epsilon^a_{bc} A^b \xi^c). \qquad \text{(b)}\ldots(4.17)$$

The freedom of transformation (4.17a) allows us to put ξ^a into one of several distinct forms; this is then followed by the use of (4.17b) to simplify α^a. Of course, $\alpha^a \xi_a$ remains invariant. It is then seen that the pair (ξ^a, α^a) can be carried by suitable E(2,1) transformations to one of the following mutually exclusive configurations

$$\xi^a, \alpha^a \rightarrow (u, 0, 0), (\alpha, 0, 0), \; u \neq 0, \; \alpha \neq 0;$$

$$\rightarrow (\epsilon, 0, \epsilon), (\alpha, 0, -\alpha), \; \epsilon = \pm 1, \; \alpha \neq 0;$$

$$\rightarrow (0, 0, v), (0, 0, \alpha), \; v > 0, \; \alpha \neq 0. \qquad \ldots(4.18)$$

Now what remains is the action by elements of SO (3,2) outside of E (2,1). Here one can convince oneself that only transformations of consequence are those generated by J_{35}, and these help to normalise α in any one of the cases (4.18) to $\epsilon' = \pm 1$:

$$[J_{35}, J_a] = 0,$$

$$[J_{35}, P_a] = P_a. \qquad \ldots(4.19)$$

The final result is that on a rank 4 orbit in **SO (3,2)** of type l, there is a unique representative element of the following mutually exclusive forms :

$$uJ_0 + \epsilon' P_0 = u\, J_{12} + \epsilon' (J_{03} + J_{05}), \quad u \neq 0, \epsilon' = \pm 1;$$

$$\epsilon\, (J_0 + J_2) + \epsilon' (P_0 - P_2) = \epsilon\, (J_{12} + J_{10}) + \epsilon' (J_{03} + J_{05} - J_{23} - J_{25}),$$
$$\epsilon = \pm 1, \epsilon' = \pm 1;$$

$$vJ_2 + \epsilon' P_2 = vJ_{10} + \epsilon' (J_{23} + J_{25}), v > 0, \epsilon' = \pm 1. \quad .. (4.20)$$

The third and last case of a rank 4 orbit in SO (3,2) is when ζ^A is of type s. If on such an orbit $J(\xi)$ is a point where the invariant vector is e_3, then $J(\xi) \in$ SO (2,2) associated with the dimensions 0 1 2 5. From the table of representative elements on rank 4 orbits in SO (2,2) we know that by SO (2,2) transformations $J(\xi)$ can be brought to one of the following standard froms labelled as in the table :

$$A : uJ_{12} + u'J_{50}, \quad u \neq 0, u' \neq 0;$$

$$B : vJ_{10} + v'J_{52}, \quad v \geqslant |v'| > 0;$$

$$C, \epsilon' : u\, (J_{12} + \epsilon' J_{50}) + v\, (J_{10} - \epsilon' J_{52}), u \neq 0, v > 0, \epsilon' = \pm 1;$$

$$D, \epsilon' : u\, (J_{12} + \epsilon' J_{50}) + \epsilon\, (J_{10} + J_{12} - \epsilon' J_{50} - \epsilon' J_{52}),$$
$$u \neq 0, \epsilon = \pm 1, \epsilon' = \pm 1;$$

$$E, \epsilon' : v\, (J_{10} - \epsilon' J_{52}) + \epsilon\, (J_{10} + J_{12} + \epsilon' J_{50} + \epsilon' J_{52}),$$
$$v > 0, \epsilon = \pm 1, \epsilon' = \pm 1. \quad ...(4.21)$$

Now these various possibilities, inequivalent within SO (2,2), can to some extent be related to one another by suitable SO (3,2) transformations. Thus a rotation of amount π in the 2–3 plane carries: u, u' under A to $-u, u'$; v, v'; under B to $v_1 - v'$: u, v, ϵ' under C to $-u, v, -\epsilon', -\epsilon'$; and a rotation of amount π in the 1-3 plane carries ϵ', u, ϵ under D to $-\epsilon', -u, -\epsilon$. In these four cases, then, we can restrict the ranges of the parameters when finding unique representative points on the concerned SO (3,2) orbits : under A, $u > 0$ and $u' \neq 0$; under B, $v \geqslant v' > 0$; under C, $\epsilon' = +1$ and $u \neq 0$, $v > 0$; under D, $\epsilon' = +1$ and $u \neq 0$, $\epsilon = \pm 1$. The type E in (4.21) does not, however admit such a reduction or havling of distinct possibilities. All we may do to reduce the number of labels is to restrict ϵ' to the value $+1$ but allow v to be nonzero positive or negative. This can be seen by tracing the effect of a rotation of amount π in the 1—3 plane on the generator in the last line of (4.21).

No simplifications beyond those described above are possible by considering elements in SO (3,2) outside SO (2,2).

Collecting all the results pertaining to the three categories t, l, s of rank 4 orbits in SO (3,2) the complete listing of possibilities can be drawn up as on p 121.

The only point of notation here requiring explanation pertains to the second, third and fourth entries. While the first letter l within $\vartheta_{3,2}(\ldots)$ refers to the nature of ζ^A, the second letter t, l or s refers to the nature of the 3-vector ξ^a in the pair (ξ^a, α^a) describing an element in the subalgebra **E** (2,1).

Rank 4 orbit structure in **SO (3,2)** :

Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative point	Invariant vector
$\vartheta_{3,2}(t; u, v)$	$u, v > 0$	$u^2 - v^2$	$-u^2v^2$	$uJ_{12} + vJ_{03}$	e_5
$\vartheta_{3,2}(l;t, u, \epsilon')$	$u \neq 0, \epsilon' = \pm 1$	u^2	0	$uJ_{12} + \epsilon'(J_{03} + J_{05})$	$e_3 + e_5$
$\vartheta_{3,2}(l; l, \epsilon, \epsilon')$	$\epsilon, \epsilon' = \pm 1$	0	0	$\epsilon(J_{10}+J_{12})+\epsilon'(J_{03}+J_{05}-J_{23}-J_{25})$	$e_3 + e_5$
$\vartheta_{3,2}(l; s, v, \epsilon')$	$v > 0, \epsilon' = \pm 1$	$-v^2$	0	$v J_{10} + \epsilon'(J_{23} + J_{25})$	$e_3 + e_5$
$\vartheta_{3,2}(s; A, u, u')$	$u > 0, u' \neq 0$	$u^2 + u'^2$	$u^2 + u'^2$	$uJ_{12} + u' J_{50}$	e_3
$\vartheta_{3,2}(s; B, v, v')$	$v \geq v' > 0$	$-v^2 - v'^2$	$v^2 v'^2$	$vJ_{10} + v' J_{52}$	e_3
$\vartheta_{3,2}(s; C, u, v)$	$u \neq 0, v > 0$	$2(u^2 - v^2)$	$(u^2 + v^2)^2$	$u(J_{12} + J_{50}) + v(J_{10} - J_{52})$	e_3
$\vartheta_{3,2}(s; D, u, \epsilon)$	$u \neq 0, \epsilon = \pm 1$	$2u^2$	u^4	$u(J_{12} + J_{50}) + \epsilon(J_{10} + J_{12} - J_{50} - J_{52})$	e_3
$\vartheta_{3,2}(s; E, v, \epsilon)$	$v \neq 0, \epsilon = \pm 1$	$-2v^2$	v^4	$v(J_{10} - J_{52}) + \epsilon(J_{10} + J_{12} + J_{50} + J_{52})$	e_3

5. Concluding Remarks

In the preceding Sections we have exhausitively classified all the orbits under the adjoint action in each of the Lie algebras SO (p, q) for $p + q \leqslant 5$. Particular care has been taken, in view of the complexity of some of the results, to develop a suggestive and systematric notation. For each orbit, we have calculated the values of the algebraic invariants, displayed a representative element, and described the geometric nature of the latter by listing a complete set of independent vectors invariant under it.

By definition, every orbit in any of the Lie algebras admits a transitive action by the corresponding group G. Therefore it is a realisation of a certain coset space G/H, where H is the subgroups of G leaving invariant the representative point on the orbit. In the case of SO (3), as is well known this can be expressed by

$$\vartheta_3 (u) \simeq \text{SO (3)/SO (2)}. \qquad ...(5.1)$$

In the SO (2,1) each orbit $\vartheta_{2,1}$ $(t; u)$ is a model for SO (2,1)/SO (2); each $\vartheta_{2,1}$ $(s; v)$ realises SO (2,1)/SO (1,1); and the two orbits $\vartheta_{2,1}$ $(l; \epsilon)$ are realisations of SO $(2,1)/H$ where H is a "parabolic" subgroup generated by $J_{12} + J_{10}$. The situation with **SO (3,1)** is actually simpler than with **SO (2,1)**. Here, each of the orbits $\vartheta_{3,1}$ $(ts; u)$, $\vartheta_{3,1}$ $(ss; v)$, $\vartheta_{3,1}$ (u, v) is a realisation of one and the same coset space SO (3,1)/SO (2) $\times$ SO (1,1) where SO (2) is generated by J_{12} and SO (1,1) by J_{03}. The single and somewhat exceptional orbit $\vartheta_{3,1}$ (ls) is the coset space SO $(3,1)/N$ where is a two-parameter abelian group generated by $J_3 - K_1$ and $J_1 + K_3$. For the orbits in the other Lie algebras a similar through sometimes tedious analysis can be carried out by finding the stability group of the representative element in each case.

Wtih the general representation $\vartheta \simeq G/H$, the dimension of an orbit ϑ is that of G minus that of H. It is a general result that dim. ϑ is always even. For both **SO (3)** and **SO (2,1)** it is geometrically clear that each nontrivial orbit is two-dimensional. For **SO (3,1)** all orbits are of dimension four, but the situation is more complicated for both **SO (4)** and **SO (2,2)**, and also for **SO (5)**. The rank 2 orbits ϑ_5 (u) in **SO (5)** are six dimensional, since the stability group of the representative element uJ_{12} is easily seen to be the four-parameter group SO(2) $\times$ SO (3) generated by J_{12}, J_{34}, J_{45}, J_{53}. The "generic" rank 4 orbits ϑ_5 $(u, u') \subset$ **SO (5)** for $u > u' > 0$ are eight dimensional (H generated by J_{12} and J_{34}), but if $u = u'$ the dimension drops to six (H now generated by $J_{12} + J_{43}, J_{12} - J_{43}, J_{23} - J_{41}, J_{31} - J_{42}$). For **SO (4)** as well as for **SO(2,2)**, "most" orbits are four-dimenssonal, but there are some two-dimensional ones as well, in case in the cartesian product representation of an orbit one factor is trivial. Thus for example in the case of **SO (4)** family of orbits ϑ_4 (u, u'), the orbit dimension is a discontinuous function of the parameters u, u', since dim ϑ_4 $(u, u') = 4$ if $u > | u' |$ and dim. ϑ_4 $(u, u') = 2$ if $u = | u' |$. A similar situation occurs in **SO (5)** too.

We must note another source of discontinuity in some families of orbits as we have listed them. Thus while the presence of labelling parameters $\epsilon, \epsilon' = \pm 1$ automatically means that the family of orbits concerned consists of several disjoint pieces, if a parameter u or v is only restricted to be nonzero that too results in there being several disjoint components in the family. It would have made our tables inordinately lengthy if we had insisted that each listed family of orbits form a connected set.

The availability of the vector ζ^A in the five-dimensional groups SO (5), SO (4,1) and SO (3,2) is fortunate since it immediately determines whether rank $J(\xi)$ is 2 or 4. In fact the squares of the various components ζ^A are simply the determinants of the various principal 4 × 4 submatrices within the 5 × 5 matrix (ξ_{AB}), so a vanishing (non-vanishing) ζ^A must imply rank (ξ_{AB}) is 2 (respectively 4). Further for these five-dimensional groups any orbit in the Lie algebra can be studied in the context of some suitable subalgebra since there is always at least one invariant vector. This kind of simplification does not always occur with **SO (4)** and **SO(3,1)**.

We conclude by remarking that the complete set of results obtained for the largest group we have analysed, and indeed the most intricate one, namely SO (3,2), has been used to the fullest extent in a study of a special class of optical fields[10]. We refer here to the action of general first order optical systems on the so-called Gaussian Schell-model beams, in which context the two-fold covering group Sp (4,R) of SO (3,2) plays a primary role. We refer the reader to the appropriate reference for details[10]. It is quite likely that the treatment of squeezed coherent states[1-15,43] two-photon coherent state[16-18], and generally the discussion of processes involving two modes of the photon field will be clarified as a result of our analysis.

The interested reader will find much relevant material on orbits in Sitaram and Tripathy[44], Auslander and Kostant[45] and Kirillor[46].

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38. Throughout the paper, in the interest of brevity, the rank of the matrix $(\xi\cdots)$ will often be called the rank of the generator $J(\xi)$. Null vectors of $(\xi\cdots)$ will also often be called null vectors of $J(\xi)$. Since such vectors are invariant under the one-parameter subgroup generated by $J(\xi)$, they will also be characterised as being "invariant under $J(\xi)$".
39. In parametrising representative elements, u and u' always accompany elliptic SO(2) generrotors, v and v' always multiply hyperbolic SO(1,1) generators, and the sign parameters $\epsilon, \epsilon' = \pm 1$ multiply parabolic generators. These conventions are adopted for all the Lie algebras studied. The u terms (v terms) always contribute positively (negatively) to $\mathscr{C}_1(\xi)$, while the ϵ terms contribnte zero. The possibility of scaling the the coefficient of a parabolic generator to $\epsilon = \pm 1$ is a general feature.
40. It should be emphasized that the SO (3) transformation used here is not an element of any SO (3) subgroup of the SO (5) acting on the indices $A, B, \cdots$ Thus a similar argument is available even when one is dealing with rank 2 generators of SO (4,1) or SO (3,2), except that one loses the positive-definiteness of M in these cases.
41. Of course in the 2+2 space time like (space like) vectors have negative (positive squared norm, through the number of dimensions of each type is two.
42. The two orbits $\vartheta_{2,2}$ $(ll; \epsilon)$ in the following table are mapped onto the two orbits $\vartheta'_{2,2}$ $(ll; \epsilon)$ by the outer automorphism $M_a \leftrightarrow N_a$ of **SO (2,2)**. This operation is used later on in simplifying the catalogue of rank 4 orbits in **SO (2,2)**.
43. The role played by the pseudo-orthogonal group in squeezing becomes transparent when the problem is analysed using the Wigner distribution—R. Simon in *Symmetries in Science II* (Ed. B. Gruber), Plenum, New York, 1987.
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